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On the Annihilator of the Complete Symmetric Form

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Abstract

In this thesis, we study a conjecture concerning the annihilator ideal of a complete symmetric form of odd degree. More precisely, let H_{2d+1} be the complete symmetric form of degree $2d+1$, and consider its annihilator under the apolar action. The conjecture predicts a parity-dependent pattern for the minimal generators of this ideal: when d is odd, no additional minimal generator should appear in degree $d+2$, while when d is even, exactly one such generator should appear.

We prove the conjecture in codimension 3, using a parity argument together with a theorem of Buchsbaum–Eisenbud on grade three Gorenstein ideals. We also prove the conjecture for every $n \geq 3$ when d is odd. Finally, we establish a general reduction result showing that, in the remaining even case, the possible degree $d+2$ generators span a vector space of dimension at most one.

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Contents

1 Introduction

Macaulay inverse systems provide a correspondence between homogeneous forms and graded Artinian Gorenstein algebras. Let

$$R = k[x_1, \dots, x_n] \qquad \text{and} \qquad S = k[X_1, \dots, X_n],$$

where k is a field of characteristic 0. We let R act on S by differentiation, so that x_i acts as $\partial/\partial X_i$. For a homogeneous form $F \in S$, its annihilator is the homogeneous ideal

$$\operatorname{Ann}(F) = \{f \in R : f \circ F = 0\}.$$

The quotient $R/\operatorname{Ann}(F)$ is then an Artinian Gorenstein algebra. Conversely, every graded Artinian Gorenstein quotient of R arises in this way. This correspondence makes it possible to study a form through the commutative algebra of its annihilator; see [IK99, Chapter 1] for a general account of inverse systems.

An important problem is to understand the minimal generators and the minimal free resolution of $\operatorname{Ann}(F)$. These record substantially more information than the Hilbert function of the quotient. For a generic form F of fixed degree, the algebra $R/\operatorname{Ann}(F)$ is compressed, meaning that its Hilbert function is as large as possible among Artinian Gorenstein algebras with the same socle degree and embedding dimension [IK99, Proposition 3.12]. Generic forms are useful because their behavior is typical, but they are not usually given by an explicit formula. It is therefore natural to look for explicit and structured forms whose annihilator algebras retain properties of the generic case.

The forms considered in this thesis are the complete symmetric forms

$$H_e = \sum_{\substack{\underline{a} \in \mathbb{N}^n \\ |\underline{a}| = e}} X^{\underline{a}},$$

that is, the sums of all monomials of degree e in n variables. Unlike a generic form, H_e is invariant under every permutation of its variables and has all its coefficients equal to one. Despite this strong symmetry, its annihilator algebra behaves in several respects like the algebra associated with a generic form.

Boij, Migliore, Miró-Roig, and Nagel proved that $R/\operatorname{Ann}(H_e)$ is compressed and has the strong Lefschetz property, and they gave a description of its annihilator ideal [BMMRN22]. The compressedness of these algebras was also obtained independently by Gesmundo and Landsberg through the study of spaces of partial derivatives [GL19]. Thus the complete symmetric forms give an explicit family attaining the maximal Hilbert function. The strong Lefschetz property places this family within the broader study of multiplication maps in Artinian algebras; for background on Lefschetz

properties, see [HMM+13].

More recently, Boij, Duarte, and Lundqvist determined a minimal Gröbner basis of $\mathrm{Ann}(H_e)$ with respect to degree reverse lexicographic order [BDL25]. They showed, in particular, that its initial ideal agrees with the initial ideal of the annihilator of a generic form of degree e. Consequently, the two annihilator algebras have the same Hilbert function and the same leading monomials in their minimal Gröbner bases. This result gives a particularly strong connection between the explicit form H_e and the generic case.

However, a minimal Gröbner basis of an ideal need not be a minimal generating set of the ideal itself. Some elements that are required to generate the initial ideal may be consequences of lower-degree generators before leading terms are taken. Hence neither the Hilbert function nor the minimal generators of the initial ideal determine the number of minimal generators of $\mathrm{Ann}(H_e)$. It is precisely this distinction that gives rise to the conjecture studied in this thesis.

We now focus on complete symmetric forms of odd degree. Write the degree as $2d+1$ and set

$$I_{2d+1} = \mathrm{Ann}(H_{2d+1}).$$

The results of [BMMRN22, BDL25] show that I_{2d+1} is generated in degrees $d+1$ and $d+2$. Its degree $d+1$ minimal generators are known explicitly, and the Gröbner basis in [BDL25] provides a collection of candidates for additional minimal generators in degree $d+2$. The number of such additional generators is measured by

$$\dim_k \frac{[I_{2d+1}]_{d+2}}{R_1[I_{2d+1}]_{d+1}}. \tag{1}$$

Indeed, the denominator consists of all degree $d+2$ elements obtained by multiplying elements of $[I_{2d+1}]_{d+1}$ by linear forms. Thus the quotient records exactly which degree $d+2$ elements are genuinely new minimal generators.

For $n \geq 3$, computations in [BDL25] led Boij, Duarte, and Lundqvist to conjecture that the dimension in (1) depends only on the parity of d. Their Conjecture 3.7 predicts that

$$\dim_k \frac{[I_{2d+1}]_{d+2}}{R_1[I_{2d+1}]_{d+1}} = \begin{cases} 0, & \text{if } d \text{ is odd,} \\ 1, & \text{if } d \text{ is even.} \end{cases} \tag{2}$$

Equivalently, when d is odd, the known generators in degree $d+1$ generate the entire annihilator. When d is even, exactly one additional minimal generator should occur in degree $d+2$. A precise statement, including explicit formulas for the generators, is given in Conjecture 3.3.5.

Here the distinction between odd and even refers to the parity of d, while the degree $2d+1$ of the complete symmetric form remains odd. There is no analogous problem for complete symmetric

forms of even degree. Indeed, when $e = 2d$, the description of [BMMRN22] shows that $\mathrm{Ann}(H_{2d})$ is equigenerated in degree $d+1$, so its minimal generators are already completely determined.

The parity dependence in (2) is not visible from the Hilbert function or from the initial ideal, both of which agree with those of the generic case. The conjecture therefore asks for finer information about the structure of the annihilator. It also provides a concrete setting in which one can compare an explicit compressed Artinian Gorenstein algebra with the algebra associated with a generic form. Understanding the minimal generators is the first step toward understanding the graded Betti numbers and the minimal free resolution of the annihilator.

Computational experimentation in Macaulay2 played an important role in the development of this thesis [GS]. Computations for small values of n and d were used to examine the candidate generators, test possible relations among them, and identify patterns that suggested approaches to the conjecture. These experiments did not replace the theoretical arguments, but they provided evidence, helped rule out unsuccessful approaches, and guided the formulation of the reduction lemmas used in the proofs.

The main results of this thesis are the following.

(i) For every $n \geq 3$ and every d, the quotient in (1) has dimension at most one. Although the Gröbner basis yields many possible generators in degree $d+2$, their classes are therefore all controlled by a single class; see Corollary 4.2.1.

(ii) When $n = 3$, Conjecture 3.3.5 holds for every d; see Theorem 4.2.2. The proof combines the preceding dimension bound with the theorem of Buchsbaum–Eisenbud that a grade three Gorenstein ideal has an odd number of minimal generators [BE77, Corollary 2.2].

(iii) When d is odd, Conjecture 3.3.5 holds for every $n \geq 3$; see Corollary 4.2.3. This is proved by using the three-variable result inside a suitable three-variable subring and then applying the general dimension bound.

Consequently, the only unresolved case is

$$n > 3 \qquad \text{and} \qquad d \text{ even}.$$

In this remaining case, the dimension bound reduces the conjecture to proving that one distinguished class is nonzero. Thus the original collection of candidate generators is replaced by a single concrete nonvanishing problem.

The remainder of the thesis is organized as follows. Section 2 develops the background needed to formulate and prove the main results. We begin with graded rings, modules, and homogeneous ideals. In particular, the graded version of Nakayama's lemma identifies the minimal generators of a homogeneous ideal with a basis of $I/\mathfrak{m}I$; its degree-wise form explains the quotient appearing in (1). We then introduce Hilbert functions, which measure the dimensions of the graded pieces of an

algebra, and Gröbner bases, which allow an ideal to be replaced by a monomial initial ideal without changing its Hilbert function.

The remainder of Section 2 specializes this material to Artinian Gorenstein algebras. Their perfect pairing gives the symmetry of the Hilbert function and, in three variables, the structure theorem of Buchsbaum–Eisenbud supplies the parity statement used in the proof of the codimension-three case. We then develop Macaulay inverse systems and the apolar action, which translate the study of the form H_e into the study of the quotient $R/\operatorname{Ann}(H_e)$. The section concludes with the weak and strong Lefschetz properties, needed to place the known results about complete symmetric forms in their appropriate algebraic setting.

Section 3 studies the complete symmetric forms and their annihilators. We first develop the results of [BMMRN22] showing that the associated Artinian Gorenstein algebras are compressed and strong Lefschetz and describing their annihilator ideals. We then develop the Gröbner-basis results of [BDL25], compare the resulting initial ideal with that of a generic form, and arrive at Conjecture 3.3.5. Proofs of the parts of these results used later are included so that the notation and the origin of the candidate generators are explicit.

Section 4 contains the new results of the thesis. A sequence of reduction lemmas first shows that all candidate generators in degree $d+2$ are scalar multiples of one class modulo the degree $d+1$ generators. This proves the general dimension bound. We then use the Buchsbaum–Eisenbud parity theorem to settle the conjecture in three variables, and use the odd three-variable case to prove the conjecture for every $n \geq 3$ when d is odd. The section ends by isolating the remaining even case. Section 5 summarizes the results and discusses possible approaches to this nonvanishing problem and to the comparison with the minimal free resolution of a generic Artinian Gorenstein algebra. The appendix records selected Macaulay2 computations used to investigate the conjecture and illustrates the computational methods employed in the thesis.

2 Preliminaries

2.1 Graded algebras

The basic terminology concerning graded rings, graded modules, and homogeneous ideals in the first part of this subsection follows [Eis95, Section 1.5]. We include the details needed later in order to fix notation.

We begin with the definition of a graded ring, which underlies the notion of a graded algebra.

Definition 2.1.1. A graded ring is a ring R together with a direct sum decomposition

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \text{ as abelian groups,}$$

such that $R_i R_j \subset R_{i+j}$ for $i, j \geq 0$.

A graded k-algebra is a graded ring whose graded pieces are k-vector spaces and whose multiplication is compatible with the grading. Throughout this thesis, k will denote a field of characteristic 0, and we will mainly work with the polynomial ring

$$R = k[x_1, \ldots, x_n].$$

We use the standard grading

$$R = \bigoplus_{i \geq 0} R_i,$$

where R_i is the k-vector space spanned by the monomials of total degree i. We call a polynomial in R_i a homogeneous polynomial of degree i.

For elements $v_1, \ldots, v_r$ of a k-vector space, we write $\langle v_1, \ldots, v_r \rangle_k$ for their k-linear span. When the field is clear, we omit the subscript k.

We will also need the corresponding notion for modules.

Definition 2.1.2. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring. An R-module M is called graded if it has a direct sum decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

such that

$$R_i M_j \subseteq M_{i+j}$$

for all $i \geq 0$ and all $j \in \mathbb{Z}$. The module M_i is called the degree i part of M.

Since $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field, it is Noetherian by Hilbert's basis theorem [HH11, Corollary 2.1.9]. Hence every ideal $I \subseteq R$ is finitely generated. This means that

there exist polynomials $f_1, \ldots, f_r \in I$ such that

$$I = (f_1, \ldots, f_r).$$

Such a set $\{f_1, \ldots, f_r\}$ is called a generating set of I. A generating set is called minimal if no proper subset of it generates I.

For homogeneous ideals, it is natural to consider generating sets consisting of homogeneous polynomials. We first recall that a homogeneous polynomial is a polynomial in which every term has the same degree.

Definition 2.1.3. An ideal $I \subseteq R$ is called homogeneous if it can be generated by homogeneous polynomials.

This does not mean that every element of a homogeneous ideal is homogeneous. For example, the ideal $(xy, x^3) \subseteq k[x, y]$ is homogeneous, but the element $xy + x^3 \in (xy, x^3)$ is not homogeneous.

If $I \subseteq R$ is any ideal, we may define

$$[I]_i = I \cap R_i.$$

However, an arbitrary ideal does not necessarily decompose as the direct sum of these pieces. For example, if $I = (x + 1) \subseteq k[x]$, then $x + 1 \in I$, but neither of its homogeneous components x and 1 lies in I.

For homogeneous ideals, the situation is different.

Proposition 2.1.4. *Let $I \subseteq R$ be a homogeneous ideal. Then*

$$I = \bigoplus_{i \geq 0} [I]_i,$$

where $[I]_i = I \cap R_i$.

Proof. Since $R = \bigoplus_{i \geq 0} R_i$, every polynomial $f \in R$ has a unique decomposition $f = f_0 + f_1 + \cdots + f_m$, where $f_i \in R_i$. We need to show that if $f \in I$, then each homogeneous component f_i also lies in I. Since I is homogeneous and R is Noetherian, I has a finite homogeneous generating set. Thus we may write $I = (g_1, \ldots, g_r)$, where each g_j is homogeneous. If $f \in I$, then

$$f = \sum_{j=1}^{r} h_j g_j$$

for some $h_j \in R$. Write each h_j as a sum of homogeneous components,

$$h_j = \sum_t h_{j,t}.$$

Then

$$f = \sum_{j,t} h_{j,t} g_j.$$

Write $\deg(g_j) = d_j$. Since $h_{j,t}$ is homogeneous of degree t, the product $h_{j,t}g_j$ is homogeneous of degree $t + d_j$. Therefore the degree i component of f is

$$f_i = \sum_{\substack{j,t \\ t+d_j=i}} h_{j,t} g_j.$$

Each term in this sum belongs to I, so $f_i \in I$. Since also $f_i \in R_i$, we have $f_i \in [I]_i$.

Thus, every element of I is a finite sum of elements from the spaces $[I]_i$. The sum is direct because the grading of R is direct. Therefore

$$I = \bigoplus_{i \geq 0} [I]_i.$$

□

Thus, by Proposition 2.1.4, if $I \subseteq R$ is a homogeneous ideal, then I decomposes into its homogeneous parts. Hence, we get a grading of homogeneous ideals

$$I = \bigoplus_{i \geq 0} [I]_i.$$

The quotient $A = R/I$ then inherits a grading

$$A = \bigoplus_{i \geq 0} A_i, \quad A_i = R_i/[I]_i.$$

In this way, R/I becomes a graded k-algebra.

We now discuss minimal generators of homogeneous ideals. This notion is central in the thesis, since Conjecture 3.7 concerns whether new minimal generators appear in a particular degree.

Let

$$\mathfrak{m} = (x_1, \ldots, x_n) \subseteq R$$

be the irrelevant maximal ideal. The polynomial ring R is not a local ring, but it is standard graded and $\mathfrak{m}$ is its unique homogeneous maximal ideal. Thus, for finitely generated graded R-modules, one has a graded version of Nakayama's lemma; see [AM16, Proposition 2.6] for the usual Nakayama lemma.

In the case of a homogeneous ideal $I \subseteq R$, viewed as a graded R-module, this says that homogeneous elements $f_1, \ldots, f_r \in I$ generate I if and only if their images generate

$$I/\mathfrak{m}I$$

as a k-vector space. Thus the quotient $I/\mathfrak{m}I$ measures which elements of I are needed as generators and which ones are obtained from lower-degree elements by multiplication with positive-degree forms.

The following standard characterization is the key point.

Proposition 2.1.5. *Let $I \subseteq R$ be a homogeneous ideal, and let $f_1, \dots, f_r \in I$ be homogeneous elements. Then $f_1, \dots, f_r$ generate I if and only if their images generate $I/\mathfrak{m}I$ as a k-vector space.*

Moreover, $f_1, \dots, f_r$ form a minimal homogeneous generating set of I if and only if their images form a k-basis of $I/\mathfrak{m}I$.

Proof. If $f_1, \dots, f_r$ generate I, then their images clearly generate $I/\mathfrak{m}I$.

Conversely, suppose that the images of $f_1, \dots, f_r$ generate $I/\mathfrak{m}I$, and let

$$J = (f_1, \dots, f_r) \subseteq I.$$

We show that $J = I$. Let $f \in I$ be homogeneous of degree i. Since the images of the f_j's generate $I/\mathfrak{m}I$, the class of f can be written as a k-linear combination of the classes of the f_j's. Therefore

$$f - g \in \mathfrak{m}I$$

for some $g \in J$. Taking degree i parts, we get

$$f - g \in [\mathfrak{m}I]_i.$$

Every element of $[\mathfrak{m}I]_i$ is a sum of products of linear forms with elements of $[I]_{i-1}$. Thus

$$[\mathfrak{m}I]_i = R_1[I]_{i-1}.$$

By induction on i, the elements of $[I]_{i-1}$ lie in J, and hence $R_1[I]_{i-1} \subseteq J$. Therefore $f - g \in J$. Since $g \in J$, it follows that $f \in J$. Hence $I = J$.

Finally, a generating set is minimal precisely when no element can be removed while still generating I. By the first part, this is equivalent to saying that no image can be removed while still spanning $I/\mathfrak{m}I$. Since $I/\mathfrak{m}I$ is a k-vector space, this is equivalent to the images forming a k-basis. □

This proposition shows that the minimal number of homogeneous generators of I is well-defined and equals

$$\dim_k I/\mathfrak{m}I.$$

Since I is graded, the quotient $I/\mathfrak{m}I$ is also graded. Its degree i part is

$$(I/\mathfrak{m}I)_i = [I]_i/[\mathfrak{m}I]_i.$$

Moreover,

$$[\mathfrak{m}I]_i = R_1[I]_{i-1},$$

where $R_1[I]_{i-1}$ denotes the k-vector space spanned by products ℓf, with $\ell \in R_1$ and $f \in [I]_{i-1}$. Therefore

$$(I/\mathfrak{m}I)_i = [I]_i/R_1[I]_{i-1}.$$

We obtain the following degree-wise characterization.

Corollary 2.1.6. *Let $I \subseteq R$ be a homogeneous ideal. The number of minimal generators of I in degree i is*

$$\dim_k \frac{[I]_i}{R_1[I]_{i-1}}.$$

Equivalently, a homogeneous element $f \in [I]_i$ contributes a new minimal generator in degree i if and only if

$$f \notin R_1[I]_{i-1}.$$

Proof. This follows immediately from Proposition 2.1.5 and the preceding discussion. □

2.2 Hilbert functions

When studying graded algebraic structures, it is useful to keep track of the dimensions of their graded pieces. The Hilbert function does this by assigning to each degree the dimension of the corresponding homogeneous component. We use the following definition from [Eis95, p. 42].

