SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Formality of Compact Kähler Manifolds and Bigraded Notions of Formality

av

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2026 - No M2

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Alexander Berglund

2026

Abstract

In this thesis, we study the notion of formality from rational homotopy theory. We give a modern treatment, using the language of commutative bidifferential bigraded algebras (cbba's), of an influential theorem of Deligne-Griffiths-Morgan-Sullivan, which states that compact Kähler manifolds are formal. We show that if $(A^{*,*}, \partial, \bar{\partial})$ is a cbba with the $\partial\bar{\partial}$-property, then its total commutative differential graded algebra (cdga) (A^*, d) is formal. The fact that compact Kähler manifolds are formal will then follow due to the $\partial\bar{\partial}$-Lemma, which we prove using the necessary material from complex geometry. We also discuss a further result: if the cohomology algebra of a cdga is of complete intersection type then it is formal. This result can be generalized to cbba's: If a cbba A satisfies the $\partial\bar{\partial}$-property and has cohomology algebra of bigraded complete intersection type, then A is strongly formal. As a consequence, we prove that compact homogeneous Kähler manifolds are strongly formal. Lastly, we discuss obstructions to formality and weak formality, given by Massey products and their bigraded generalization ABC-Massey products. Examples are provided, mostly of nilmanifolds, both of computational importance and as counterexamples.

Sammanfattning

I denna uppsats studeras begreppet formalitet från rationell homotopiteori. Vi diskuterar en inflytelserik sats, tillskriven Deligne-Griffiths-Morgan-Sullivan, som säger att kompakta Kählermångfalder är formella, med hjälp av den modernare teorin av kommutativa bidifferentiella bigraderade algebror (cbba:s). Vi bevisar att om $(A^{*,*}, \partial, \bar{\partial})$ är en cbba med $\partial\bar{\partial}$-egenskapen så är den totala kommutativa differentiella graderade algebran (cdga:n) (A^*, d) formell. Att kompakta Khälermångfalder är formella blir därefter en följd av $\partial\bar{\partial}$-Lemmat, som vi bevisar efter att ha etablerat den nödvändiga bakgrunden inom komplex geometri. Vi diskuterar även ett vidare resultat: Om kohomologialgebran av en cdga är en komplett skärning, så är den formell. Detta resultat generaliseras till bigraderade algebror och vi bevisar att en cbba A, vars kohomologialgebra är en bigraderad komplett skärning, är starkt formell, givet att A har $\partial\bar{\partial}$-egenskapen. Som följdsats får vi att kompakta homogena Kählermångfalder är starkt formella. Hinder för formalitet diskuteras också. Dessa ges av Massey-produkter och den bigraderade generaliseringen ABC-Massey-produkter. Exempel är försedda, främst nilmångfalder, både av beräkningsintresse och som motexempel.

Acknowledgements

I would like to first and foremost thank my advisor Alexander Berglund for all the help and guidance he has given me during the writing of this thesis. I would also like to thank him for introducing me to most of the subjects presented in this thesis.

As this thesis marks the end of my master's program in mathematics, I would like to thank all the wonderful people I have met on this Journey. Some people worth mentioning by name are Elis, Anders, August, Ronny, Sejad, Florence, Anton, Sofia, Benjamin and Nami. A special thanks goes out to Matilda for being there for me essentially every day.

Lastly, I would like to thank my sister, father and mother for their unending support and love for me, for without it I would never have come this far.

Contents

Introduction

This thesis is focused on the notion of formality, introduced in the 1970's, most notably in [DGMS75]. The slogan for formality is that if a space is formal then the rational cohomology algebra contains all the rational homotopy information of the space. In the same article as the notion's introduction, Deligne, Griffiths, Morgan and Sullivan give the first large class of formal spaces, compact Kähler manifolds.

The proof of this fact is purely algebraic after establishing the, so called, dd^c-lemma, or more commonly phrased as the $\partial\bar{\partial}$-lemma today, which motivates one to extract this information. In this thesis we prove that any commutative bidifferential bigraded algebra that satisfies the conclusion of the $\partial\bar{\partial}$-lemma is formal, giving us the conclusion in a more general framework and highlighting the crucial step.