Definition 2.2.1. Let M be a finitely generated graded module over $R = k[x_1, \ldots, x_n]$, with grading

$$M = \bigoplus_{i \in \mathbb{Z}} M_i.$$

The Hilbert function of M is the numerical function

$$\mathrm{HF}_M(i) = \dim_k M_i.$$

Although Definition 2.2.1 is stated for graded modules, it applies in particular to the quotient algebras considered in this thesis. Indeed, if $I \subseteq R$ is a homogeneous ideal and $A = R/I$, then A is a graded k-algebra, but also a finitely generated graded R-module. Therefore its Hilbert function is defined by

$$\mathrm{HF}_A(i) = \dim_k A_i.$$

More generally, every finitely generated standard graded k-algebra can be realized as a quotient of a polynomial ring by a homogeneous ideal.

As a first example, consider the polynomial ring $R = k[x_1, \ldots, x_n]$ with the standard grading. The degree i piece R_i is spanned by all monomials of total degree i. Hence $\mathrm{HF}_R(i)$ is the number of monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 + \cdots + a_n = i$. By the stars and bars formula, this number is

$$\mathrm{HF}_R(i) = \binom{n+i-1}{i}.$$

We will mostly be interested in graded k-algebras of the form R/I, where $I \subseteq R$ is a homogeneous ideal. Since

$$(R/I)_i = R_i/[I]_i,$$

we have

$$\mathrm{HF}_{R/I}(i) = \dim_k (R/I)_i = \dim_k R_i - \dim_k [I]_i.$$

Thus the Hilbert function of R/I measures how many degree i forms remain after quotienting by the degree i part of I.

Example 2.2.2. Let

$$R = k[x, y] \quad \text{and} \quad A = R/I,$$

where $I = (x^2 + y^2, y^3)$. Since I is homogeneous, the quotient A is graded. We compute its Hilbert function using

$$\mathrm{HF}_A(i) = \dim_k (R/I)_i = \dim_k R_i - \dim_k [I]_i.$$

In degree 0, we have $R_0 = \langle 1 \rangle$ and $[I]_0 = 0$. Hence

$$\mathrm{HF}_A(0) = \dim_k R_0 - \dim_k [I]_0 = 1 - 0 = 1.$$

In degree 1, we have $R_1 = \langle x, y \rangle$ and $[I]_1 = 0$. Hence

$$\mathrm{HF}_A(1) = \dim_k R_1 - \dim_k [I]_1 = 2 - 0 = 2.$$

In degree 2, we have $R_2 = \langle x^2, xy, y^2 \rangle$. The degree 2 part of I is $[I]_2 = \langle x^2 + y^2 \rangle$, so

$$\mathrm{HF}_A(2) = \dim_k R_2 - \dim_k [I]_2 = 3 - 1 = 2.$$

In degree 3, we have $R_3 = \langle x^3, x^2y, xy^2, y^3 \rangle$. The degree 3 part of I is generated by the degree 3 multiples of the generators of I:

$$[I]_3 = \langle x(x^2 + y^2), y(x^2 + y^2), y^3 \rangle = \langle x^3 + xy^2, x^2y + y^3, y^3 \rangle.$$

These three forms are linearly independent, so

$$\mathrm{HF}_A(3) = \dim_k R_3 - \dim_k [I]_3 = 4 - 3 = 1.$$

In degree 4 and higher, every monomial lies in I, so $A_i = 0$ for all $i \geq 4$. Therefore the Hilbert function is

$$\mathrm{HF}_A = (1, 2, 2, 1).$$

Here the notation means that

$$\mathrm{HF}_A(0) = 1, \quad \mathrm{HF}_A(1) = 2, \quad \mathrm{HF}_A(2) = 2, \quad \mathrm{HF}_A(3) = 1,$$

and $\mathrm{HF}_A(i) = 0$ for all $i \geq 4$.

2.3 Gröbner bases and initial ideals

Gröbner bases provide a way to study ideals in a polynomial ring by replacing them with monomial ideals. This is useful because monomial ideals are controlled by divisibility of monomials and are therefore easier to understand combinatorially. Before defining initial ideals and Gröbner bases, we first recall what monomial ideals are.

The standard definitions and results reviewed in this subsection are drawn primarily from [HH11, Chapters 1 and 2] and [CLO98, Chapter 1, Sections 2–3]. In particular, the definitions of a monomial order, lexicographic order, graded lexicographic order, and graded reverse lexicographic order are taken from [CLO98, Chapter 1, Section 2]. We include proofs of the specific consequences that will be used later in the thesis.

Definition 2.3.1. An ideal $J \subseteq R = k[x_1, \ldots, x_n]$ is called a monomial ideal if it can be generated by monomials.

If

$$J = (x^{\alpha_1}, \ldots, x^{\alpha_r})$$

is a monomial ideal, then a monomial x^β lies in J if and only if x^{α_j} divides x^β for some j. Equivalently, $\beta_i \geq \alpha_{j,i}$ for all i and for some j. Thus membership in a monomial ideal is determined by divisibility.

By [HH11, Theorem 1.1.2], the monomials contained in J form a k-basis of J. Consequently, if

$$f = \sum_\alpha c_\alpha x^\alpha \in R,$$

then uniqueness of the expansion of f in the monomial basis of R gives

$$f \in J \quad \Longleftrightarrow \quad x^\alpha \in J \text{ for every } \alpha \text{ such that } c_\alpha \neq 0.$$

This equivalence is also stated explicitly in [HH11, Corollary 1.1.3].

Definition 2.3.2. Let $J \subseteq R$ be a monomial ideal. A monomial x^α is called standard with respect to J if

$$x^\alpha \notin J.$$

The following proposition records the key property of standard monomials.

Proposition 2.3.3. *Let $J \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal. The residue classes of the standard monomials with respect to J form a k-basis of R/J.*

Proof. Since J is a monomial ideal, it is spanned as a k-vector space by the monomials it contains. Every polynomial $f \in R$ can be written uniquely as a finite k-linear combination of monomials. Modulo J, all monomials lying in J vanish. Hence every class in R/J can be represented by a k-linear combination of standard monomials. Thus the residue classes of the standard monomials span R/J.

It remains to show that they are linearly independent. Suppose that

$$\sum_\alpha c_\alpha x^\alpha$$

is a k-linear combination of standard monomials whose class is zero in R/J. Then

$$\sum_\alpha c_\alpha x^\alpha \in J.$$

But J is spanned by monomials lying in J, while every x^α appearing in the sum is standard. Since the monomials of R are k-linearly independent, this is possible only if all $c_\alpha = 0$. Therefore the residue classes of the standard monomials are linearly independent. □

Example 2.3.4. Let

$$J = (x^2, xy, y^3) \subseteq k[x, y].$$

Then $x^2y^4 \in J$, since it is divisible by x^2, while $x \notin J$, since it is not divisible by any of x^2, xy, or y^3. The standard monomials with respect to J are

$$1,\ x,\ y,\ y^2.$$

Hence these monomials form a k-basis of R/J.

The goal of an initial ideal is to associate a monomial ideal to an arbitrary ideal $I \subseteq R$. To do this, one must choose which monomial of a polynomial should be regarded as its largest monomial. This is done using a monomial order.

Definition 2.3.5. A monomial order on the polynomial ring $k[x_1, \ldots, x_n]$ is a relation $>$ defined on the set of monomials x^α that satisfies the following properties:

(i) $>$ is a total order on the set of monomials.

(ii) If $x^\alpha > x^\beta$, then $x^\alpha x^\gamma > x^\beta x^\gamma$ for every monomial x^γ.

(iii) Every nonempty set of monomials has a smallest element with respect to $>$.

The third condition implies, in particular, that 1 is the smallest monomial. Indeed, if some monomial $x^\alpha \neq 1$ satisfied $x^\alpha < 1$, then multiplying repeatedly by x^α would give an infinite descending chain

$$1 > x^\alpha > x^{2\alpha} > x^{3\alpha} > \cdots,$$

contradicting condition (iii).

Three common examples of monomial orders are lexicographic order, graded lexicographic order, and graded reverse lexicographic order.

Definition 2.3.6 (Lexicographic Order)**.** Fix an ordering of the variables $x_1 > \cdots > x_n$. For monomials x^α and x^β, we say

$$x^\alpha >_{\text{lex}} x^\beta$$

if, in the difference $\alpha - \beta \in \mathbb{Z}^n$, the leftmost nonzero entry is positive.

Lexicographic order is one of the simplest monomial orders and, as stated in [CLO98, p. 8], is analogous to the ordering of words used in dictionaries. Since grading plays a central role in what follows, we also introduce two monomial orders that compare total degree first.

Definition 2.3.7 (Graded Lexicographic Order)**.** Let x^α and x^β be monomials in $k[x_1, \ldots, x_n]$. We say $x^\alpha >_{grlex} x^\beta$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$, and $x^\alpha >_{lex} x^\beta$.

Definition 2.3.8 (Graded Reverse Lexicographic Order)**.** Let x^α and x^β be monomials in $k[x_1, \ldots, x_n]$. We say $x^\alpha >_{grevlex} x^\beta$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$, and in the difference $\alpha - \beta \in \mathbb{Z}^n$, the rightmost nonzero entry is negative.

All three orders depend on the chosen ordering of the variables. Unless otherwise stated, we use

$$x_1 > x_2 > \cdots > x_n.$$

Once a monomial order has been fixed, every nonzero polynomial has a distinguished largest monomial.

Definition 2.3.9. For a nonzero polynomial $f \in k[x_1, \ldots, x_n]$, the initial monomial, denoted $\mathrm{in}_>(f)$, is the largest monomial appearing in f with respect to the chosen monomial order. We call the coefficient of $\mathrm{in}_>(f)$ in f the leading coefficient.

By convention, we set $\mathrm{in}_>(0) = 0$. If c is the leading coefficient of f, then $f = c\,\mathrm{in}_>(f) + g$, where every monomial appearing in g is smaller than $\mathrm{in}_>(f)$. Since k is a field, one may divide by c and replace f by the monic polynomial $c^{-1}f$ when needed. This observation is one of the basic ingredients in the multivariable division algorithm: one cancels leading monomials step by step until no further cancellation is possible.

Example 2.3.10. Let

$$I = (x_1x_3 + x_2^2,\ x_1x_3^2 + x_2^2x_3) \subseteq k[x_1, x_2, x_3].$$

We compare the leading monomials of its generators under two different monomial orders.

With graded lexicographic order $x_1 > x_2 > x_3$, we have

$$\mathrm{in}_{\mathrm{grlex}}(x_1x_3 + x_2^2) = x_1x_3, \quad \mathrm{in}_{\mathrm{grlex}}(x_1x_3^2 + x_2^2x_3) = x_1x_3^2.$$

With graded reverse lexicographic order $x_1 > x_2 > x_3$, we instead have

$$\mathrm{in}_{\mathrm{grevlex}}(x_1x_3 + x_2^2) = x_2^2, \quad \mathrm{in}_{\mathrm{grevlex}}(x_1x_3^2 + x_2^2x_3) = x_2^2x_3.$$

Thus the two graded orders can select different leading monomials.

We now define the initial ideal.

Definition 2.3.11. Let $I \subseteq k[x_1, \ldots, x_n]$ be a nonzero ideal. The initial ideal of I with respect to a monomial order $>$ is

$$\mathrm{in}_>(I) = (\mathrm{in}_>(f) : f \in I).$$

Although the definition ranges over all elements of I, the same ideal is generated by the initial monomials of finitely many nonzero elements of I. More precisely, Dickson's lemma implies that every monomial ideal has a finite minimal set of monomial generators; see [HH11, Theorem 2.1.1]. Let $m_1, \ldots, m_s$ be the minimal monomial generators of $\mathrm{in}_>(I)$. Since $\mathrm{in}_>(I)$ is generated by the initial monomials of elements of I, for each m_j there is an element $g_j \in I$ such that $\mathrm{in}_>(g_j)$ divides m_j. The minimality of m_j then implies that $m_j = \mathrm{in}_>(g_j)$. This application to initial ideals is also stated explicitly in [HH11, Section 2.1.3]. Consequently,

$$\mathrm{in}_>(I) = (\mathrm{in}_>(g_1), \ldots, \mathrm{in}_>(g_s)).$$

This observation motivates the definition of a Gröbner basis, which is a finite set of elements of I whose initial monomials generate the full initial ideal.

Definition 2.3.12. Let I be a nonzero ideal in R. A finite set of nonzero polynomials $\{g_1, \ldots, g_s\}$ with each $g_i \in I$ is said to be a Gröbner basis of I with respect to $>$ if the initial ideal $\mathrm{in}_>(I)$ of I is generated by the monomials $\mathrm{in}_>(g_1), \ldots, \mathrm{in}_>(g_s)$.

Gröbner bases are important not only because they define initial ideals, but also because they make ideal membership computable. Given an ordered list of polynomials, one has a multivariable division algorithm, which divides a polynomial by that list and produces a remainder. For an arbitrary generating set, the remainder may depend on the chosen order of the generators and need not give a reliable ideal membership test.

The situation is better for Gröbner bases. If $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I, then division by $g_1, \ldots, g_s$ gives a normal form for each polynomial $f \in R$. In particular,

$$f \in I \quad \Longleftrightarrow \quad \text{the remainder of } f \text{ on division by } g_1, \ldots, g_s \text{ is zero.}$$

Thus Gröbner bases turn ideal membership into an effective computation. For the multivariable division algorithm and this ideal-membership criterion, see [CLO98, Chapter 1, Sections 2–3].

There is also an algorithm for constructing Gröbner bases. Buchberger's algorithm starts from a finite generating set of an ideal and repeatedly adds new polynomials until the resulting set is a Gröbner basis. We will not need the details of the algorithm here, but it explains why Gröbner bases are central in computational commutative algebra; see [CLO98, Chapter 1, Section 3] or [HH11, Chapter 2].

The following example illustrates that a Gröbner basis need not have the same number of elements as the generating set one starts with.

Example 2.3.13. Let

$$I = (xz + y^2,\ xy) \subseteq k[x, y, z].$$

This ideal is generated by two polynomials. Note that $x(xz + y^2) - y(xy) = x^2z \in I$, thus we know that $\mathrm{in}(I)$ has to contain x^2z even though it is not a inital monomial of the generators of I. With graded reverse lexicographic order $x > y > z$, a Gröbner basis computation gives

$$\{y^2 + xz,\ xy,\ x^2z\}.$$

Thus

$$\mathrm{in}_{\mathrm{grevlex}}(I) = (y^2,\ xy,\ x^2z).$$

This example shows that a Gröbner basis may contain more elements than the original generating set of the ideal. The computation was carried out in Macaulay2; the code is included in Appendix A.1.

Since $\mathrm{in}_>(I)$ is a monomial ideal, its standard monomials form a k-basis of $R/\mathrm{in}_>(I)$ by Propo-

sition 2.3.3. A fundamental result of Gröbner basis theory says that the same standard monomials also form a k-basis of R/I; see [HH11, Chapter 2].

One important consequence is that an ideal and its initial ideal have the same Hilbert function.

Proposition 2.3.14. *Let $I \subseteq R$ be a homogeneous ideal and let $>$ be a monomial order. Then*

$$\mathrm{HF}_{R/I}(i) = \mathrm{HF}_{R/\operatorname{in}_>(I)}(i)$$

for every $i \geq 0$.

Proof. The standard monomials with respect to $\operatorname{in}_>(I)$ form a k-basis of both R/I and $R/\operatorname{in}_>(I)$. Therefore the degree i standard monomials give bases for the degree i pieces of both quotients. Hence the two Hilbert functions agree in every degree. □

The Hilbert function can also be used to verify that a proposed generating set is a Gröbner basis.

Corollary 2.3.15. *Let $I \subseteq R$ be a homogeneous ideal, and let*

$$\{g_1, \ldots, g_s\} \subseteq I$$

be a finite set of homogeneous polynomials. Set

$$J = (\operatorname{in}_>(g_1), \ldots, \operatorname{in}_>(g_s)).$$

If

$$\mathrm{HF}_{R/J}(i) = \mathrm{HF}_{R/I}(i)$$

for all $i \geq 0$, then $\{g_1, \ldots, g_s\}$ is a Gröbner basis of I with respect to $>$.

Proof. Since each $g_j \in I$, we have $J \subseteq \operatorname{in}_>(I)$, and hence there is a natural graded surjection $R/J \twoheadrightarrow R/\operatorname{in}_>(I)$. By Proposition 2.3.14 and the assumption, we have $\mathrm{HF}_{R/J}(i) = \mathrm{HF}_{R/I}(i) = \mathrm{HF}_{R/\operatorname{in}_>(I)}(i)$ for every $i \geq 0$. The surjection is therefore an isomorphism on every graded component, which forces $J = \operatorname{in}_>(I)$. Thus

$$\operatorname{in}_>(I) = (\operatorname{in}_>(g_1), \ldots, \operatorname{in}_>(g_s)),$$

so $\{g_1, \ldots, g_s\}$ is a Gröbner basis of I. □

2.4 Artinian algebras, socle, and socle degree

In the previous subsections, we introduced graded quotients $A = R/I$ and their Hilbert functions. In many of the cases considered later, the quotient will be finite-dimensional as a k-vector space. Equivalently, its Hilbert function eventually becomes zero. This type of finiteness is captured algebraically by the notion of an Artinian ring. For Artinian graded algebras, it is also natural to study the elements in the highest nonzero degrees; this leads to the notions of the socle and socle degree.

For standard background on Artinian rings, see [AM16, Chapter 8]. For the terminology concerning socles and socle degree, we refer to [HMM+13, Chapter 2].

Definition 2.4.1. Let A be a ring. We say that A is Artinian if every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

stabilizes. That is, there exists an integer N such that $I_j = I_N$ for all $j \geq N$.

Equivalently, an Artinian ring is a ring satisfying the descending chain condition on ideals. Similarly, a module is called Artinian if it satisfies the descending chain condition on submodules.

We will mostly be interested in graded quotients of the polynomial ring $R = k[x_1, \ldots, x_n]$. In this setting, being Artinian is equivalent to being finite-dimensional as a k-vector space.

Proposition 2.4.2. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a homogeneous ideal, and let $A = R/I$. Then A is Artinian if and only if* $\dim_k A < \infty$.

Proof. Assume first that A is Artinian. Since A is graded, we can write

$$A = \bigoplus_{i \geq 0} A_i.$$

For each $t \geq 0$, define

$$A_{\geq t} = \bigoplus_{i \geq t} A_i.$$

Each $A_{\geq t}$ is an ideal of A, and we get a descending chain

$$A = A_{\geq 0} \supseteq A_{\geq 1} \supseteq A_{\geq 2} \supseteq \cdots .$$

Since A is Artinian, this chain stabilizes. Hence there exists N such that $A_{\geq N} = A_{\geq N+1}$. Because the grading is a direct sum, this implies that $A_N = 0$. Since the chain stabilizes from this point onward, we also have $A_{\geq i} = A_{\geq i+1}$ for all $i \geq N$, and hence $A_i = 0$ for all $i \geq N$. Therefore

$$A = \bigoplus_{i=0}^{N-1} A_i.$$

Each A_i is finite-dimensional over k, since it is a quotient of the finite-dimensional vector space R_i. Hence

$$\dim_k A = \sum_{i=0}^{N-1} \dim_k A_i < \infty.$$

Conversely, assume that $\dim_k A < \infty$. Let

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$$

be a descending chain of ideals of A. Since each J_i is also a k-vector subspace of the finite-dimensional vector space A, the sequence

$$\dim_k J_1 \geq \dim_k J_2 \geq \dim_k J_3 \geq \cdots$$

is a descending sequence of nonnegative integers. Hence it stabilizes. If $J_i \supseteq J_{i+1}$ and $\dim_k J_i = \dim_k J_{i+1}$, then $J_i = J_{i+1}$. Therefore the chain stabilizes, and so A is Artinian. □

Equivalently, for a graded quotient $A = R/I$, the proposition says that A is Artinian if and only if $A_i = 0$ for all sufficiently large i. In particular, the Hilbert function of an Artinian graded algebra eventually becomes zero.