The framework of commutative bidifferential bigraded algebras (cbba's), which we base of the recent work of Stelzig, et. al. [Ste25, MS24, PSZ25], leads to many generalizations of the notions of commutative differential graded algebras (cdga's). In particular, two new notions of formality; weak formality and strong formality. Strongly formal cbba's are both formal and weakly formal, but formality and weak formality are independent. These notions are of interest to the recent introduction of pluripotential homotopy theory [Ste25] which is trying to answer the broad question of what the holomorphic extra structure on the homotopy type of a complex manifold is.

Spaces with cohomology algebras of complete intersection type gives us another class of formal spaces. We give a detailed proof of this fact, following [Smi95], as well as a proof of a bigraded generalization of this, which gives us our first class of strongly formal manifolds, compact homogeneous Kähler manifolds. This generalization is due to Placini, Stelzig and Zoller [PSZ25], so, while our proof follows theirs closely, we write out a few more details and provide a proof of a lemma they cite.

Lastly, we discuss obstructions to formality. For cdga's the classic obstruction is the existence of non-trivial Massey products [Mas58, Kra66]. For cbba's the notion of Aeppli-Bott-Chern-Massey products has recently been established, see [AT15, MS24]. We study the first two such, the triple and quadruple ABC-Massey products, through examples.

The main examples throughout the thesis will be nilmanifolds. Since a nilmanifold is only formal if it is a torus, these will give us examples of non-formal spaces and show existence of non-trivial Massey products. By giving nilmanifolds complex structure we will also see examples of non-trivial ABC-Massey products. In particular, in this thesis we will work out some examples provided in literature in detail, see example 7.5 and 7.11, as well as provide our own computational examples, see example 4.9 and 7.9.

A reader familiar with the basics of algebraic topology and smooth manifolds, e.g. familiar with the material in [Hat01] and [Lee13], will hopefully be able to follow the thesis, thanks to the preliminary material in both rational homotopy theory and complex manifolds provided. While introductions to these subjects are plenty, this thesis provides both as well as a complete proof of the influential theorem of Deligne-Griffiths-Morgan-Sullivan. Moreover, the newer results included are picked out to be comprehensible without much background knowledge in the area.

Structure

The structure of the thesis is as follows: In Section 1 we recall the necessary material of rational homotopy theory, mostly following [FHT01]. Section 2 recalls Lie groups and nilmanifolds, as well as their minimal models. Section 3 concerns complex manifolds. We build up the theory from the ground, assuming knowledge of Riemannian manifolds, and concluding with the necessary properties of Kähler manifolds needed in later sections.

The theory of cbba's is introduced in section 4 following Stelzig's recent work [Ste25, MS24, PSZ25]. We note how the algebra of complex differential forms on a complex manifold makes up a cbba. Some of the various cohomologies of a cbba are collected here as well as the cohomology diamond, connecting them all. Strong and weak formality is also introduced. The section ends by discussing the pluripotential minimal model of complex nilmanifolds and an example of computational and historical importance.

Formality of compact Kähler manifolds is proven in section 5. The section is mostly concerned with the $\partial\bar{\partial}$-property. We prove that if a cbba A satisfies the $\partial\bar{\partial}$-property then all the natural morphisms in the cohomology diamond are isomorphism. Using the Green's operator, we show the $\partial\bar{\partial}$-Lemma, i.e. that the cbba of complex differential forms on a compact Kähler manifold satisfies the $\partial\bar{\partial}$-Lemma. Finally, we prove that if A satisfies the $\partial\bar{\partial}$-property then it is formal giving us the main theorem as a consequence.