We will also use the following standard consequence. If R/I is Artinian, then I has height n. Since an Artinian ring has Krull dimension zero, an Artinian quotient R/I satisfies $\dim(R/I) = 0$. As $R = k[x_1, \ldots, x_n]$ has dimension n, this means that I has height n; see [Eis95, Chapter 8]. In particular, if $R = k[x_1, x_2, x_3]$ and R/I is Artinian, then I has height 3.

For a standard graded polynomial ring, the homogeneous ideal generated by the variables plays an important role.

Definition 2.4.3. The ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ is called the irrelevant maximal ideal.

It is maximal because $R/\mathfrak{m} \cong k$.

Let $A = R/I$ be a standard graded quotient, and let $\mathfrak{m}_A = \mathfrak{m}A = (x_1, \ldots, x_n)A$ be the image of the irrelevant maximal ideal in A.

Definition 2.4.4. The socle of A is

$$\operatorname{Soc}(A) = 0 :_A \mathfrak{m}_A = \{a \in A : \mathfrak{m}_A a = 0\}.$$

Thus, the socle consists of the elements of A that are annihilated by all variables $x_1, \ldots, x_n$.

From Proposition 2.4.2 we know that $\dim_k A < \infty$, thus we know that there will be a point in which the Hilbert function stabilizes to zero. More formally, we have

Definition 2.4.5. Let $A = \bigoplus_{i \geq 0} A_i$ be a standard graded Artinian algebra. The socle degree of A is the largest integer s such that $A_s \neq 0$, or equivalently such that $\mathrm{HF}_A(s) \neq 0$.

Indeed, if s is the largest degree with $A_s \neq 0$, then $\mathfrak{m}_A A_s \subseteq A_{s+1} = 0$. Hence $A_s \subseteq \mathrm{Soc}(A)$, which explains the terminology. The socle may also contain elements in lower degrees, but the top nonzero graded piece is always contained in the socle.

The following simple example illustrates the case where the socle is not concentrated in the top nonzero degree.

Example 2.4.6. Let $A = k[x, y]/(x^2, xy, y^3)$. Then

$$A_0 = \langle 1 \rangle, \quad A_1 = \langle x, y \rangle, \quad A_2 = \langle y^2 \rangle.$$

The irrelevant maximal ideal is $\mathfrak{m}_A = (x, y)$, and the socle is generated by x and y^2, since $x \cdot x = 0$, $x \cdot y = 0$, $y^2 \cdot x = 0$ and $y^2 \cdot y = 0$ in A. Thus

$$\mathrm{Soc}(A) = \langle x, y^2 \rangle$$

and the socle degree is 2.

2.5 Artinian Gorenstein algebras

We now introduce Artinian Gorenstein algebras, which are the Artinian algebras most relevant to the ideals studied later in the thesis. We use the standard characterization of a standard graded Artinian Gorenstein algebra by a one-dimensional socle; see [Eis95, Chapter 21]. Gorenstein rings can be defined in greater generality, but the algebras appearing later are Artinian quotients, so we restrict to the Artinian case.

Definition 2.5.1. Let $A = R/I$ be a standard graded Artinian k-algebra. We say that A is Artinian Gorenstein if

$$\dim_k \mathrm{Soc}(A) = 1.$$

Thus an Artinian Gorenstein algebra is an Artinian algebra whose socle is generated by a single element. If $I \subseteq R$ is a homogeneous ideal such that R/I is Artinian Gorenstein, then we say that I is an Artinian Gorenstein ideal.

One important consequence of the Gorenstein property is that the Hilbert function is symmetric. We first recall the relevant linear-algebra terminology.

Definition 2.5.2. Let V and W be finite-dimensional k-vector spaces. A bilinear map

$$\langle -, - \rangle : V \times W \longrightarrow k$$

is called a *perfect pairing* if the induced linear map

$$V \longrightarrow \mathrm{Hom}_k(W, k), \qquad v \longmapsto \Big(w \mapsto \langle v, w\rangle\Big),$$

is an isomorphism. Equivalently, the pairing is nondegenerate in both factors: for every nonzero $v \in V$ there exists $w \in W$ such that $\langle v, w\rangle \neq 0$, and for every nonzero $w \in W$ there exists $v \in V$ such that $\langle v, w\rangle \neq 0$. In particular, a perfect pairing implies that $\dim_k V = \dim_k W$.

The following standard perfect-pairing characterization is a graded form of Gorenstein duality; see [Eis95, Chapter 21].

Proposition 2.5.3. *Let $A = \bigoplus_{i\geq 0} A_i$ be a standard graded Artinian algebra with socle degree s. Then A is Gorenstein if and only if $\dim_k A_s = 1$ and, for every $0 \leq i \leq s$, the multiplication map*

$$A_i \times A_{s-i} \longrightarrow A_s, \qquad (a, b) \longmapsto ab$$

is a perfect pairing, after identifying $A_s \cong k$.

Proof. Suppose first that A is Gorenstein. Since A_s is the top nonzero graded piece, we have $A_s \subseteq \mathrm{Soc}(A)$. Since A is Gorenstein, $\mathrm{Soc}(A)$ is one-dimensional. Hence $\mathrm{Soc}(A) = A_s$ and $\dim_k A_s = 1$.

We now show that the multiplication pairing is perfect. Let $0 \neq a \in A_i$. Since $a \neq 0$, there exists a largest integer t such that

$$aA_t \neq 0.$$

Because A has socle degree s, we must have $t \leq s - i$. If $t < s - i$, then aA_t is a nonzero subspace of A in degree $i + t < s$. By maximality of t, this subspace is killed by A_1, and hence lies in $\mathrm{Soc}(A)$, contradicting $\mathrm{Soc}(A) = A_s$. Therefore $t = s - i$. Hence there exists $b \in A_{s-i}$ such that

$$ab \neq 0.$$

Thus no nonzero element of A_i annihilates all of A_{s-i}.

By the same argument with the roles of A_i and A_{s-i} interchanged, no nonzero element of A_{s-i} annihilates all of A_i. Since A_s is one-dimensional, the multiplication map

$$A_i \times A_{s-i} \longrightarrow A_s$$

is therefore a perfect pairing.

Conversely, suppose that $\dim_k A_s = 1$ and that, for every $0 \leq i \leq s$, the multiplication map

$$A_i \times A_{s-i} \longrightarrow A_s$$

is a perfect pairing. Since A_s is the top nonzero graded piece, we have

$$A_s \subseteq \mathrm{Soc}(A).$$

We claim that the socle has no nonzero elements in lower degree.

Indeed, let $0 \neq a \in \mathrm{Soc}(A)$ be homogeneous of degree $i < s$. Since the pairing

$$A_i \times A_{s-i} \longrightarrow A_s$$

is perfect, there exists $b \in A_{s-i}$ such that

$$ab \neq 0.$$

But $s - i > 0$, so b has positive degree. Since $a \in \mathrm{Soc}(A)$, it is annihilated by all positive-degree elements of A. Hence $ab = 0$ giving a contradiction. Therefore

$$\mathrm{Soc}(A) = A_s.$$

Since $\dim_k A_s = 1$, the socle is one-dimensional. Thus A is Gorenstein. □

Corollary 2.5.4. *Let A be a standard graded Artinian Gorenstein algebra with socle degree s. Then*

$$\mathrm{HF}_A(i) = \mathrm{HF}_A(s - i)$$

for all $0 \leq i \leq s$.

Proof. By Proposition 2.5.3, the multiplication map

$$A_i \times A_{s-i} \longrightarrow A_s$$

is a perfect pairing for every $0 \leq i \leq s$. Since $A_s \cong k$, this implies $\dim_k A_i = \dim_k A_{s-i}$. Therefore

$$\mathrm{HF}_A(i) = \mathrm{HF}_A(s - i)$$

for all $0 \leq i \leq s$. □

Besides the symmetry of the Hilbert function, we will need one structural result about Gorenstein ideals in three variables. This result concerns the number of minimal generators of a grade three Gorenstein ideal.

Recall that the grade of an ideal is the length of a maximal regular sequence contained in it. We will only use this notion in the following situation: if $R = k[x_1, x_2, x_3]$ and R/I is Artinian, then I

has height 3. Since polynomial rings are Cohen–Macaulay, height and grade agree in this case; see [Eis95, Chapter 18]. Thus I has grade 3.

We will use the following corollary of Buchsbaum–Eisenbud [BE77, Corollary 2.2].

Corollary 2.5.5 (Buchsbaum–Eisenbud)**.** *Let I be a grade 3 Gorenstein ideal. Then the number of minimal generators of I is odd.*

2.6 Inverse systems and annihilator ideals

We now introduce inverse systems, which provide a way to construct Artinian Gorenstein quotients from polynomial forms. The idea is to let one polynomial ring act on another by differentiation, and then study the differential operators that annihilate a given form.

This subsection gives a self-contained account of the parts of Macaulay inverse-system theory needed later in the thesis. The definitions and standard correspondences follow [IK99, Chapter 1], but we include proofs in the standard graded characteristic-zero setting to record the precise forms that will be used. We also include the proofs because references presenting these results together in this elementary graded formulation are not readily available.

Let $R = k[x_1, \dots, x_n]$ and $S = k[X_1, \dots, X_n]$. We think of R as a ring of differential operators acting on the ring S of forms. More precisely, the variable x_i acts as partial differentiation with respect to X_i. This gives the following action on monomials.

Definition 2.6.1. For monomials $x^\alpha \in R$ and $X^\beta \in S$, define

$$x^\alpha \circ X^\beta = \begin{cases} \dfrac{\beta!}{(\beta-\alpha)!} X^{\beta-\alpha}, & \text{if } \alpha_i \le \beta_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\beta! = \beta_1! \cdots \beta_n!$. The action is extended k-linearly in both variables.

This action makes S into an R-module. The R-submodules of S arising in this way are called inverse systems.

Definition 2.6.2. An inverse system is an R-submodule $M \subseteq S$ with respect to the differential action. If $F \in S$, the cyclic inverse system generated by F is

$$\langle F \rangle_R = \{ f \circ F : f \in R \}.$$

Thus a cyclic inverse system consists of all forms obtained from a fixed form F by applying differential operators from R. We now associate to each inverse system an ideal of differential operators that kill it.

Definition 2.6.3. Let $M \subseteq S$ be an inverse system. The annihilator of M is the ideal

$$\mathrm{Ann}(M) = \{f \in R : f \circ G = 0 \text{ for all } G \in M\}.$$

If $M = \langle F \rangle_R$ is cyclic, we write

$$\mathrm{Ann}(F) = \{f \in R : f \circ F = 0\}.$$

In other words, $\mathrm{Ann}(F)$ consists of all differential operators that annihilate F.

If $I \subseteq R$ is a homogeneous ideal, define its inverse system by

$$I^{\perp} = \{G \in S : f \circ G = 0 \text{ for all } f \in I\}.$$

Thus $I^{\perp}$ is a graded R-submodule of S.

For each $j \geq 0$, the differential action induces a bilinear pairing

$$R_j \times S_j \longrightarrow k, \qquad (f, G) \longmapsto f \circ G. \tag{3}$$

This pairing is perfect. Indeed, with respect to the monomial bases

$$\{x^{\alpha} : |\alpha| = j\} \quad \text{and} \quad \{X^{\beta} : |\beta| = j\},$$

its matrix is diagonal, since

$$x^{\alpha} \circ X^{\beta} = \begin{cases} \alpha!, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta, \end{cases}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. Every diagonal entry is nonzero because $\mathrm{char}(k) = 0$, so this matrix is invertible. Equivalently, if $0 \neq f = \sum_{|\alpha|=j} c_{\alpha} x^{\alpha} \in R_j$ and $c_{\gamma} \neq 0$, then

$$f \circ X^{\gamma} = c_{\gamma} \gamma! \neq 0.$$

The analogous argument for a nonzero element of S_j shows that the pairing is nondegenerate in both factors, and hence is perfect by Definition 2.5.2.

To distinguish linear-algebra orthogonal complements from inverse systems, we use the superscript $\vee$ for orthogonal complements with respect to the degree pairing (3). Thus, for a subspace $U \subseteq R_j$ and a subspace $V \subseteq S_j$, define

$$U^{\vee} = \{G \in S_j : f \circ G = 0 \text{ for every } f \in U\},$$
$$V^{\vee} = \{f \in R_j : f \circ G = 0 \text{ for every } G \in V\}.$$

Since the degree pairing is perfect and the spaces are finite-dimensional, taking the orthogonal complement twice recovers the original subspace:

$$(U^\vee)^\vee = U \qquad \text{and} \qquad (V^\vee)^\vee = V.$$

The notation $I^\perp$ will continue to denote the inverse system associated with a homogeneous ideal $I \subseteq R$.

The assignments $I \mapsto I^\perp$ and $M \mapsto \mathrm{Ann}(M)$ are inclusion-reversing. This means that if $I \subseteq J$ are homogeneous ideals of R, then

$$J^\perp \subseteq I^\perp,$$

since a form annihilated by every element of J is, in particular, annihilated by every element of I. Similarly, if $M \subseteq N$ are graded R-submodules of S, then

$$\mathrm{Ann}(N) \subseteq \mathrm{Ann}(M),$$

because an element of R that annihilates every form in N also annihilates every form in M. The following proposition is Macaulay's inverse system correspondence; see [IK99, Chapter 1].

Proposition 2.6.4 (Macaulay inverse systems)**.** *The maps*

$$I \longmapsto I^\perp \qquad \textit{and} \qquad M \longmapsto \mathrm{Ann}(M)$$

give mutually inverse bijections between the homogeneous ideals of R and the graded R-submodules of S. Moreover, for every homogeneous ideal $I \subseteq R$ and every $j \geq 0$,

$$\mathrm{HF}_{R/I}(j) = \dim_k (I^\perp)_j.$$

Proof. Fix $j \geq 0$. We first claim that

$$(I^\perp)_j = [I]_j^\vee. \tag{4}$$

By the definition of $I^\perp$, we have

$$(I^\perp)_j \subseteq [I]_j^\vee,$$

since every element of $I^\perp$ is annihilated by every element of I. For the reverse inclusion, let $G \in S_j$ satisfy $f \circ G = 0$ for every $f \in [I]_j$, and let $h \in I$ be homogeneous of degree d. If $d > j$, then $h \circ G = 0$ for degree reasons. If $d = j$, the equality holds by assumption. If $d < j$, then for every $p \in R_{j-d}$,

$$p \circ (h \circ G) = (ph) \circ G = 0,$$

because $ph \in [I]_j$. The perfect pairing between R_{j-d} and S_{j-d} therefore implies that $h \circ G = 0$. Thus

$G \in (I^\perp)_j$, proving the claim.

It follows from finite-dimensional linear algebra that

$$\dim_k (I^\perp)_j = \dim_k R_j - \dim_k [I]_j = \mathrm{HF}_{R/I}(j).$$

Now let $M \subseteq S$ be a graded R-submodule. We claim that

$$[\mathrm{Ann}(M)]_j = M_j^\vee. \tag{5}$$

The inclusion $[\mathrm{Ann}(M)]_j \subseteq M_j^\vee$ follows from the definition of $\mathrm{Ann}(M)$. For the reverse inclusion, let $f \in M_j^\vee$. It is enough to show that f annihilates all of M. Let $G \in M_d$ be homogeneous. If $d < j$, then $f \circ G = 0$ for degree reasons, while if $d = j$, the equality follows from $f \in M_j^\vee$. If $d > j$, then for every $p \in R_{d-j}$ we have $p \circ G \in M_j$, and hence

$$p \circ (f \circ G) = f \circ (p \circ G) = 0.$$

The perfect pairing between R_{d-j} and S_{d-j} implies that $f \circ G = 0$. This proves (5). Applying (4) to the homogeneous ideal $\mathrm{Ann}(M)$ and then taking the degreewise orthogonal complement gives

$$\begin{aligned} (\mathrm{Ann}(M)^\perp)_j &= [\mathrm{Ann}(M)]_j^\vee \\ &= (M_j^\vee)^\vee \\ &= M_j. \end{aligned}$$

Thus $\mathrm{Ann}(M)^\perp = M$. Applying the orthogonal-complement description to $M = I^\perp$ also gives

$$\begin{aligned} [\mathrm{Ann}(I^\perp)]_j &= ((I^\perp)_j)^\vee \\ &= ([I]_j^\vee)^\vee \\ &= [I]_j, \end{aligned}$$

where we used (5) in the first equality and (4) in the second. Hence $\mathrm{Ann}(I^\perp) = I$, completing the proof. □

The correspondence also detects the Artinian property.

Proposition 2.6.5. *The maps in Proposition 2.6.4 restrict to mutually inverse bijections between homogeneous ideals $I \subseteq R$ for which R/I is Artinian and finitely generated graded R-submodules of S. Equivalently, R/I is Artinian if and only if $I = \mathrm{Ann}(M)$ for some finitely generated graded R-submodule $M \subseteq S$.*

Proof. By Proposition 2.4.2, R/I is Artinian if and only if $\mathrm{HF}_{R/I}(j) = 0$ for all sufficiently large j. Proposition 2.6.4 shows that this is equivalent to $(I^\perp)_j = 0$ for all sufficiently large j.

A graded R-submodule $M \subseteq S$ is finitely generated if and only if $M_j = 0$ for all sufficiently large j. Indeed, differentiation lowers degree, so finitely many homogeneous generators give an upper bound on the nonzero degrees of M. Conversely, if the graded pieces vanish in high degree, then M is finite-dimensional over k, and any k-basis generates it as an R-module. The result now follows from Proposition 2.6.4. □

Artinian Gorenstein algebras correspond precisely to cyclic inverse systems. The following standard theorem is the Gorenstein case of Macaulay's correspondence; see [IK99, Chapter 1].

Theorem 2.6.6. *Let $s \geq 0$. The maps in Proposition 2.6.4 give mutually inverse bijections between homogeneous ideals $I \subseteq R$ for which R/I is Artinian Gorenstein of socle degree s and cyclic submodules $\langle F \rangle_R \subseteq S$ generated by nonzero forms $F \in S_s$. Equivalently, R/I is Artinian Gorenstein of socle degree s if and only if*

$$I = \mathrm{Ann}(F)$$

for some nonzero $F \in S_s$.

Proof. Suppose first that $A = R/I$ is Artinian Gorenstein of socle degree s. By Proposition 2.6.4,

$$\dim_k (I^\perp)_s = \mathrm{HF}_A(s) = 1.$$

Choose a nonzero $F \in (I^\perp)_s$, so $(I^\perp)_s = kF$, and set $M = \langle F \rangle_R$. Since $F \in I^\perp$ and $I^\perp$ is an R-submodule, we have $M \subseteq I^\perp$.

Fix $0 \leq j \leq s$, and choose $f_1, \ldots, f_r \in R_{s-j}$ whose residue classes form a basis of A_{s-j}. We claim that $f_1 \circ F, \ldots, f_r \circ F$ are linearly independent in M_j. Let f be a linear combination of the f_i whose residue class in A_{s-j} is nonzero. By Proposition 2.5.3, there exists $g \in R_j$ such that fg is nonzero in A_s. Since $(I^\perp)_s = kF$ is the dual of A_s, we have

$$g \circ (f \circ F) = (fg) \circ F \neq 0.$$

Thus $f \circ F \neq 0$, proving the claim. Consequently,

$$\dim_k M_j \geq \dim_k A_{s-j} = \dim_k A_j = \dim_k (I^\perp)_j,$$

where the middle equality follows from the symmetry of the Hilbert function. Since $M_j \subseteq (I^\perp)_j$, equality holds in every degree, so $I^\perp = M$. Proposition 2.6.4 now gives $I = \mathrm{Ann}(F)$.

Conversely, let $0 \neq F \in S_s$ and set $I = \mathrm{Ann}(F)$. By Proposition 2.6.4, $I^\perp = \langle F \rangle_R$. This cyclic module is concentrated in degrees at most s, so Proposition 2.6.5 shows that $A = R/I$ is Artinian.

Moreover, A_s is one-dimensional and $A_i = 0$ for $i > s$.

For $0 \leq j \leq s$, consider the multiplication pairing

$$A_j \times A_{s-j} \longrightarrow A_s.$$

If $0 \neq \overline{f} \in A_j$, then $f \circ F \neq 0$. By the perfect pairing (3), there exists $g \in R_{s-j}$ such that

$$(fg) \circ F = g \circ (f \circ F) \neq 0.$$

Therefore $fg \notin I = \operatorname{Ann}(F)$, so $\overline{fg} = \overline{f}\,\overline{g} \neq 0$ in A_s. Thus the multiplication pairing is nondegenerate in the first factor; the same argument applies after interchanging j and $s - j$. Hence it is a perfect pairing. Proposition 2.5.3 shows that A is Gorenstein of socle degree s. □

Example 2.6.7. Let $R = k[x_1, x_2, x_3]$, let $S = k[X_1, X_2, X_3]$, and set $F = X_1 + X_2 + X_3$. Then

$$\operatorname{Ann}(F) = (x_1 - x_3, x_2 - x_3, x_3^2) \subseteq R.$$

Indeed,

$$(x_1 - x_3) \circ F = 1 - 1 = 0$$

and

$$(x_2 - x_3) \circ F = 1 - 1 = 0.$$

Moreover, every differential operator of degree at least 2 annihilates F, since F has degree 1. Modulo the relations $x_1 - x_3$ and $x_2 - x_3$, every quadratic monomial is congruent to x_3^2. Hence the annihilator is generated by $x_1 - x_3$, $x_2 - x_3$, and x_3^2.

Moreover, $A = R/\operatorname{Ann}(F) \cong k[x_3]/(x_3^2)$. Thus A has basis $\overline{1}, \overline{x_3}$, with $\overline{x_3}^2 = 0$. Its homogeneous maximal ideal is $\mathfrak{m}_A = (\overline{x_3})$, so

$$\operatorname{Soc}(A) = 0 :_A \mathfrak{m}_A = \langle \overline{x_3} \rangle.$$

Hence the socle is one-dimensional, and therefore A is Artinian Gorenstein.

Theorem 2.6.6 shows that studying standard graded Artinian Gorenstein quotients of R is equivalent to studying annihilator ideals of homogeneous forms in S.

Example 2.6.8. Let $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and set

$$F = X^{\underline{a}} = X_1^{a_1} \cdots X_n^{a_n} \in S.$$

Then

$$\operatorname{Ann}(F) = (x_1^{a_1+1}, \ldots, x_n^{a_n+1}).$$

Indeed, each $x_i^{a_i+1}$ annihilates F, since it differentiates more times with respect to X_i than the exponent of X_i allows. Hence

$$(x_1^{a_1+1}, \dots, x_n^{a_n+1}) \subseteq \operatorname{Ann}(F).$$

Conversely, let x^α be a monomial differential operator. If $\alpha_i > a_i$ for some i, then $x^\alpha \circ F = 0$, and x^α is divisible by $x_i^{a_i+1}$. If $\alpha_i \leq a_i$ for every i, then

$$x^\alpha \circ F = \prod_{i=1}^{n} \frac{a_i!}{(a_i - \alpha_i)!} X^{\underline{a}-\alpha} \neq 0.$$

Distinct monomial differential operators that do not annihilate F produce distinct monomials in S, so no cancellation yields additional elements of the annihilator. Therefore the reverse inclusion also holds.

Hence

$$R/\operatorname{Ann}(F) \cong k[x_1, \dots, x_n]/(x_1^{a_1+1}, \dots, x_n^{a_n+1}),$$

which is a monomial complete intersection. For example, if

$$F = X_1 X_2^2 X_3,$$

then

$$\operatorname{Ann}(F) = (x_1^2, x_2^3, x_3^2).$$

We now record a technical variant of the differential action. The results above were stated using differentiation, but another common convention is to let R act on S by contraction. Since the two actions differ by factorial coefficients, one must be slightly careful when comparing results stated in the two conventions.

The purpose of the next few definitions and results is to show how to pass back and forth between contraction and differentiation. This is done using a factorial rescaling map $\varphi : S \to S$, which converts contraction into differentiation.

Definition 2.6.9. For monomials $x^\alpha \in R$ and $X^\beta \in S$, define

$$x^\alpha \circ_c X^\beta = \begin{cases} X^{\beta-\alpha}, & \text{if } \alpha_i \leq \beta_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

The action is extended k-linearly in both variables.

Definition 2.6.10. Following [BDL25, Definition 3.1], define the k-linear map $\varphi : S \to S$ on monomials by

$$\varphi(X_1^{i_1} \cdots X_n^{i_n}) = \frac{1}{i_1! \cdots i_n!} X_1^{i_1} \cdots X_n^{i_n}.$$

Since $\mathrm{char}(k) = 0$, this map is a k-linear isomorphism.

Lemma 2.6.11. *For every $f \in R$ and every $G \in S$, we have*

$$f \circ \varphi(G) = \varphi(f \circ_c G).$$

Proof. Both sides are k-linear in f and G, so it is enough to take $f = x^\alpha$ and $G = X^\beta$. If $\alpha_i \leq \beta_i$ for all i, then

$$x^\alpha \circ \varphi(X^\beta) = x^\alpha \circ \left(\frac{1}{\beta!} X^\beta\right) = \frac{1}{(\beta - \alpha)!} X^{\beta - \alpha}.$$

On the other hand,

$$\varphi(x^\alpha \circ_c X^\beta) = \varphi(X^{\beta-\alpha}) = \frac{1}{(\beta - \alpha)!} X^{\beta - \alpha}.$$

If $\alpha_i > \beta_i$ for some i, then both sides are zero. Hence the equality holds in all cases. □

Proposition 2.6.12. *Let $M \subseteq S$ be a k-subspace, and set*

$$\varphi(M) = \{\varphi(G) : G \in M\}.$$

Let Ann *denote annihilation with respect to the differential action, and let* Ann_c *denote annihilation with respect to contraction. Then*

$$\mathrm{Ann}(\varphi(M)) = \mathrm{Ann}_c(M).$$

Proof. Let $f \in R$. By Lemma 2.6.11, for every $G \in M$ we have

$$f \circ \varphi(G) = \varphi(f \circ_c G).$$

Since φ is a k-linear isomorphism, this is zero if and only if $f \circ_c G = 0$. Therefore f annihilates $\varphi(M)$ with respect to the differential action if and only if f annihilates M with respect to contraction. Hence

$$\mathrm{Ann}(\varphi(M)) = \mathrm{Ann}_c(M).$$

□

As a first application of Proposition 2.6.12, we compute the contraction annihilator of the complete symmetric form. This example also illustrates why the contraction action is often convenient.

Example 2.6.13. For $e \geq 1$, let

$$H_e = \sum_{\substack{\underline{a} \in \mathbb{N}^n \\ |\underline{a}| = e}} X^{\underline{a}} \in S = k[X_1, \ldots, X_n]$$

be the complete symmetric form of degree e. Then, with respect to contraction,

$$\mathrm{Ann}_c(H_e) = (x_1 - x_n, \ldots, x_{n-1} - x_n, x_n^{e+1}).$$

Indeed, for each $1 \leq i \leq n$, contraction by x_i lowers the exponent of X_i by one in every monomial where this is possible. Hence $x_i \circ_c H_e = H_{e-1}$ for every i. Therefore

$$(x_i - x_n) \circ_c H_e = 0$$

for all $1 \leq i \leq n-1$. Also, since H_e has degree e, we have $x_n^{e+1} \circ_c H_e = 0$. Thus

$$J := (x_1 - x_n, \ldots, x_{n-1} - x_n, x_n^{e+1}) \subseteq \mathrm{Ann}_c(H_e).$$

It remains to show equality. The quotient R/J has k-basis

$$1, x_n, x_n^2, \ldots, x_n^e,$$

so $\dim_k R/J = e + 1$. On the other hand, the cyclic inverse system generated by H_e is spanned by

$$H_e, H_{e-1}, \ldots, H_0,$$

because for any monomial x^α with $|\alpha| \leq e$, we have

$$x^\alpha \circ_c H_e = H_{e-|\alpha|}.$$

These forms have distinct degrees and are therefore linearly independent. The action map $R \to S$ given by $f \mapsto f \circ_c H_e$ has kernel $\mathrm{Ann}_c(H_e)$ and image of dimension $e + 1$. Hence

$$\dim_k(R/\mathrm{Ann}_c(H_e)) = e + 1.$$

Since $J \subseteq \mathrm{Ann}_c(H_e)$ and both quotients have dimension $e + 1$, we conclude that

$$\mathrm{Ann}_c(H_e) = J.$$

Moreover,

$$R/\mathrm{Ann}_c(H_e) \cong k[x_n]/(x_n^{e+1}).$$

Hence this quotient is Artinian Gorenstein with socle degree e, and its Hilbert function is

$$\mathrm{HF}_{R/\mathrm{Ann}_c(H_e)} = (1, 1, \ldots, 1),$$

with one nonzero entry in each degree $0, \dots, e$.

Now Proposition 2.6.12 gives, for any single form $F \in S$,

$$\operatorname{Ann}(\varphi(F)) = \operatorname{Ann}_c(F),$$

where Ann denotes the annihilator with respect to the differential action. Applying this to $F = H_e$, we obtain

$$\operatorname{Ann}(\varphi(H_e)) = \operatorname{Ann}_c(H_e) = (x_1 - x_n, \dots, x_{n-1} - x_n, x_n^{e+1}).$$

Thus the differential and contraction versions of the annihilator are equivalent after applying the factorial rescaling map φ. This explains how the two common conventions for inverse systems are related. The example also introduces the complete symmetric form, which is the main object studied in this thesis.

2.7 Lefschetz Properties

We now introduce Lefschetz properties, which give strong restrictions on the Hilbert function of a graded Artinian algebra. The basic idea is to study multiplication maps induced by a linear form and ask whether these maps have the largest possible rank in every degree.

Let

$$A = \bigoplus_{i \geq 0} A_i$$

be a standard graded Artinian k-algebra. For a linear form $\ell \in A_1$, multiplication by powers of ℓ gives k-linear maps

$$\times \ell^j : A_i \longrightarrow A_{i+j}.$$

We say that such a map has maximal rank if it is either injective or surjective, equivalently if its rank is

$$\min\{\dim_k A_i, \dim_k A_{i+j}\}.$$

We follow the terminology of [HMM+13].

Definition 2.7.1. We say that A has the weak Lefschetz property if there exists a linear form $\ell \in A_1$ such that

$$\times \ell : A_i \longrightarrow A_{i+1}$$

has maximal rank for every $i \geq 0$. Such a linear form is called a weak Lefschetz element.

Definition 2.7.2. We say that A has the strong Lefschetz property if there exists a linear form $\ell \in A_1$ such that

$$\times \ell^j : A_i \longrightarrow A_{i+j}$$

has maximal rank for all $i \geq 0$ and all $j \geq 1$. Such a linear form is called a strong Lefschetz element.

By taking $j = 1$, we see immediately that the strong Lefschetz property implies the weak Lefschetz property.

The Lefschetz properties also impose conditions on the Hilbert function. If A has the weak Lefschetz property and ℓ is a weak Lefschetz element, then the maps

$$\times \ell : A_i \longrightarrow A_{i+1}$$

have maximal rank for all i. Moreover, once one of these maps is surjective, all the subsequent maps are surjective. Indeed, suppose that

$$\ell A_i = A_{i+1}.$$

Since A is standard graded, we have $A_{i+2} = A_1 A_{i+1}$, and hence

$$A_{i+2} = A_1(\ell A_i) = \ell(A_1 A_i) = \ell A_{i+1}.$$

Thus multiplication by ℓ from A_{i+1} to A_{i+2} is also surjective, and the assertion follows by induction.

Write $h_i = \dim_k A_i$. Before the multiplication maps become surjective, their maximal-rank property forces them to be injective, so the sequence h_i is nondecreasing. From the first surjective map onward, the sequence is nonincreasing. Recall that a finite sequence $(h_0, \ldots, h_s)$ is called *unimodal* if there is an index p such that

$$h_0 \leq h_1 \leq \cdots \leq h_p \qquad \text{and} \qquad h_p \geq h_{p+1} \geq \cdots \geq h_s.$$

Consequently, the Hilbert function of an algebra with the weak Lefschetz property is unimodal.

The following theorem is often attributed to Stanley, whose proof uses the hard Lefschetz theorem [Sta80]. We give instead the elementary algebraic proof of Reid, Roberts, and Roitman [RRR91, Theorem 5].

Theorem 2.7.3 (Stanley)**.** *Let*

$$A = k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$$

where $a_i \geq 1$ for all i. Then A has the strong Lefschetz property. More precisely,

$$\ell = x_1 + \cdots + x_n$$

is a strong Lefschetz element.

Proof. Set

$$t = \sum_{r=1}^{n} (a_r - 1) \qquad \text{and} \qquad \ell = x_1 + \cdots + x_n.$$

Thus t is the socle degree of A. We first prove the following estimate: if $0 \neq f \in A_q$ and $\ell^m f = 0$ for some $m \geq 1$, then

$$q \geq \frac{t - m + 1}{2}. \tag{6}$$

Following [RRR91, Theorem 5], we argue by induction on q. If $q = 0$, then f is a nonzero scalar. If $q < (t-m+1)/2$, then $m \leq t$. We can therefore choose integers $0 \leq b_r \leq a_r - 1$ such that $\sum_r b_r = m$. The monomial $x_1^{b_1} \cdots x_n^{b_n}$ occurs in the expansion of ℓ^m with a nonzero coefficient because k has characteristic 0. Hence $\ell^m \neq 0$ in A, contradicting $\ell^m f = 0$.

Now suppose that $q > 0$. Choose the standard monomial representative F of f, and, after relabeling the variables if necessary, assume that $F' = \partial F / \partial x_n$ is nonzero. Write

$$B = k[x_1, \ldots, x_{n-1}]/(x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}}), \qquad T = B[x_n].$$

Since $\ell^m f = 0$ in $A = T/(x_n^{a_n})$, there is an element $G \in T$ such that

$$\ell^m F = G x_n^{a_n}.$$

Differentiating with respect to x_n gives

$$\ell^{m-1}(mF + \ell F') = G' x_n^{a_n} + a_n G x_n^{a_n - 1}.$$

Multiplying by ℓ and reducing modulo $x_n^{a_n - 1}$, while using $\ell^m F = 0$ modulo $x_n^{a_n - 1}$, yields

$$\ell^{m+1} F' = 0 \qquad \text{in} \qquad T/(x_n^{a_n - 1}).$$

The latter algebra has socle degree $t-1$, while F' has degree $q-1$. The induction hypothesis therefore gives

$$q - 1 \geq \frac{(t-1) - (m+1) + 1}{2} = \frac{t - m - 1}{2},$$

which proves (6).

It remains to translate this estimate into the maximal-rank property. The one-dimensional socle A_t is spanned by

$$x_1^{a_1 - 1} \cdots x_n^{a_n - 1},$$

and multiplication gives a perfect pairing

$$A_q \times A_{t-q} \longrightarrow A_t.$$

Consider multiplication by ℓ^m from A_q to A_{q+m}. If $2q+m \leq t$, then $q < (t-m+1)/2$, so (6) shows that this map is injective. If $q+m > t$, then $A_{q+m} = 0$, so the map is automatically surjective. Finally, if $2q+m \geq t$ and $q+m \leq t$, its dual under the perfect pairing is multiplication by ℓ^m from A_{t-q-m} to A_{t-q}. Since

$$2(t-q-m)+m \leq t,$$

the first case shows that this dual map is injective, and hence the original map is surjective. Thus every map

$$\times \ell^m : A_q \longrightarrow A_{q+m}$$

has maximal rank, proving that ℓ is a strong Lefschetz element. □

Example 2.7.4. Consider

$$A = k[x,y]/(x^2, y^2).$$

This is a monomial complete intersection, so by Theorem 2.7.3 it has the strong Lefschetz property. In this case, one can see the Lefschetz maps directly.

The Hilbert function is

$$\mathrm{HF}_A = (1,2,1),$$

with

$$A_0 = \langle 1 \rangle, \qquad A_1 = \langle x, y \rangle, \qquad A_2 = \langle xy \rangle.$$

Let $\ell = x+y$. Then

$$\times \ell : A_0 \to A_1, \qquad 1 \mapsto x+y,$$

is injective, while

$$\times \ell : A_1 \to A_2, \qquad ax+by \mapsto (a+b)xy,$$

is surjective. Finally,

$$\times \ell^2 : A_0 \to A_2, \qquad 1 \mapsto (x+y)^2 = 2xy,$$

is an isomorphism. Hence $\ell = x+y$ is a strong Lefschetz element in this example.

3 Complete symmetric forms and their annihilators

In this section we develop the results about complete symmetric forms that lead to Conjecture 3.7 of [BDL25]. Rather than only quoting the two papers on which the conjecture is based, we prove the parts of their arguments that will be used later. We first follow Section 2 of [BMMRN22] to prove compressedness, the strong Lefschetz property, and a description of the annihilator. We then follow Sections 2 and 3 of [BDL25] to determine a Gröbner basis. Throughout the section, annihilators are taken with respect to the differential action from Definition 2.6.1.