In section 6 we study cdga's and cbba with certain cohomology algebras. We give a detailed proof of the classic theorem stating that topological spaces with cohomology algebra of complete intersection type are formal. The Theorem generalizes to bigraded algebra, and we prove that a cbba satisfying the $\partial\bar{\partial}$-property and with cohomology algebra of bigraded complete intersection type is strongly formal. The geometric corollary is that compact homogeneous Kähler manifolds are strongly formal, giving us a class of strongly formal manifolds.

We end the thesis by considering Massey products and their bigraded generalizations ABC-Massey products in section 7. Examples are provided as well as proofs that the triple Massey and ABC-Massey products are invariants under quasi-isomorphism and pluripotential quasi-isomorphisms respectively, giving us the results that they are obstructions to formality and weak formality.

1 Commutative differential graded algebras

Given a graded R-module $M = \bigoplus_{k\in\mathbb{Z}} M^k$ we say that $m \in M$ is of degree k if $m \in M^k$ and write $|m| = k$. If M and N are two graded R-modules then a homomorphism $f : M \to N$ is said to be of degree n if $f(M^k) \subset M^{n+k}$ for all $k \in \mathbb{Z}$. If A is a graded R-algebra then A is called (graded) commutative if $x\cdot y = (-1)^{k\ell} y\cdot x$ for $x \in A^k$ and $y \in A^\ell$. One can prove that the (graded) tensor product is the coproduct in the category of (graded) commutative R-algebras.

Recall that a cochain complex C over a ring R is a sequence of R-modules C^k ($k \in \mathbb{Z}$) and homomorphisms $d^k : C^k \to C^{k+1}$ such that $d^{k+1} \circ d^k = 0$ for all k. Alternatively, viewed as a graded R-module $C = \bigoplus_{k\in\mathbb{Z}} C^k$ together with a morphism $d : C \to C$ of degree 1 such that $d^2 = 0$. The cohomology of a cochain complex $H^*(C)$ is the graded R-module given by

$$H^k(C) = \ker(d^k : C^k \to C^{k+1}) / \operatorname{im}(d^{k-1} : C^{k-1} \to C^k).$$

Definition 1.1. A differential graded algebra over $\Bbbk$ ($\Bbbk$-dga) (A, d), or just A, is a graded $\Bbbk$-algebra $A = \bigoplus_{k\in\mathbb{Z}} A^k$ together with a degree one homomorphism $d : A \to A$ called a differential such that

$$d^2 = 0 \quad \text{and} \quad d(x \cdot y) = dx \cdot y + (-1)^k x \cdot dy \quad \text{if } x \in A^k.$$

Intuitively, (A, d) is a cochain complex that is also a graded algebra such that d satisfies the Leibniz rule.

A commutative differential graded algebra over $\Bbbk$ ($\Bbbk$-cdga) (A, d) is a $\Bbbk$-dga such that A is (graded) commutative.

A morphism of $\Bbbk$-(c)dga's $f : (A, d) \to (B, d)$ is a degree 0 $\Bbbk$-algebra homomorphism $f : A \to B$ such that $df = fd$, i.e. a multiplicative homomorphism of cochain complexes.

We will mostly be concered with cdga's moving forward, we denote the category of cdga's (over some field $\Bbbk$) and their morphisms by **CDGA**. Note that the cohomology of a cdga $H^*(A)$ inherits the commutative multiplication and becomes a graded commutative algebra. We will even view it as a cdga with trivial differential $d = 0$. Note also that any cdga, or in general any cochain complex, with trivial differential is its own cohomology. Recall that any morphism of cochain complexes $f : C \to D$ induces a morphism on cohomology $f^* : H^*(C) \to H^*(D)$. Thus cohomology defines a functor $H^* : \mathbf{CDGA} \to \mathbf{CDGA}$ with $(H^*)^2 = H^*$.