3.1 The associated Artinian Gorenstein algebras

For an integer $e \geq 0$, the *complete symmetric form* of degree e in n variables is

$$H_e = \sum_{\substack{\underline{a} \in \mathbb{N}^n \\ |\underline{a}|=e}} X^{\underline{a}} \in S_e.$$

Thus H_e is the sum of all monomials of degree e, each with coefficient one. We write

$$I_e = \operatorname{Ann}(H_e) \subseteq R \qquad \text{and} \qquad A_e = R/I_e.$$

By Theorem 2.6.6, A_e is an Artinian Gorenstein algebra with socle degree e.

The Hilbert function of any Artinian Gorenstein quotient of R with socle degree e satisfies

$$\mathrm{HF}_A(i) \leq \min\{\dim_k R_i, \dim_k R_{e-i}\}.$$

Indeed, the first bound follows from A_i being a quotient of R_i, and the second follows from the symmetry of the Hilbert function.

Definition 3.1.1. Let $A = R/I$ be an Artinian Gorenstein algebra with socle degree e. We say that A is *compressed* if

$$\mathrm{HF}_A(i) = \min\{\dim_k R_i, \dim_k R_{e-i}\}$$

for every $0 \leq i \leq e$.

Thus, among Artinian Gorenstein quotients of R with socle degree e, a compressed algebra has the largest possible Hilbert function in every degree. The intuition is that its Hilbert function agrees with that of the polynomial ring R for as long as possible. Around the middle degree, the symmetry of the Gorenstein Hilbert function, established in Corollary 2.5.4, forces the values to decrease by reflecting the first half. In other words, the Hilbert function grows as freely as the ambient polynomial ring permits until Gorenstein symmetry forces it back down.

Since $\dim_k R_i = \binom{n+i-1}{n-1}$, a compressed Artinian Gorenstein algebra of socle degree e has Hilbert function

$$\mathrm{HF}_A(i) = \min\left\{\binom{n+i-1}{n-1}, \binom{n+e-i-1}{n-1}\right\}. \tag{7}$$

A generic form defines a compressed Artinian Gorenstein algebra [IK99, Proposition 3.12]. Here, a *generic form* of degree e means a form belonging to a suitable nonempty Zariski-open subset of the parameter space S_e. The form H_e is far from generic, since it is fixed by every permutation of the variables, but it nevertheless has the same Hilbert function. We now prove this result, following [BMMRN22, Lemma 2.2, Lemmas 2.7–2.8, and Proposition 2.10].

Let

$$\ell = x_1 + \cdots + x_n$$

Differentiating a monomial of H_m in all possible variables gives the basic identity

$$\ell \circ H_m = (n+m-1)H_{m-1}. \tag{8}$$

Indeed, the coefficient of $X_1^{b_1}\cdots X_n^{b_n}$, where $|\underline{b}| = m-1$, in the left-hand side is

$$\sum_{j=1}^{n}(b_j+1) = n+m-1.$$

To formulate the remaining computations, we use the inverse of the factorial rescaling map from Definition 2.6.10.

Definition 3.1.2. Let $\varphi^{-1} : S \to S$ be the inverse of φ. Thus

$$\varphi^{-1}(X_1^{a_1}\cdots X_n^{a_n}) = a_1!\cdots a_n! X_1^{a_1}\cdots X_n^{a_n}.$$

We identify the monomial bases of R and S by $x^{\underline{a}} \leftrightarrow X^{\underline{a}}$. Under this identification, we use the same symbols φ and φ^{-1} for the corresponding maps on R, or between R and S, whenever no confusion can arise.

Definition 3.1.3. For $\underline{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$, define

$$\begin{aligned}\widehat{F}_{\underline{a}} &= (X_1 - X_n)^{a_1}\cdots(X_{n-1}-X_n)^{a_{n-1}} \in S,\\ f_{\underline{a}} &= \varphi(\widehat{F}_{\underline{a}}) \in R.\end{aligned}$$

After identifying the variables X_i and x_i, the element $f_{\underline{a}}$ is the factorially rescaled polynomial $\varphi(F_{\underline{a}})$, where $F_{\underline{a}}$ is introduced below in (11).

The following consequence of [BMMRN22, Lemmas 2.6 and 2.7] is the main ingredient in the proof. We state only the two parts that will be used below.

Lemma 3.1.4. *Let $q = |\underline{a}|$. Then*

(i) $f_{\underline{a}} \circ H_m = 0$ *for* $m < 2q$;

(ii) $f_{\underline{a}} \circ H_{2q} = \widehat{F}_{\underline{a}}$.

Proof sketch. Lemma 2.6 of [BMMRN22] computes the action of $f_{\underline{a}}$ on the generating series of the complete symmetric forms. That computation shows that the lowest nonzero homogeneous component of the result has degree q and is equal to $\widehat{F}_{\underline{a}}$. Since $f_{\underline{a}}$ has degree q, the contribution coming from H_m has degree $m-q$. It must therefore vanish when $m-q < q$, while for $m-q = q$ it is exactly $\widehat{F}_{\underline{a}}$. These are statements (i) and (ii); see [BMMRN22, Lemma 2.7]. □

Definition 3.1.5. For $q \geq 0$, define

$$M_q = \operatorname{span}_k\{f_{\underline{a}} : |\underline{a}| = q\} \subseteq R_q.$$

Since the forms $\widehat{F}_{\underline{a}}$ form a basis of the degree q part of $k[X_1 - X_n, \ldots, X_{n-1} - X_n]$, we have

$$\dim_k M_q = \binom{n+q-2}{n-2}. \tag{9}$$

Lemma 3.1.6. *If $|\underline{a}| = |\underline{b}| = q$, then*

$$f_{\underline{a}} \circ \widehat{F}_{\underline{b}} = \prod_{i=1}^{n-1} \binom{a_i + b_i}{a_i}.$$

Proof. By the binomial theorem,

$$\widehat{F}_{\underline{a}} = \sum_{\substack{\underline{i} \in \mathbb{N}^{n-1} \\ \underline{i} \leq \underline{a}}} (-1)^{q-|\underline{i}|} \left(\prod_{r=1}^{n-1} \binom{a_r}{i_r} \right) X_1^{i_1} \cdots X_{n-1}^{i_{n-1}} X_n^{q-|\underline{i}|},$$

where $\underline{i} \leq \underline{a}$ means that $i_r \leq a_r$ for every r. Applying the factorial rescaling map gives

$$f_{\underline{a}} = \sum_{\substack{\underline{i} \in \mathbb{N}^{n-1} \\ \underline{i} \leq \underline{a}}} (-1)^{q-|\underline{i}|} \left(\prod_{r=1}^{n-1} \binom{a_r}{i_r} \right) \frac{x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} x_n^{q-|\underline{i}|}}{i_1! \cdots i_{n-1}! (q - |\underline{i}|)!}.$$

Similarly,

$$\widehat{F}_{\underline{b}} = \sum_{\substack{\underline{j} \in \mathbb{N}^{n-1} \\ \underline{j} \leq \underline{b}}} (-1)^{q-|\underline{j}|} \left(\prod_{r=1}^{n-1} \binom{b_r}{j_r} \right) X_1^{j_1} \cdots X_{n-1}^{j_{n-1}} X_n^{q-|\underline{j}|}.$$

Consider the action of the term indexed by $\underline{i}$ in $f_{\underline{a}}$ on the term indexed by $\underline{j}$ in $\widehat{F}_{\underline{b}}$. For this action

to be nonzero, we must have

$$i_r \le j_r \quad \text{for every } 1 \le r \le n-1$$

and

$$q - |\underline{i}| \le q - |\underline{j}|.$$

The first condition implies $|\underline{i}| \le |\underline{j}|$, whereas the second implies $|\underline{i}| \ge |\underline{j}|$. Hence $|\underline{i}| = |\underline{j}|$, and the coordinatewise inequalities force $\underline{i} = \underline{j}$.

For the matching terms, differentiation produces the factor

$$i_1! \cdots i_{n-1}!(q - |\underline{i}|)!,$$

which cancels the factorial denominator in $f_{\underline{a}}$. Moreover, the two signs $(-1)^{q-|\underline{i}|}$ cancel. We therefore obtain

$$f_{\underline{a}} \circ \widehat{F}_{\underline{b}} = \sum_{\underline{i} \in \mathbb{N}^{n-1}} \prod_{r=1}^{n-1} \binom{a_r}{i_r} \binom{b_r}{i_r}.$$

Here a binomial coefficient is understood to be zero when its lower index is larger than its upper index. The multi-index sum now factors as

$$\begin{aligned} f_{\underline{a}} \circ \widehat{F}_{\underline{b}} &= \prod_{r=1}^{n-1} \left(\sum_{s \ge 0} \binom{a_r}{s} \binom{b_r}{s} \right) \\ &= \prod_{r=1}^{n-1} \binom{a_r + b_r}{a_r}, \end{aligned}$$

where the last equality follows from Vandermonde's identity. □

Proposition 3.1.7 (Boij–Migliore–Miró-Roig–Nagel)**.** *For every $d \ge 0$, the pairing*

$$R_d \times R_d \longrightarrow k, \qquad (f, g) \longmapsto (fg) \circ H_{2d},$$

is zero on $\ell^i M_{d-i} \times \ell^j M_{d-j}$ when $i \ne j$, and restricts to a perfect pairing on $\ell^i M_{d-i}$ when $i = j$. Consequently,

$$R_d = M_d \oplus \ell M_{d-1} \oplus \cdots \oplus \ell^d M_0. \tag{10}$$

Proof. Take $f \in M_{d-i}$ and $g \in M_{d-j}$. Repeated use of (8) gives a nonzero scalar c such that

$$(\ell^i f\, \ell^j g) \circ H_{2d} = c(fg) \circ H_{2d-i-j}.$$

If $i < j$, then $2(d-i) > 2d-i-j$, so Lemma 3.1.4(i), applied by linearity, shows that $f \circ H_{2d-i-j} = 0$. The same argument with f and g interchanged handles $j < i$. Thus the pairing is zero between distinct summands.

If $i = j$, put $q = d - i$. By Lemma 3.1.4(ii) and linearity, every $f \in M_q$ satisfies

$$f \circ H_{2q} = \varphi^{-1}(f),$$

under the identification of the monomial bases of R and S used above. Consequently,

$$(fg) \circ H_{2q} = g \circ \varphi^{-1}(f).$$

By [BMMRN22, Proposition 2.10], the restriction of this pairing to M_q is perfect. Multiplication by the powers of ℓ only changes the pairing by the nonzero scalar appearing above, so the restriction to $\ell^i M_{d-i}$ is also perfect.

To see that the sum in (10) is direct, suppose that

$$h_0 + \cdots + h_d = 0, \qquad h_i \in \ell^i M_{d-i}.$$

Pairing this equality with an arbitrary element of $\ell^j M_{d-j}$ eliminates all terms except h_j, because the pairing is zero between distinct summands. Since the restriction to $\ell^j M_{d-j}$ is perfect, we obtain $h_j = 0$ for every j. Hence the sum is direct. Finally, the hockey-stick identity and (9) give

$$\sum_{q=0}^{d} \dim_k M_q = \sum_{q=0}^{d} \binom{n+q-2}{n-2} = \binom{n+d-1}{n-1} = \dim_k R_d,$$

so the direct sum is all of R_d. □

Theorem 3.1.8 (Boij–Migliore–Miró-Roig–Nagel)**.** *For every $e \geq 0$, the algebra*

$$A_e = R/\operatorname{Ann}(H_e)$$

is compressed and has the strong Lefschetz property. More precisely,

$$\ell = x_1 + \cdots + x_n$$

is a strong Lefschetz element of A_e.

Proof. First suppose that $e = 2d$. Proposition 3.1.7 says that the middle pairing on R_d is perfect. Therefore no nonzero element of R_d annihilates H_{2d}, so

$$[I_{2d}]_d = 0.$$

It follows that A_{2d} agrees with R through degree d. The Gorenstein symmetry from Corollary 2.5.4 then gives the compressed Hilbert function (7).

Now let $e = 2d+1$ and suppose that $f \in [I_{2d+1}]_d$. By (8),

$$(\ell f) \circ H_{2d+2} = f \circ (\ell \circ H_{2d+2}) = (n+2d+1)(f \circ H_{2d+1}) = 0.$$

The even case applied to H_{2d+2} shows that its annihilator is zero in degree $d+1$. Hence $\ell f = 0$, and therefore $f = 0$. Thus A_{2d+1} is also compressed.

It remains to prove the strong Lefschetz property. For an Artinian Gorenstein algebra of socle degree e, it is enough to prove that the central maps

$$\times \ell^{e-2i} : (A_e)_i \longrightarrow (A_e)_{e-i}$$

are isomorphisms for $0 \leq i \leq e/2$: the injectivity and surjectivity of all other powers follow by factoring through these maps and using the perfect pairing of Proposition 2.5.3.

Suppose that $f \in R_i$ belongs to the kernel of the displayed map. Then $\ell^{e-2i} f \circ H_e = 0$. Repeated use of (8) gives a nonzero scalar c such that

$$\ell^{e-2i} f \circ H_e = c(f \circ H_{2i}).$$

The even case, applied to H_{2i}, shows that its annihilator has zero degree i part. Hence $f = 0$. The central map is injective, and it is bijective because its source and target have the same dimension by Gorenstein symmetry. Thus ℓ is a strong Lefschetz element in the sense of Definition 2.7.2. □

Compressedness of A_e was also obtained independently in [GL19] by studying spaces of partial derivatives. In particular, its Hilbert function is given by (7).

For a compressed Artinian Gorenstein algebra, standard results on its minimal free resolution restrict the possible degrees of the minimal generators. In odd socle degree, the required two-degree bound follows from [Boi99, Proposition 3.2]. In even socle degree, the sharper equigeneration statement follows from [Iar84, Proposition 4.1B and Example 4.7]. If $e = 2d$, then I_{2d} has no elements through degree d, and it is generated in degree $d+1$. If $e = 2d+1$, then I_{2d+1} has no elements through degree d, while generators may occur in degrees $d+1$ and $d+2$. We next describe these generator spaces more explicitly.

3.2 A description of the annihilator

We use the factorial rescaling map φ from Definition 2.6.10, identifying the variables X_i and x_i as above.

Definition 3.2.1. For $\underline{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$, define

$$F_{\underline{a}} = (x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}, \qquad |\underline{a}| = a_1 + \cdots + a_{n-1}, \tag{11}$$

Under the identification of the variables X_i and x_i, Definition 3.1.3 gives

$$f_{\underline{a}} = \varphi(F_{\underline{a}}).$$

Consequently, the spaces from Definition 3.1.5 may equivalently be written as

$$M_q = \operatorname{span}_k\{\varphi(F_{\underline{a}}) : |\underline{a}| = q\} \subseteq R_q.$$

The forms $F_{\underline{a}}$ constitute a basis of the degree q part of

$$k[x_1 - x_n, \ldots, x_{n-1} - x_n],$$

in agreement with the dimension formula (9).

The following theorem is [BMMRN22, Theorem 2.12]. If $V \subseteq R$ is a vector space of forms, we write (V) for the ideal generated by V. We reproduce the main computations that identify the relevant graded pieces of the annihilators. We also use the general degree bounds for compressed Artinian Gorenstein algebras recalled above.

Theorem 3.2.2 (Boij–Migliore–Miró-Roig–Nagel)**.** *Let $d \geq 0$ and let $\ell = x_1 + \cdots + x_n$. Then*

$$\operatorname{Ann}(H_{2d}) = (M_{d+1} \oplus \ell M_d),$$
$$\operatorname{Ann}(H_{2d+1}) = (M_{d+1} + \ell^2 M_d).$$

Proof sketch. For $d = 0$, both formulas can be checked directly: modulo the linear forms $x_i - x_n$, the element ℓ is a nonzero scalar multiple of x_n, so the claims reduce to the annihilators of $H_0 = 1$ and $H_1 = X_1 + \cdots + X_n$. We therefore assume $d \geq 1$.

We first treat the even case. If $|\underline{a}| = d + 1$, then Lemma 3.1.4(i) gives

$$\varphi(F_{\underline{a}}) \circ H_{2d} = 0.$$

If $|\underline{a}| = d$, then (8) and the same lemma give

$$(\ell\varphi(F_{\underline{a}})) \circ H_{2d} = (n + 2d - 1)\varphi(F_{\underline{a}}) \circ H_{2d-1} = 0.$$

Thus $M_{d+1} + \ell M_d \subseteq [I_{2d}]_{d+1}$.

The sum is direct. Indeed, apply Proposition 3.1.7 to the pairing defined by H_{2d+2}: the restriction to M_{d+1} is perfect, while M_{d+1} and ℓM_d are paired to zero. If an element belongs to their intersection, it therefore pairs to zero with every element of M_{d+1}. Since it also belongs to M_{d+1} and the pairing

on that space is perfect, the element must be zero. By compressedness,

$$\begin{aligned}\dim_k [I_{2d}]_{d+1} &= \dim_k R_{d+1} - \dim_k R_{d-1} \\ &= \binom{n+d-1}{n-2} + \binom{n+d-2}{n-2} \\ &= \dim_k M_{d+1} + \dim_k M_d.\end{aligned}$$

Therefore

$$[I_{2d}]_{d+1} = M_{d+1} \oplus \ell M_d.$$

Since in even socle degree all generators have degree $d+1$, it follows that

$$I_{2d} = (M_{d+1} \oplus \ell M_d).$$

Now let $e = 2d+1$. Lemma 3.1.4(i) shows that $M_{d+1} \subseteq [I_{2d+1}]_{d+1}$. Moreover,

$$\dim_k [I_{2d+1}]_{d+1} = \dim_k R_{d+1} - \dim_k R_d = \binom{n+d-1}{n-2} = \dim_k M_{d+1}.$$

Hence

$$[I_{2d+1}]_{d+1} = M_{d+1}. \tag{12}$$

Also, if $f \in M_d$, then

$$(\ell^2 f) \circ H_{2d+1} = c(f \circ H_{2d-1}) = 0$$

for a nonzero scalar c, so $\ell^2 M_d \subseteq I_{2d+1}$.