The tensor product of two cdga's is defined by taking the graded tensor product of their underlying algebras and the differential is defined on simple tensors by the Leibniz rule $d_{A\otimes B}(a\otimes b) = d_A(a)\otimes b + (-1)^{|a|}a\otimes d_B(b)$ and on general tensors by linearity. It follows that $H^*(A\otimes B)\cong H^*(A)\otimes H^*(B)$ by the Künneth Theorem. The base field $\Bbbk$ will be viewed as a cdga concentrated in degree 0 with $d(\lambda)=0$ for all $\lambda\in\Bbbk$. When viewed as a cdga we write either $(\Bbbk,0)$ or just $\Bbbk$ if the context is sufficient. A cdga A is called contractible if $H^*(A)\cong\Bbbk$.

Example 1.2. If M is a smooth manifold then the de Rham complex Ω^*M is a $\mathbb{R}$-cdga with multiplication given by wedge product, since

$$d(\omega\wedge\tau) = d\omega\wedge\tau + (-1)^k\omega\wedge d\tau \quad \text{if } \omega\in\Omega^kM.$$

In the context of cdga's it is often denoted $A_{dR}(M)$. If $f: M\to N$ is a smooth map then the pullback $f^*: A_{dR}(N)\to A_{dR}(M)$ is a morphism of cdga's. Thus A_{dR} defines a contravariant functor from the category of smooth manifolds to **CDGA**.

Definition 1.3. A morphism of cdga's $f: A\to B$ is called a quasi-isomorphism if the induced map on cohomology $f^*: H^*(A)\to H^*(B)$ is an isomorphism of graded algebras.

The name suggests that we define a homotopy category by formally inverting quasi-isomorphisms: $\mathrm{Ho}(\mathbf{CDGA}) = \mathbf{CDGA}[\text{quasi-iso}^{-1}]$. Recall that the morphisms in such a category are zig-zags where all the arrows going the opposite way are quasi-isomorphisms

$$A_1 \longrightarrow A_2 \xleftarrow{\sim} A_3 \longrightarrow A_4 \xleftarrow{\sim} A_5.$$

If V is a graded vector space over $\Bbbk$ then the free graded commutative algebra over V is the qoutient algebra TV/I where TV is the tensor algebra and I is the ideal generated by $v\otimes w - (-1)^{k\ell}w\otimes v$ for $v\in V^k$ and $w\in V^\ell$. We denote it by $\wedge V$ with multiplication written by simply $\cdot$ or omitted. Note that if V^{even} denotes the sum of the even degree summands of V and V^{odd} the odd degree summands then

$$\wedge V = \mathrm{Sym}(V^{\text{even}})\otimes\mathrm{Ext}(V^{\text{odd}})$$

where $\mathrm{Sym}(V)$ denotes the symmetric algebra of V and $\mathrm{Ext}(V)$ the exterior algebra of V. The grading of $\wedge V$ is given by

$$|v_1\cdots v_k| = |v_1|+\cdots+|v_k|.$$

The free graded commutative algebra over V can also be viewed as the subspace of Σ invariant tensors of TV under the action defined on transpositions $\tau=(ij)$ by

$$\tau(v_1\otimes\cdots\otimes v_i\otimes\cdots\otimes v_j\otimes\cdots\otimes v_p) = (-1)^{|v_i||v_j|}(v_1\otimes\cdots\otimes v_j\otimes\cdots\otimes v_i\otimes\cdots\otimes v_p)$$

and then extended to arbitrary permutations. In this thesis a free algebra means a free graded commutative algebra. By $\wedge(v_1, \dots, v_n)$, we mean the free algebra of the vector space spanned by $v_1, \dots, v_n$.

For free graded commutative algebras we also define the following subspaces

- $\wedge^p V$ is the subspace generated by products $v_1 \cdots v_p \in \wedge V$ for $v_1, \dots, v_p \in V$. We identify $\wedge^0 V$ with the underlying field $\Bbbk$ and $\wedge^1 V$ with V.
- $\wedge^+ V = \bigoplus_{n \geq 1} \wedge^n V$
- $\wedge^{\geq p} V = \bigoplus_{n \geq p} \wedge^n V$

If $v \in \wedge^{\geq 2} V$ we call v decomposable. Let us define the main object of interest.