It remains to account for all elements in degree $d+2$. The decomposition (10), first with $d+1$ and then with d, gives

$$R_{d+1} = M_{d+1} \oplus \ell R_d.$$

Consequently every $g \in R_{d+2} = R_1 R_{d+1}$ can be written as

$$g = \ell f_1 + f_2, \qquad f_1 \in R_{d+1}, \quad f_2 \in R_1 M_{d+1}.$$

Suppose that $g \in [I_{2d+1}]_{d+2}$. Since $M_{d+1} \subseteq I_{2d+1}$, the element f_2 already belongs to the ideal. Thus $\ell f_1 \in I_{2d+1}$, and

$$0 = (\ell f_1) \circ H_{2d+1} = (n+2d) f_1 \circ H_{2d}.$$

The even case shows that

$$f_1 \in [I_{2d}]_{d+1} = M_{d+1} \oplus \ell M_d.$$

It follows that

$$g \in (M_{d+1}) + \ell^2 M_d.$$

We have proved that the degree $d+1$ and $d+2$ parts of the annihilator are generated by $M_{d+1}+\ell^2 M_d$. Since in odd socle degree all generators have degree $d+1$ or $d+2$, this gives the claimed equality of ideals. □

For even degree, both summands in Theorem 3.2.2 lie in degree $d+1$, so the annihilator is equigenerated in that degree. Its minimal generators may therefore be chosen as

$$\{\varphi(F_{\underline{a}}) : |\underline{a}| = d+1\} \cup \{\ell\varphi(F_{\underline{a}}) : |\underline{a}| = d\}.$$

In the odd case, M_{d+1} is precisely $[I_{2d+1}]_{d+1}$, whereas the term $\ell^2 M_d$ only says that I_{2d+1} is generated in degrees $d+1$ and $d+2$. It does not determine how many elements of degree $d+2$ remain minimal modulo $R_1[I_{2d+1}]_{d+1}$. This is the point refined by the Gröbner basis in [BDL25].

3.3 The Gröbner basis and Conjecture 3.7

We now equip R with the degree reverse lexicographic order induced by

$$x_1 > \cdots > x_n.$$

This is the order from Definition 2.3.8. We begin with the generic calculation on which the proof in [BDL25] is based.

We will use generic forms only as comparison objects. A statement is said to hold for a generic form of degree e if it holds whenever the coefficients of the form are chosen outside the zero set of finitely many nonzero polynomial conditions. Equivalently, it holds on a nonempty Zariski open subset of the vector space S_e. No further theory of generic forms will be needed here.

The following proposition is [BDL25, Theorem 2.1].

Proposition 3.3.1 (Boij–Duarte–Lundqvist)**.** *Let $G \in S_e$ be a generic form, let $>$ be any monomial order, and let $e/2 \le q \le e$. Then $[\mathrm{in}(\mathrm{Ann}(G))]_q$ is spanned by the first*

$$\dim_k R_q - \dim_k R_{e-q}$$

monomials of degree q in the chosen order.

Proof. See [BDL25, Theorem 2.1]. □

For $r \ge 0$, let $\mathcal{A}_r$ denote the set of monomials of degree r in $k[x_1, \ldots, x_{n-1}]$. The preceding proposition gives the following explicit minimal monomial generators in degree reverse lexicographic

order.

Corollary 3.3.2. *Let $G \in S_e$ be generic.*

(i) *If $e = 2d+1$, the minimal monomial generators of* $\mathrm{in}(\mathrm{Ann}(G))$ *are*

$$x_n^{2i}\mathcal{A}_{d+1-i}, \qquad 0 \le i \le d+1.$$

(ii) *If $e = 2d$, its minimal monomial generators are*

$$\mathcal{A}_{d+1} \cup x_n\mathcal{A}_d$$

in degree $d+1$, and

$$x_n^{2i+1}\mathcal{A}_{d-i}, \qquad 1 \le i \le d,$$

in the higher degrees.

Proof. Put $J = \mathrm{in}(\mathrm{Ann}(G))$. Proposition 3.3.1 shows that, in every degree $q \ge e/2$, the standard monomials are the last $\dim_k R_{e-q}$ monomials. In degree reverse lexicographic order these are exactly the monomials divisible by x_n^{2q-e}, because division by this power gives all monomials of degree $e-q$. Thus $[J]_q$ contains precisely the monomials not divisible by x_n^{2q-e}. For $q = e+1$, the same description is understood with no standard monomials, since every differential operator of degree $e+1$ annihilates G.

For q beyond the first nonzero degree of J, the monomials of $R_1[J]_{q-1}$ are precisely those not divisible by x_n^{2q-e-1}. Hence the new minimal generators in degree q are divisible by x_n^{2q-e-1} but not by x_n^{2q-e}. They are therefore

$$x_n^{2q-e-1}\mathcal{A}_{e-q+1}.$$

Substituting $q = d+1+i$ gives the two displayed lists. In the first nonzero degree, Proposition 3.3.1 gives $\mathcal{A}_{d+1}$ in the odd case and $\mathcal{A}_{d+1} \cup x_n\mathcal{A}_d$ in the even case. □

The next step is to construct elements of $\mathrm{Ann}(H_e)$ with exactly these leading monomials. This is [BDL25, Proposition 3.3]; its proof uses the annihilator description from Theorem 3.2.2 in one additional variable.

Proposition 3.3.3. *If $e = 2d$, then*

$$\varphi(x_n^{2i+1}F_{\underline{a}}) \in \mathrm{Ann}(H_{2d}) \quad \textit{for } 0 \le i \le d, \quad |\underline{a}| = d-i.$$

If $e = 2d+1$, then

$$\varphi(x_n^{2i}F_{\underline{a}}) \in \mathrm{Ann}(H_{2d+1}) \quad \textit{for } 0 \le i \le d+1, \quad |\underline{a}| = d+1-i.$$

Proof. We argue by strong induction on e, giving the even case in detail. The case $e = 0$ is immediate, since every positive-degree differential operator annihilates $H_0 = 1$. The case $e = 1$ amounts to checking that $x_j - x_n$ and $\frac{1}{2}x_n^2$ annihilate $H_1 = X_1 + \cdots + X_n$.

Now let $e \geq 2$, and assume that both assertions of the proposition hold for every integer m with $0 \leq m < e$.

Suppose $e = 2d$ and choose i and $\underline{a}$ as in the statement. Add variables x_0 and X_0, and use primes for the corresponding objects in $n + 1$ variables. The polynomial

$$P = \varphi'\Big(F_{\underline{a}}(x_0 - x_n)^{2i+1}\Big)$$

belongs to M'_{d+i+1}. The odd part of Theorem 3.2.2, applied in $n + 1$ variables, therefore gives

$$P \circ H'_{2d+2i+1} = 0.$$

Expand

$$P = \sum_{j=0}^{2i+1} b_j x_0^{2i+1-j} \varphi(x_n^j F_{\underline{a}}),$$

where every b_j is nonzero, and use

$$H'_{2d+2i+1} = \sum_{t=0}^{2d+2i+1} X_0^t H_{2d+2i+1-t}.$$

The coefficient of X_0^{2i+1} in $P \circ H'_{2d+2i+1}$ is

$$\sum_{r=0}^{2i+1} c_r\Big(\varphi(x_n^{2i+1-r} F_{\underline{a}}) \circ H_{2d-r}\Big), \tag{13}$$

with all $c_r \neq 0$. For $r \geq 1$, each summand vanishes by the induction hypothesis. To verify the indices, first suppose that $r = 2u$ is even. Then

$$H_{2d-r} = H_{2(d-u)}, \qquad 2i + 1 - r = 2(i - u) + 1,$$

and $|\underline{a}| = d - i = (d - u) - (i - u)$, so the term belongs to the even family for the smaller degree $2(d - u)$. If $r = 2u + 1$ is odd, then

$$H_{2d-r} = H_{2(d-u-1)+1}, \qquad 2i + 1 - r = 2(i - u),$$

and

$$|\underline{a}| = d - i = (d - u - 1) + 1 - (i - u),$$

so the term belongs to the odd family for the smaller degree $2(d-u-1)+1$. The range of r ensures that the new indices $i-u$ are nonnegative. Hence all terms with $r \geq 1$ vanish. Since (13) is zero, its $r=0$ term is zero, proving

$$\varphi(x_n^{2i+1} F_{\underline{a}}) \circ H_{2d} = 0.$$

For $e = 2d+1$ and $i \geq 1$, start instead with

$$\varphi'\Big(F_{\underline{a}}(x_0 - x_n)^{2i}\Big) \circ H'_{2d+2i} = 0.$$

Taking the coefficient of X_0^{2i-1} gives

$$\sum_{r=0}^{2i} c'_r \Big(\varphi(x_n^{2i-r} F_{\underline{a}}) \circ H_{2d+1-r}\Big) = 0.$$

Again, the indices can be checked according to the parity of r. If $r = 2u$ is even, then

$$H_{2d+1-r} = H_{2(d-u)+1}, \qquad 2i - r = 2(i-u),$$

and

$$|\underline{a}| = d + 1 - i = (d-u) + 1 - (i-u),$$

so the odd induction hypothesis applies. If $r = 2u+1$ is odd, then

$$H_{2d+1-r} = H_{2(d-u)}, \qquad 2i - r = 2(i-u-1) + 1,$$

and

$$|\underline{a}| = d + 1 - i = (d-u) - (i-u-1),$$

so the even induction hypothesis applies. The permitted range of r again ensures that the new indices are nonnegative. Thus all terms with $r \geq 1$ vanish, and consequently the $r = 0$ term vanishes as well. The case $i = 0$ is already contained in (12). This completes the induction. □

We can now prove the main Gröbner-basis result of [BDL25, Theorem 3.4].

Theorem 3.3.4 (Boij–Duarte–Lundqvist)**.** *Let $G \in S_e$ be a generic form. Then*

$$\mathrm{in}(\mathrm{Ann}(H_e)) = \mathrm{in}(\mathrm{Ann}(G)).$$

More explicitly, if $e = 2d$, then

$$\mathcal{G}_{2d} = \{\varphi(F_{\underline{a}}) : |\underline{a}| = d+1\} \cup \{\varphi(x_n^{2i+1} F_{\underline{a}}) : 0 \leq i \leq d,\ |\underline{a}| = d - i\}$$

is a minimal Gröbner basis of $\operatorname{Ann}(H_{2d})$. *If* $e = 2d+1$, *then*

$$\mathcal{G}_{2d+1} = \{\varphi(x_n^{2i} F_{\underline{a}}) : 0 \le i \le d+1,\ |\underline{a}| = d+1-i\}$$

is a minimal Gröbner basis of $\operatorname{Ann}(H_{2d+1})$.

Proof. We give the odd case first. Proposition 3.3.3 shows that every element of $\mathcal{G}_{2d+1}$ belongs to I_{2d+1}. Factorial rescaling changes coefficients but not monomial supports. Moreover, among the terms of $x_n^{2i} F_{\underline{a}}$, the term with the smallest exponent of x_n is largest in degree reverse lexicographic order. Therefore

$$\operatorname{in}\Big(\varphi(x_n^{2i} F_{\underline{a}})\Big) = x_n^{2i} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

By Corollary 3.3.2, these are precisely the minimal monomial generators of $\operatorname{in}(\operatorname{Ann}(G))$.

$$\operatorname{in}(\operatorname{Ann}(G)) \subseteq \operatorname{in}(I_{2d+1}).$$

Both $R/\operatorname{Ann}(G)$ and R/I_{2d+1} are compressed Artinian Gorenstein algebras of socle degree $2d+1$: the first by the generic compressedness result [IK99, Proposition 3.12], and the second by Theorem 3.1.8. Hence they have the same Hilbert function. Proposition 2.3.14 gives

$$\operatorname{HF}_{R/\operatorname{in}(\operatorname{Ann}(G))} = \operatorname{HF}_{R/\operatorname{Ann}(G)} = \operatorname{HF}_{R/I_{2d+1}}.$$

The Hilbert-function criterion in Corollary 2.3.15 now shows that $\mathcal{G}_{2d+1}$ is a Gröbner basis of I_{2d+1} and that

$$\operatorname{in}(I_{2d+1}) = \operatorname{in}(\operatorname{Ann}(G)).$$

It is a minimal Gröbner basis because its leading monomials form the minimal monomial generating set in Corollary 3.3.2.

For $e = 2d$, Lemma 3.1.4 shows that the elements $\varphi(F_{\underline{a}})$ with $|\underline{a}| = d+1$ annihilate H_{2d}, and Proposition 3.3.3 handles the remaining elements. Their leading monomials are

$$\mathcal{A}_{d+1} \cup \bigcup_{i=0}^{d} x_n^{2i+1} \mathcal{A}_{d-i},$$

which is exactly the even list in Corollary 3.3.2. The same Hilbert-function argument proves the even case. □

The leading monomials in the odd case are

$$x_n^{2i} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}, \qquad |\underline{a}| = d+1-i, \quad 0 \le i \le d+1.$$

Hence the initial ideal is reverse lexicographic: in each degree, if it contains a monomial, then it contains every larger monomial of that degree. Notice, however, that a minimal Gröbner basis need not be a minimal generating set of the original ideal. The elements with $i \geq 2$ in $\mathcal{G}_{2d+1}$ have degree at least $d+3$ and cannot be minimal ideal generators, since Theorem 3.2.2 shows that I_{2d+1} is generated in degrees $d+1$ and $d+2$. Thus the degree $d+1$ elements

$$\varphi(F_{\underline{a}}), \qquad |\underline{a}| = d+1,$$

are minimal generators, and the only candidates for additional minimal generators are

$$\varphi(x_n^2 F_{\underline{a}}), \qquad |\underline{a}| = d. \tag{14}$$

There are $\binom{n+d-2}{n-2}$ candidates in (14), but some or all of them may belong to $R_1[I_{2d+1}]_{d+1}$. Equivalently, the unresolved quantity is

$$\dim_k \frac{[I_{2d+1}]_{d+2}}{R_1[I_{2d+1}]_{d+1}}.$$

Computations in [BDL25] predict that this dimension depends only on the parity of d. Understanding this parity-dependent behavior was the initial motivation for this thesis, whose original goal was to prove the following conjecture.

Conjecture 3.3.5 (Boij–Duarte–Lundqvist, Conjecture 3.7)**.** Let $n \geq 3$ and let $I_{2d+1} = \mathrm{Ann}(H_{2d+1})$.

(i) If d is odd, then I_{2d+1} has no minimal generator in degree $d+2$, and

$$I_{2d+1} = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1).$$

(ii) If d is even, then I_{2d+1} has exactly one minimal generator in degree $d+2$, and it may be chosen as

$$\varphi\Big(x_n^2(x_{n-1} - x_n)^d\Big).$$

Hence

$$I_{2d+1} = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1) + \left(\varphi\Big(x_n^2(x_{n-1} - x_n)^d\Big)\right).$$

The next section develops reductions among the candidate classes (14). These reductions show that their images modulo $R_1[I_{2d+1}]_{d+1}$ span a space of dimension at most one; the remaining question is whether that space vanishes, which is where the parity of d enters.

4 Results towards solving Conjecture 3.7

This section establishes three main results concerning Conjecture 3.3.5, which is Conjecture 3.7 of [BDL25]. We first prove Corollary 4.2.1, showing that the possible minimal generators in degree $d+2$, modulo those obtained from the generators in degree $d+1$, span a vector space of dimension at most one. We then combine this bound with the structure of grade three Gorenstein ideals to prove the conjecture in codimension 3 for every d in Theorem 4.2.2. Finally, we use the codimension-3 case together with the general dimension bound to prove the conjecture in every codimension when d is odd in Corollary 4.2.3.

The proof begins with a collection of reduction lemmas. Since φ is a linear isomorphism, relations among the candidate generators may be studied after applying φ^{-1}, where they take a more convenient form. We derive relations that reduce the candidates in degree $d+2$ modulo products of the degree $d+1$ generators. An induction then shows that all their classes are scalar multiples of a single class, yielding the dimension bound. The remaining arguments determine whether this class vanishes: first in codimension 3 by a parity argument, and then, for odd d, in arbitrary codimension by reducing to a suitable three-variable subring.

4.1 General reduction lemmas

Recall from Definition 3.2.1 that $F_{\underline{a}} = (x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}$. By Theorems 3.2.2 and 3.3.4, the forms $\varphi(F_{\underline{a}})$ with $|\underline{a}| = d+1$, are the minimal generators of $\mathrm{Ann}(H_{2d+1})$ in degree $d+1$, while the only candidates for additional minimal generators in degree $d+2$ are $\varphi(x_n^2 F_{\underline{a}})$ with $|\underline{a}| = d$. By Corollary 2.1.6, the problem is therefore to determine which of these candidates remain nonzero modulo

$$R_1[\mathrm{Ann}(H_{2d+1})]_{d+1}.$$

The lemmas in this subsection provide the relations needed to answer this question. We first describe how multiplication behaves after applying φ^{-1}. We then specialize this description to the forms $F_{\underline{a}}$ and combine the resulting formulas to obtain relations among the candidate degree $d+2$ generators.

Lemma 4.1.1. *For a polynomial $f \in R = k[x_1, \ldots, x_n]$ and a monomial x^α where $\alpha \in \mathbb{N}^n$, the following equality holds:*

$$\varphi^{-1}(x^\alpha \cdot \varphi(f)) = x^\alpha (x^\alpha \circ (x^\alpha f)).$$

Proof. Since both sides are linear in f, it is enough to prove it when f is a monomial. Thus, we need to show that

$$\varphi^{-1}(x^\alpha \cdot \varphi(x^\beta)) = x^\alpha (x^\alpha \circ (x^\alpha x^\beta)), \quad \beta \in \mathbb{N}^n.$$

Starting with the left-hand side of the equality, we obtain,

$$\varphi^{-1}(x^\alpha \cdot \varphi(x^\beta)) = \varphi^{-1}\left(\frac{1}{\beta!}x^{\alpha+\beta}\right) = \frac{(\alpha+\beta)!}{\beta!}x^{\alpha+\beta}.$$

Then let us compare it to the right-hand side of the equality,

$$x^\alpha(x^\alpha \circ (x^{\alpha+\beta})) = x^\alpha\left(\frac{(\alpha+\beta)!}{\beta!}x^\beta\right) = \frac{(\alpha+\beta)!}{\beta!}x^{\alpha+\beta}.$$

This completes the proof. □

Since R_1 is spanned by the variables $x_1, \ldots, x_n$, studying products of the degree $d+1$ generators reduces to understanding expressions of the form

$$\varphi^{-1}(x_i\varphi(F_{\underline{a}})).$$

Applying Lemma 4.1.1 with $x^\alpha = x_i$ gives the following explicit formulas.

Lemma 4.1.2. *For $n > 1$, let $\underline{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$. Then, for $1 \leq i \leq n-1$, we have the following equalities:*

(i) $\varphi^{-1}(x_i\varphi(F_{\underline{a}})) = x_iF_{\underline{a}} + a_ix_i^2F_{\underline{a}-e_i} = (a_i+1)x_i^2F_{\underline{a}-e_i} - x_ix_nF_{\underline{a}-e_i}, \quad a_i \geq 1.$

(ii) $\varphi^{-1}(x_i\varphi(F_{\underline{a}})) = x_iF_{\underline{a}}, \quad a_i = 0.$

(iii) $\varphi^{-1}(x_n\varphi(F_{\underline{a}})) = x_nF_{\underline{a}} - \sum_{a_j\neq 0} a_jx_n^2F_{\underline{a}-e_j}.$

Proof. We begin with (i). Assume that $a_i \geq 1$. Using Lemma 4.1.1, we obtain

$$\begin{aligned}\varphi^{-1}(x_i\varphi(F_{\underline{a}})) &= x_i(x_i \circ (x_iF_{\underline{a}}))\\ &= x_i\left(F_{\underline{a}} + a_ix_iF_{\underline{a}-e_i}\right).\end{aligned}$$

Since $F_{\underline{a}} = (x_i - x_n)F_{\underline{a}-e_i}$, this becomes

$$\begin{aligned}\varphi^{-1}(x_i\varphi(F_{\underline{a}})) &= x_i\left((x_i-x_n)F_{\underline{a}-e_i} + a_ix_iF_{\underline{a}-e_i}\right)\\ &= (a_i+1)x_i^2F_{\underline{a}-e_i} - x_ix_nF_{\underline{a}-e_i}.\end{aligned}$$

This proves (i).

For (ii), assume that $a_i = 0$. Then $F_{\underline{a}}$ does not depend on x_i. Hence, again using Lemma 4.1.1,

we get

$$\begin{aligned}\varphi^{-1}(x_i\varphi(F_{\underline{a}})) &= x_i(x_i \circ (x_i F_{\underline{a}})) \\ &= x_i F_{\underline{a}}.\end{aligned}$$

This proves (ii).

Finally, we prove (iii). Using Lemma 4.1.1, we obtain

$$\begin{aligned}\varphi^{-1}(x_n\varphi(F_{\underline{a}})) &= x_n(x_n \circ (x_n F_{\underline{a}})) \\ &= x_n\left(F_{\underline{a}} + x_n(x_n \circ F_{\underline{a}})\right).\end{aligned}$$

Now

$$x_n \circ F_{\underline{a}} = -\sum_{a_j \neq 0} a_j F_{\underline{a}-e_j},$$

since differentiating the factor $(x_j - x_n)^{a_j}$ with respect to x_n gives

$$-a_j(x_j - x_n)^{a_j-1}.$$

Therefore,

$$\begin{aligned}\varphi^{-1}(x_n\varphi(F_{\underline{a}})) &= x_n\left(F_{\underline{a}} - \sum_{a_j \neq 0} a_j x_n F_{\underline{a}-e_j}\right) \\ &= x_n F_{\underline{a}} - \sum_{a_j \neq 0} a_j x_n^2 F_{\underline{a}-e_j}.\end{aligned}$$

Thus we have shown (iii), and this concludes the proof. □

The formulas in Lemma 4.1.2 contain terms involving $x_r^2 F_{\underline{b}-e_r}$ and $x_r x_n F_{\underline{b}-e_r}$, in addition to terms of the desired form $x_n^2 F_{\underline{c}}$. We now choose suitable linear combinations in order to complete the square. This produces relations involving only a form $F_{\underline{b}+e_r}$ and terms of the form $x_n^2 F_{\underline{c}}$.

When $|\underline{b}| = d+1$, the left-hand sides of these relations belong to $\varphi^{-1}(R_1[I]_{d+1})$. The relations can therefore be used to compare the candidate degree $d+2$ generators modulo products of the degree $d+1$ generators.

Lemma 4.1.3. *Let $n \geq 3$, let $\underline{b} = (b_1, \ldots, b_{n-1}) \in \mathbb{N}^{n-1}$, and let $1 \leq r \leq n-1$.*

If $b_r \geq 1$, then

$$\begin{aligned}
&-\frac{1}{2b_r+1}\varphi^{-1}(x_r\varphi(F_{\underline{b}})) + \varphi^{-1}(x_n\varphi(F_{\underline{b}})) \\
&= -\frac{b_r+1}{2b_r+1}F_{\underline{b}+e_r} - \frac{2b_r(b_r+1)}{2b_r+1}x_n^2F_{\underline{b}-e_r} - \sum_{\substack{1\leq j\leq n-1\\ j\neq r,\ b_j\neq 0}} b_jx_n^2F_{\underline{b}-e_j}.
\end{aligned}$$

If $b_r = 0$, then

$$-\varphi^{-1}(x_r\varphi(F_{\underline{b}})) + \varphi^{-1}(x_n\varphi(F_{\underline{b}})) = -F_{\underline{b}+e_r} - \sum_{\substack{1\leq j\leq n-1\\ b_j\neq 0}} b_jx_n^2F_{\underline{b}-e_j}.$$

Proof. Assume first that $b_r \geq 1$. By Lemma 4.1.2, we have

$$\begin{aligned}
\varphi^{-1}(x_r\varphi(F_{\underline{b}})) &= (b_r+1)x_r^2F_{\underline{b}-e_r} - x_rx_nF_{\underline{b}-e_r}, \\
\varphi^{-1}(x_n\varphi(F_{\underline{b}})) &= x_nF_{\underline{b}} - \sum_{b_j\neq 0} b_jx_n^2F_{\underline{b}-e_j}.
\end{aligned}$$

Since

$$x_nF_{\underline{b}} = x_rx_nF_{\underline{b}-e_r} - x_n^2F_{\underline{b}-e_r},$$

we get, for any $\alpha \in k$,

$$\begin{aligned}
&\alpha\varphi^{-1}(x_r\varphi(F_{\underline{b}})) + \varphi^{-1}(x_n\varphi(F_{\underline{b}})) \\
&= \alpha(b_r+1)x_r^2F_{\underline{b}-e_r} + (1-\alpha)x_rx_nF_{\underline{b}-e_r} - (b_r+1)x_n^2F_{\underline{b}-e_r} \\
&\quad - \sum_{\substack{1\leq j\leq n-1\\ j\neq r,\ b_j\neq 0}} b_jx_n^2F_{\underline{b}-e_j}.
\end{aligned}$$

Choose

$$\alpha = -\frac{1}{2b_r+1}.$$

Then, $1-\alpha = -2\alpha(b_r+1)$. Therefore

$$\begin{aligned}
&\alpha(b_r+1)x_r^2F_{\underline{b}-e_r} + (1-\alpha)x_rx_nF_{\underline{b}-e_r} - (b_r+1)x_n^2F_{\underline{b}-e_r} \\
&= \alpha(b_r+1)x_r^2F_{\underline{b}-e_r} - 2\alpha(b_r+1)x_rx_nF_{\underline{b}-e_r} + \alpha(b_r+1)x_n^2F_{\underline{b}-e_r} - \Big(\alpha(b_r+1)+(b_r+1)\Big)x_n^2F_{\underline{b}-e_r} \\
&= -\frac{b_r+1}{2b_r+1}(x_r-x_n)^2F_{\underline{b}-e_r} - \frac{2b_r(b_r+1)}{2b_r+1}x_n^2F_{\underline{b}-e_r} \\
&= -\frac{b_r+1}{2b_r+1}F_{\underline{b}+e_r} - \frac{2b_r(b_r+1)}{2b_r+1}x_n^2F_{\underline{b}-e_r}.
\end{aligned}$$

Substituting this into the previous expression proves the first formula.

Now assume that $b_r = 0$. By Lemma 4.1.2, we have

$$\varphi^{-1}(x_r\varphi(F_{\underline{b}})) = x_r F_{\underline{b}},$$
$$\varphi^{-1}(x_n\varphi(F_{\underline{b}})) = x_n F_{\underline{b}} - \sum_{b_j\neq 0} b_j x_n^2 F_{\underline{b}-e_j}.$$

Hence

$$\begin{aligned}
&-\varphi^{-1}(x_r\varphi(F_{\underline{b}})) + \varphi^{-1}(x_n\varphi(F_{\underline{b}})) \\
&= -x_r F_{\underline{b}} + x_n F_{\underline{b}} - \sum_{b_j\neq 0} b_j x_n^2 F_{\underline{b}-e_j} \\
&= -(x_r - x_n) F_{\underline{b}} - \sum_{b_j\neq 0} b_j x_n^2 F_{\underline{b}-e_j} \\
&= -F_{\underline{b}+e_r} - \sum_{b_j\neq 0} b_j x_n^2 F_{\underline{b}-e_j}.
\end{aligned}$$

This proves the second formula. □

The cancellation relations can now be organized into a reduction procedure. Given a candidate $x_n^2 F_{\underline{a}}$ with some $a_s > 0$ for $s \geq 2$, we apply Lemma 4.1.3 in two different ways and eliminate the common F-term from the resulting relations. This expresses its class modulo $\varphi^{-1}(R_1[I]_{d+1})$ in terms of candidates for which the sum $a_2+\cdots+a_{n-1}$ is smaller. Repeating this process transfers all exponent weight to the first coordinate and reduces every candidate to the class represented by $x_n^2 F_{(d,0,\ldots,0)}$.

Proposition 4.1.4. *Let $n \geq 3$, let $I = \mathrm{Ann}(H_{2d+1}) \subseteq R = k[x_1, \ldots, x_n]$, and set*

$$W = \varphi^{-1}\left(R_1[I]_{d+1}\right) \subseteq R_{d+2}.$$

For $|\underline{a}| = d$, write

$$G_{\underline{a}} = x_n^2 F_{\underline{a}}.$$

Then, for every $\underline{a} \in \mathbb{N}^{n-1}$ with $|\underline{a}| = d$, we have

$$G_{\underline{a}} \in \mathrm{span}_k\{G_{(d,0,\ldots,0)}\} + W.$$

Proof. We prove the claim by induction on

$$m(\underline{a}) = a_2 + \cdots + a_{n-1}.$$

If $m(\underline{a}) = 0$, then $\underline{a} = (d, 0, \ldots, 0)$, and there is nothing to prove.

Assume now that $m(\underline{a}) > 0$. Choose an index $s \in \{2, \ldots, n-1\}$ such that $a_s > 0$. We will

show that $G_{\underline{a}}$ is congruent modulo W to a linear combination of terms $G_{\underline{b}}$ with $m(\underline{b}) < m(\underline{a})$. The induction hypothesis will then imply the claim.

First apply Lemma 4.1.3 to

$$\underline{b} = \underline{a} + e_1$$

with $r = 1$. Since $|\underline{b}| = d + 1$, we have $\varphi(F_{\underline{b}}) \in [I]_{d+1}$, and hence the left-hand side of the relation in Lemma 4.1.3 lies in W. Since $b_1 = a_1 + 1 \geq 1$, we obtain

$$-\frac{a_1+2}{2a_1+3}F_{\underline{a}+2e_1} - \frac{2(a_1+1)(a_1+2)}{2a_1+3}G_{\underline{a}} - \sum_{\substack{2\leq j\leq n-1\\ a_j\neq 0}} a_j G_{\underline{a}+e_1-e_j} \in W. \tag{15}$$

Next apply Lemma 4.1.3 to

$$\underline{b} = \underline{a} + 2e_1 - e_s$$

with $r = s$. This is well-defined because $a_s > 0$, and again $|\underline{b}| = d+1$. If $a_s > 1$, then $b_s = a_s - 1 \geq 1$, and Lemma 4.1.3 gives

$$\begin{aligned} -\frac{a_s}{2a_s-1}F_{\underline{a}+2e_1} - \frac{2(a_s-1)a_s}{2a_s-1}G_{\underline{a}+2e_1-2e_s} - (a_1+2)G_{\underline{a}+e_1-e_s} \\ - \sum_{\substack{2\leq j\leq n-1\\ j\neq s,\ a_j\neq 0}} a_j G_{\underline{a}+2e_1-e_s-e_j} \in W. \end{aligned} \tag{16}$$

If $a_s = 1$, then $b_s = 0$, and Lemma 4.1.3 gives

$$-F_{\underline{a}+2e_1} - (a_1+2)G_{\underline{a}+e_1-e_s} - \sum_{\substack{2\leq j\leq n-1\\ j\neq s,\ a_j\neq 0}} a_j G_{\underline{a}+2e_1-e_s-e_j} \in W. \tag{17}$$

Let E_1 denote the expression on the left-hand side of (15), and let E_2 denote the expression on the left-hand side of whichever of (16) or (17) applies. Define

$$\lambda = \begin{cases} \dfrac{(a_1+2)(2a_s-1)}{(2a_1+3)a_s}, & a_s > 1, \\ \dfrac{a_1+2}{2a_1+3}, & a_s = 1. \end{cases}$$

Since $E_1, E_2 \in W$, we also have

$$E_1 - \lambda E_2 \in W.$$

The choice of λ makes the coefficient of $F_{\underline{a}+2e_1}$ in this linear combination equal to zero. Indeed, if $a_s > 1$, this coefficient is

$$-\frac{a_1+2}{2a_1+3} + \lambda\frac{a_s}{2a_s-1} = 0,$$

while if $a_s = 1$, it is

$$-\frac{a_1+2}{2a_1+3}+\lambda = 0.$$

The second relation contains no term $G_{\underline{a}}$, so its coefficient in $E_1 - \lambda E_2$ remains

$$-\frac{2(a_1+1)(a_1+2)}{2a_1+3} \neq 0.$$

Therefore we may solve the resulting relation modulo W for the class of $G_{\underline{a}}$. It is a linear combination of classes $G_{\underline{b}}$ with

$$m(\underline{b}) < m(\underline{a}).$$

Indeed, all remaining G-terms are obtained from $\underline{a}$ by increasing the first coordinate and decreasing at least one of the coordinates $a_2, \ldots, a_{n-1}$.

By the induction hypothesis, each such $G_{\underline{b}}$ lies in

$$\operatorname{span}_k\{G_{(d,0,\ldots,0)}\} + W.$$

Hence

$$G_{\underline{a}} \in \operatorname{span}_k\{G_{(d,0,\ldots,0)}\} + W.$$

This completes the induction. □

4.2 Applications to Conjecture 3.7

Our first application of Proposition 4.1.4 is to translate the reduction among the candidate classes into a bound on the number of minimal generators in degree $d+2$. Together with Corollary 2.1.6, the proposition will show that there can be at most one such generator in any codimension.

This bound does not determine whether the remaining class is zero. After establishing the bound, we resolve this question in codimension 3 using the parity of the number of generators of a grade three Gorenstein ideal. Finally, we use the codimension-3 result to prove the conjecture in arbitrary codimension when d is odd.

Corollary 4.2.1. *Let $n \geq 3$ and let*

$$I = \operatorname{Ann}(H_{2d+1}) \subseteq R = k[x_1, \ldots, x_n].$$

Then

$$\dim_k \frac{[I]_{d+2}}{R_1[I]_{d+1}} \leq 1.$$

Proof. Set

$$W = \varphi^{-1}\left(R_1[I]_{d+1}\right) \subseteq R_{d+2}.$$

By Theorems 3.2.2 and 3.3.4, the possible minimal generators of I in degree $d+2$ are represented by the elements

$$\varphi(x_n^2 F_{\underline{a}}), \quad |\underline{a}| = d.$$

Since φ is a k-linear isomorphism, it is enough to study the classes of

$$G_{\underline{a}} = x_n^2 F_{\underline{a}}$$

modulo W.

By Proposition 4.1.4, for every $\underline{a} \in \mathbb{N}^{n-1}$ with $|\underline{a}| = d$, we have

$$G_{\underline{a}} \in \operatorname{span}_k\{G_{(d,0,\dots,0)}\} + W.$$

Therefore all possible degree $d+2$ generators have classes spanned by the single class of

$$G_{(d,0,\dots,0)} = x_n^2 F_{(d,0,\dots,0)}.$$

It follows that

$$\dim_k \frac{[I]_{d+2}}{R_1[I]_{d+1}} \leq 1.$$

□

We now combine this bound with the Buchsbaum–Eisenbud parity result to determine exactly what happens in codimension 3.

Theorem 4.2.2. *Let $n = 3$. Then Conjecture 3.7 of [BDL25] holds. More precisely, if d is odd, then*

$$\operatorname{Ann}(H_{2d+1}) = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1),$$

and if d is even, then

$$\operatorname{Ann}(H_{2d+1}) = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1) + (\varphi(x_3^2(x_2 - x_3)^d)).$$

Proof. Let

$$I = \operatorname{Ann}(H_{2d+1}) \subseteq R = k[x_1, x_2, x_3].$$

By Theorems 3.2.2 and 3.3.4, the ideal I is generated in degrees $d+1$ and $d+2$. Moreover, the minimal generators in degree $d+1$ are precisely

$$\varphi(F_{\underline{a}}), \quad |\underline{a}| = d+1.$$

Since $n = 3$, these are the elements

$$\varphi(F_{(i,d+1-i)}), \quad 0 \leq i \leq d+1.$$

Hence there are exactly $d+2$ minimal generators in degree $d+1$.

The only possible minimal generators in degree $d+2$ are represented by the elements

$$\varphi(x_3^2 F_{(i,d-i)}), \quad 0 \leq i \leq d.$$

By Corollary 4.2.1, we have

$$\dim_k \frac{[I]_{d+2}}{R_1[I]_{d+1}} \leq 1.$$

Therefore I has at most one minimal generator in degree $d+2$.

On the other hand, I is an Artinian Gorenstein ideal of height 3. Since $R = k[x_1, x_2, x_3]$ is Cohen–Macaulay, I has grade 3. Hence, by Corollary 2.5.5, the total number of minimal generators of I is odd.

Now consider the parity of d. If d is odd, then $d+2$ is odd. Since I already has $d+2$ minimal generators in degree $d+1$, and since there is at most one additional minimal generator in degree $d+2$, there cannot be any additional generator in degree $d+2$. Otherwise the total number of minimal generators would be $d+3$, which is even. Therefore

$$[I]_{d+2} = R_1[I]_{d+1},$$

and hence

$$I = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1).$$

If d is even, then $d+2$ is even. Therefore the $d+2$ minimal generators in degree $d+1$ cannot be the full minimal generating set of I, since the total number of minimal generators must be odd. Thus I must have at least one minimal generator in degree $d+2$. Since we already know that there is at most one such generator, there is exactly one.

It remains to identify a possible choice for this generator. By Proposition 4.1.4, the quotient

$$\frac{[I]_{d+2}}{R_1[I]_{d+1}}$$

is spanned by the class of

$$\varphi(x_3^2 F_{(d,0)}).$$

Since the quotient is nonzero in the even case, this class is nonzero. By symmetry in the variables

x_1 and x_2, the class of

$$\varphi(x_3^2 F_{(0,d)}) = \varphi(x_3^2(x_2 - x_3)^d)$$

is also nonzero and spans the same one-dimensional quotient. Therefore the unique minimal generator in degree $d+2$ may be chosen to be

$$\varphi(x_3^2(x_2 - x_3)^d).$$

This proves both cases. □

The preceding argument uses the structure of grade three Gorenstein ideals and therefore applies directly only when $n = 3$. Nevertheless, when d is odd, the codimension-3 result can be used to determine the distinguished class from Proposition 4.1.4 for every $n \geq 3$. This gives the following extension.

Corollary 4.2.3. *Let d be odd and let $n \geq 3$. Then Conjecture 3.7 of [BDL25] holds. That is,*

$$\mathrm{Ann}(H_{2d+1}) = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1).$$

Proof. Let $I = \mathrm{Ann}(H_{2d+1}) \subseteq R = k[x_1, \ldots, x_n]$, and set

$$W = \varphi^{-1}\left(R_1[I]_{d+1}\right) \subseteq R_{d+2}.$$

By Theorems 3.2.2 and 3.3.4, the ideal I is generated in degrees $d+1$ and $d+2$, and the only possible minimal generators in degree $d+2$ are represented by the elements

$$\varphi(x_n^2 F_{\underline{a}}), \quad |\underline{a}| = d.$$

Since φ is a k-linear isomorphism, it is enough to show that $x_n^2 F_{\underline{a}} \in W$ for every $\underline{a} \in \mathbb{N}^{n-1}$ with $|\underline{a}| = d$.

Write $G_{\underline{a}} = x_n^2 F_{\underline{a}}$. By Proposition 4.1.4, every such $G_{\underline{a}}$ satisfies

$$G_{\underline{a}} \in \mathrm{span}_k\{G_{(d,0,\ldots,0)}\} + W.$$

Thus it remains to show that $G_{(d,0,\ldots,0)} = x_n^2(x_1 - x_n)^d$ belongs to W.

Consider the subring $R^{(3)} = k[x_1, x_2, x_n] \subseteq R$. Let $H^{(3)}_{2d+1} \in k[X_1, X_2, X_n]$ denote the complete symmetric form of degree $2d+1$ in the three variables X_1, X_2, X_n. By Theorem 4.2.2, applied to these three variables, and since d is odd, there are no minimal generators in degree $d+2$ for

$$\mathrm{Ann}_{R^{(3)}}(H^{(3)}_{2d+1}).$$

Equivalently,

$$x_n^2(x_1 - x_n)^d \in \varphi^{-1}\left((R^{(3)})_1[\mathrm{Ann}_{R^{(3)}}(H^{(3)}_{2d+1})]_{d+1}\right).$$

The degree $d+1$ generators appearing in this three-variable relation are $\varphi((x_1-x_n)^i(x_2-x_n)^{d+1-i})$ for $0 \le i \le d+1$. Viewed inside R, these are precisely the elements $\varphi(F_{(i,d+1-i,0,\dots,0)})$, and they belong to $[I]_{d+1}$. Therefore the same relation, viewed inside R, shows that

$$x_n^2(x_1 - x_n)^d \in W.$$

Hence $G_{(d,0,\dots,0)} \in W$.

Now Proposition 4.1.4 implies that

$$G_{\underline{a}} \in W$$

for every $\underline{a} \in \mathbb{N}^{n-1}$ with $|\underline{a}| = d$. Therefore all possible degree $d+2$ generators vanish modulo $R_1[I]_{d+1}$, so

$$[I]_{d+2} = R_1[I]_{d+1}.$$

Since I is generated in degrees $d+1$ and $d+2$, it follows that

$$I = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1).$$

This proves the claim. □

4.3 The remaining even case

The preceding results prove Conjecture 3.3.5 for every d when $n = 3$, and for every $n \ge 3$ when d is odd. The only case not covered by these results is therefore

$$n > 3 \qquad \text{and} \qquad d \text{ even}.$$

We now describe precisely what remains to be proved in this case and why the argument used for odd d does not extend directly.

Let

$$I = \mathrm{Ann}(H_{2d+1}) \subseteq R = k[x_1, \dots, x_n]$$

and consider the vector space

$$Q_{n,d} = \frac{[I]_{d+2}}{R_1[I]_{d+1}}. \tag{18}$$

By Corollary 4.2.1, we have

$$\dim_k Q_{n,d} \le 1.$$

Moreover, Proposition 4.1.4 shows that all the candidate generators in degree $d+2$ have classes contained in the span of a single distinguished class. By symmetry, this class may be represented by

$$\varphi\Big(x_n^2(x_{n-1}-x_n)^d\Big).$$

Consequently, when d is even, Conjecture 3.3.5 is equivalent to proving that

$$\varphi\Big(x_n^2(x_{n-1}-x_n)^d\Big) \notin R_1[I]_{d+1}. \tag{19}$$

Equivalently, it is enough to show that $Q_{n,d}$ is nonzero. The general dimension bound would then imply that $\dim_k Q_{n,d} = 1$, and hence that there is exactly one minimal generator in degree $d+2$.

For $n = 3$, the nonvanishing in (19) follows from the Buchsbaum–Eisenbud parity argument used in Theorem 4.2.2. That argument depends on the special structure of grade three Gorenstein ideals and does not apply directly when $n > 3$. Nor does nonvanishing in three variables automatically imply nonvanishing in a larger polynomial ring. After additional variables are introduced, the space $[I]_{d+1}$ contains further generators involving those variables, and $R_1[I]_{d+1}$ therefore contains additional degree $d+2$ relations. It is possible in principle that the distinguished element becomes a combination of these new products, even though no such expression exists in three variables.

This explains an important difference between the odd and even cases. For odd d, the three-variable result provides an explicit membership relation, and the same relation remains valid after it is viewed in a larger polynomial ring. For even d, the three-variable result gives a nonmembership statement, which need not be preserved when the denominator in (18) becomes larger. A proof in the remaining case must therefore detect the distinguished class directly in higher codimension or show that the distinguished element cannot be written as a k-linear combination of products ℓf, where $\ell \in R_1$ and $f \in [I]_{d+1}$. In the remainder of this subsection, we examine this problem computationally and discuss observations that may be useful in finding such an argument.

We first give a direct computational verification for one choice of the parameters. For fixed values of n and d, the conjecture can be tested by computing the annihilator directly. In the even case, let

$$J_{n,d} = (\varphi(F_{\underline{a}}) : |\underline{a}| = d+1) + \left(\varphi\Big(x_n^2(x_{n-1}-x_n)^d\Big)\right).$$

Conjecture 3.3.5 predicts that

$$J_{n,d} = \operatorname{Ann}(H_{2d+1}).$$

The Macaulay2 code in Appendix A.2 constructs the complete symmetric form, computes its annihilator with respect to the differential action, and obtains a minimal generating matrix for this annihilator. It then constructs the ideal $J_{n,d}$ from the explicit generators in the conjecture and tests the equality of the two ideals.

As an example, take $n = 4$ and $d = 2$. Thus the computation concerns the form H_5 in four variables, which lies in the remaining case since $n > 3$ and d is even. The output for the degrees of the minimal generators consists of ten generators in degree 3 and one generator in degree 4. The final ideal comparison returns `true`. Hence, in this example,

$$\mathrm{Ann}(H_5) = (\varphi(F_{\underline{a}}) : |\underline{a}| = 3) + \Big(\varphi\Big(x_4^2(x_3 - x_4)^2\Big)\Big),$$

as predicted by the conjecture. The parameters at the beginning of the code may be changed to test other values of n and even d.

A second computational approach focuses directly on the quotient $Q_{n,d}$ from (18). Let

$$M = (\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1).$$

Since these forms constitute $[I]_{d+1}$, the degree $d + 2$ part of M is

$$[M]_{d+2} = R_1[I]_{d+1}.$$

For every $\underline{a} \in \mathbb{N}^{n-1}$ with $|\underline{a}| = d$, consider the candidate

$$\varphi(x_n^2 F_{\underline{a}}).$$

The code in Appendix A.3 computes its normal form modulo a Gröbner basis of M. It then expresses these normal forms in the monomial basis of R_{d+2} and places their coefficient vectors as the columns of a matrix, which we denote by $A_{n,d}$.

The columns of $A_{n,d}$ represent the classes of all the possible degree $d + 2$ generators modulo $R_1[I]_{d+1}$. Since these candidates span $Q_{n,d}$, it follows that

$$\mathrm{rank}(A_{n,d}) = \dim_k Q_{n,d}.$$

Thus the conjecture predicts that this rank is 0 when d is odd and 1 when d is even. The kernel of $A_{n,d}$ records the linear relations among the candidate classes. The number of columns is

$$\binom{n + d - 2}{n - 2},$$

so its kernel has dimension

$$\binom{n + d - 2}{n - 2} - \mathrm{rank}(A_{n,d}).$$

For $n = 4$ and $d = 4$, there are

$$\binom{4+4-2}{4-2} = \binom{6}{2} = 15$$

candidate classes. The computation gives $\operatorname{rank}(A_{4,4}) = 1$ and $\dim_k \ker(A_{4,4}) = 14$. In particular, the matrix is nonzero, so the distinguished class survives modulo $R_1[I]_{d+1}$ in this example. The rank agrees with both Conjecture 3.3.5 and the upper bound proved in Corollary 4.2.1.

The computations were repeated for several values of n and even d. In Table 1, the third column gives the number of minimal generators in degree $d+1$, as obtained from the output of `mingens`. The fourth column gives the number

$$\binom{n+d-2}{n-2}$$

of candidate generators in degree $d+2$. In every case, the minimal generating matrix contained exactly one generator in degree $d+2$, the direct comparison $J_{n,d} = \operatorname{Ann}(H_{2d+1})$ returned `true`, and the normal-form matrix had rank 1.

n	d	# in degree $d+1$	# candidates in degree $d+2$	$\operatorname{rank}(A_{n,d})$
4	2	10	6	1
5	2	20	10	1
6	2	35	15	1
7	2	56	21	1
4	4	21	15	1
5	4	56	35	1
6	4	126	70	1
4	6	36	28	1
5	6	120	84	1
4	8	55	45	1

Table 1: Computational tests of Conjecture 3.3.5 for $n > 3$ and even d. In every row, the direct ideal comparison returned `true` and there was one minimal generator in degree $d+2$.

The two computations have complementary purposes. The direct computation checks the full predicted generating set against the annihilator, while the normal-form matrix isolates the precise quotient that remains unresolved in the general case. These examples provide evidence for the conjecture, but a proof for all $n > 3$ and even d would require a uniform argument showing that $A_{n,d}$ always has positive rank.

5 Concluding remarks and further directions

In this thesis, we have studied Conjecture 3.3.5 on the minimal generators of the annihilator of the complete symmetric form H_{2d+1}. The conjecture predicts that the minimal generating set depends on the parity of d: when d is odd, the generators in degree $d+1$ generate the entire annihilator, whereas when d is even, one additional minimal generator appears in degree $d+2$.

The main reduction established here shows that the apparently large collection of possible degree $d+2$ generators is controlled by a single class. More precisely, for every $n \geq 3$, we proved that

$$\dim_k \frac{[\mathrm{Ann}(H_{2d+1})]_{d+2}}{R_1[\mathrm{Ann}(H_{2d+1})]_{d+1}} \leq 1.$$

Thus, although the Gröbner basis provides $\binom{n+d-2}{n-2}$ candidates in degree $d+2$, determining the minimal generators reduces to deciding whether one distinguished class vanishes. This replaces a problem involving many candidate generators by a single membership or nonmembership question.

In codimension 3, the structure theorem of Buchsbaum–Eisenbud for grade three Gorenstein ideals determines whether this class vanishes. Combined with the dimension bound, the parity of the total number of minimal generators proves Conjecture 3.3.5 for every d when $n = 3$. The relations obtained in the odd three-variable case remain valid after passing to a polynomial ring with more variables, which proves the conjecture for every $n \geq 3$ when d is odd. Consequently, the only case that remains open is

$$n > 3 \qquad \text{and} \qquad d \text{ even}.$$

As described in Subsection 4.3, the remaining case is equivalent to proving the nonvanishing

$$\varphi\Big(x_n^2(x_{n-1}-x_n)^d\Big) \notin R_1[\mathrm{Ann}(H_{2d+1})]_{d+1}.$$

The computations reported there support this prediction: in every tested even case with $n > 3$, the normal-form matrix has rank one and the annihilator has exactly one minimal generator in degree $d+2$. These calculations do not give a uniform proof, but they show that the distinguished class survives in a range of examples and that the general upper bound is attained.

The obstacle is that a membership relation in three variables can be carried to a larger polynomial ring, whereas a nonmembership statement need not be preserved after new degree $d+1$ generators and hence new degree $d+2$ relations become available. A proof of the remaining case must therefore detect nonvanishing directly in higher codimension. One possible approach is to construct a linear functional on $[\mathrm{Ann}(H_{2d+1})]_{d+2}$ which vanishes on $R_1[\mathrm{Ann}(H_{2d+1})]_{d+1}$ but not on the distinguished element. It may also be useful to study whether the class can be detected through the minimal free resolution or through additional structure of the associated Artinian Gorenstein algebra.

There is also a natural comparison with the generic case. The annihilator of H_{2d+1} has the same

initial ideal as the annihilator of a generic form of the same degree. Thus the two ideals have the same Hilbert function and the same leading monomials in their minimal Gröbner bases. Nevertheless, these facts do not determine the minimal generators of the ideals themselves: a minimal Gröbner basis need not be a minimal generating set. The parity-dependent behavior studied in this thesis therefore raises the question of how closely the graded Betti numbers and minimal free resolution of $\mathrm{Ann}(H_{2d+1})$ resemble those arising from a generic form. Investigating this comparison may help explain whether the conjectured generator is a special feature of complete symmetric forms or reflects a more general phenomenon for compressed Artinian Gorenstein algebras.

In summary, the results of this thesis prove the conjecture in codimension 3 and for every $n \geq 3$ when d is odd, while reducing the unresolved even case to a concrete nonvanishing problem involving a single class. Together with the computational evidence, this reduction suggests that the conjecture should continue to hold for every $n \geq 3$ and provides a focused starting point for further work.

A Macaulay2 computations

This appendix contains the Macaulay2 computations used in the thesis. All computations in this appendix were carried out using the Macaulay2 web interface, version `1.26.06-1665-gc42566cf7f` (vanilla).

A.1 A Gröbner basis example

The following code performs the Gröbner basis computation used in Example 2.3.13 for the ideal

$$I = (xz + y^2, xy) \subseteq \mathbb{Q}[x, y, z]$$

with respect to graded reverse lexicographic order.

```
R = QQ[x,y,z, MonomialOrder => GRevLex]

I = ideal(x*z + y^2, x*y)

gens gb I
```

The computation returns the Gröbner basis

$$\{y^2 + xz,\ xy,\ x^2z\}.$$

A.2 Direct verification of the conjecture

The parameter values used in Appendix A.2 and Appendix A.3 are representative. To reproduce any row of Table 1, replace the values of n and d at the beginning of both code blocks and rerun the computations. The first computation verifies the predicted generators, while the second computes the normal-form rank. The direct verification uses the bundled `InverseSystems` package, which is loaded explicitly below.

```
restart

-- Load the package providing fromDual and DividedPowers.
needsPackage "InverseSystems"

-- Choose an even value of d to test the remaining case.
d = 2
```

```
n = 4
R = QQ[x_1..x_n]

-- Construct the complete symmetric form H_{2d+1} in R.
completeSymmetricForm = d -> sum flatten entries basis(2*d+1,R)

-- Apply the factorial rescaling map phi.
phi = f -> sum apply(terms f, t -> (
    e := (exponents t)#0;
    c := product apply(e, u -> u!);
    (1/c)*t
));

-- Construct all exponent vectors of length r and total degree q.
exponentVectors = (r,q) -> (
    R' := QQ[y_1..y_r];
    apply(flatten entries basis(q,R'), m -> first exponents m)
);

-- Construct F_a from its exponent vector a.
polynomialFromVector = v -> (
    product for i from 0 to #v-1 list ( (R_(i) - R_(n-1))^(v#i) )
);

-- Compute Ann(H_{2d+1}) with respect to the differential action.
-- DividedPowers => false selects differentiation instead of contraction.
annihilatorH = ideal fromDual(
    completeSymmetricForm(d),
    DividedPowers => false
)

-- Compute a Gröbner basis and then a minimal generating matrix.
annihilatorGB = gb annihilatorH
minimalGenerators = mingens annihilatorGB

-- Display the degrees of the minimal generators.
degrees source minimalGenerators
```

```
-- Construct the known generators in degree d+1.
degreeDPlusOneVectors = exponentVectors(n-1,d+1)

degreeDPlusOnePolynomials = apply(
    degreeDPlusOneVectors,
    v -> polynomialFromVector(v)
)

degreeDPlusOneGenerators = apply(
    degreeDPlusOnePolynomials,
    f -> phi(f)
)

-- Construct the additional generator predicted when d is even.
degreeDPlusTwoGenerator = phi(x_n^2*(x_(n-1)-x_n)^d)

-- Form the ideal generated by the elements predicted in the conjecture.
conjecturedIdeal = ideal append(
    degreeDPlusOneGenerators,
    degreeDPlusTwoGenerator
)

-- The output true verifies the conjecture for the chosen n and d.
conjecturedIdeal == annihilatorH
```

For the displayed values $n = 4$ and $d = 2$, the output of `degrees source minimalGenerators` reports ten generators in degree 3 and one generator in degree 4, and the final comparison returns `true`.

A.3 The normal-form matrix

```
restart

n = 4
d = 4

-- Apply the factorial rescaling map phi.
```

```
phi = f -> sum apply(terms f, t -> (
    e := (exponents t)#0;
    c := product apply(e, u -> u!);
    (1/c)*t
));

-- Construct all exponent vectors of length r and total degree q.
exponentVectors = (r,q) -> (
    R' := QQ[y_1..y_r];
    apply(flatten entries basis(q,R'), m -> first exponents m)
);

-- Construct F_a from its exponent vector a.
polynomialFromVector = v -> (
    product for i from 0 to #v-1 list ( (R_(i) - R_(n-1))^(v#i) )
);

R = QQ[x_1..x_n]

-- Construct the known generators in degree d+1.
degreeDPlusOneVectors = exponentVectors(n-1,d+1);

degreeDPlusOnePolynomials = apply(
    degreeDPlusOneVectors,
    v -> polynomialFromVector(v)
);

degreeDPlusOneGenerators = apply(
    degreeDPlusOnePolynomials,
    f -> phi(f)
);

degreeDPlusOneIdeal = ideal degreeDPlusOneGenerators;

-- Compute this Gröbner basis once and reuse it below.
degreeDPlusOneGB = gb degreeDPlusOneIdeal;
```

```
-- Construct the candidate generators in degree d+2.
degreeDPlusTwoVectors = exponentVectors(n-1,d);

degreeDPlusTwoCandidates = apply(
    degreeDPlusTwoVectors,
    v -> phi(x_n^2 * polynomialFromVector(v))
);

-- Compute their normal forms modulo the degree d+1 generators.
normalForms = apply(
    degreeDPlusTwoCandidates,
    f -> f % degreeDPlusOneGB
);

-- Choose the monomial basis of R_{d+2}.
monomialBasis = flatten entries basis(d+2, R);

-- Express a polynomial as a coefficient vector in this basis.
coefficientVector = f -> apply(
    monomialBasis,
    m -> coefficient(m, f)
);

-- Form the matrix whose columns represent the normal forms.
normalFormMatrix = transpose matrix apply(
    normalForms,
    f -> coefficientVector(f)
);

normalFormRank = rank normalFormMatrix;
normalFormKernel = ker normalFormMatrix;
kernelDimension = numColumns normalFormMatrix - normalFormRank;

-- Display the dimension of the span of the candidate classes.
normalFormRank

-- Display the number of independent relations among the classes.
```

```
kernelDimension
```

For the displayed values $n = 4$ and $d = 4$, the final two outputs are `normalFormRank = 1` and `kernelDimension = 14`.

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