Definition 1.4. A Sullivan cdga is a cdga whose algebra is free graded commutative $\wedge V$ for a positively graded vector space $V = \bigoplus_{n \geq 1} V^n$ such that V admits a well-ordered basis v_α satisfying $d(v_\alpha) \in \wedge(v_\beta)_{\beta < \alpha}$.

A minimal cdga is a Sullivan cdga such that $d(V) \subset \wedge^{\geq 2} V$. In other words, the image of d consists of decomposable elements.

Given a cdga (A, d_A) and a finite dimensional graded vector space V, an elementary extension of A is a new cdga $(B = A \otimes \wedge V, d_B)$ such that $d_B|_A = d_A$ and $d_B(V) \subset A$. Omitting the differential we denote such extensions by $A \otimes_d \wedge V$. Note that if $A = \wedge W$ then $\wedge W \otimes \wedge V = \wedge(W \otimes V)$, so an elementary extension of a Sullivan algebra is a Sullivan algebra by extending the well-ordered indexing set of W with $\dim V$ elements all greater than those of W.

In the classic paper [DGMS75], a minimal cdga M is defined as a cdga that can be written as an increasing union of sub-cdgas M_i

$$\Bbbk \subset M_1 \subset M_2 \subset \cdots \quad \text{and} \quad \bigcup_{i \geq 0} M_i = M$$

such that M_{i+1} is an elementary extension of M_i and d is decomposable. Such a sequence of sub-cdga's is called a series for M. These definitions are equivalent as can be seen by what we wrote above, that an elementary extension of a Sullivan algebra is a Sullivan algebra. Conversely, from a Sullivan algebra we can build such a series using the well-ordered indexing set, which can be put in correspondence with a subset of the naturals as the dimension of V is at most countable. Both definitions also just assumes that d is decomposable.

An augmented cdga is a cdga (A, d) together with a cdga morphism $\epsilon : (A, d) \to (\Bbbk, 0)$, called an augmentation, that induces an isomorphism in degree zero homology $\epsilon^* : H^0(A, d) \xrightarrow{\cong} H^0(\Bbbk, 0) = \Bbbk$. The kernel of ϵ is denoted $\overline{A}$ and

$Q(A)$ is defined as the quotient $\overline{A}/(\overline{A}\cdot\overline{A})$. This becomes a cochain complex with the induced differential $Q(d)$, which exists as $d(\overline{A}) \subset \overline{A}$ and $d(\overline{A}\cdot\overline{A}) \subset \overline{A}\cdot\overline{A}$.

If $\wedge V$ is a Sullivan algebra then as $(\wedge V)^0 = \mathbb{k}$ and $(\wedge V)^k = 0$ for $k < 0$ there is an obvious augmentation which makes $\overline{\wedge V} = \wedge^+ V$, $Q(\wedge V)$ naturally identified with V and $Q(d)$ the "linear" part of d, i.e. in the decomposition of $d(v)$ in $\bigoplus_{p\geq 1} \wedge^p V$ the $\wedge^1 V = V$ part. If $f : \wedge V \to \wedge W$ is a cdga morphism then it induces a morphism of cochain complexes $Q(f) : Q(\wedge V) \to Q(\wedge W)$.

Proposition 1.5. A morphism of Sullivan algebras $f : \wedge V \to \wedge W$ is a quasi-isomorphism if and only if $Q(f) : Q(\wedge V) \to Q(\wedge W)$ is a quasi-isomorphism (of cochain complexes).

For a proof see Proposition 14.13 in [FHT01].

Corollary 1.6. A morphism of minimal cdga's $f : \wedge V \to \wedge W$ is a quasi-isomorphism if and only if it is an isomorphism.

Proof. If $f : \wedge V \to \wedge W$ is a quasi-isomorphism then so is $Q(f) : Q(\wedge V) \to Q(\wedge W)$. Since d is decomposable, the linear part is zero and hence $Q(d) = 0$ for both $\wedge V$ and $\wedge W$. In particular $H^*(Q(\wedge V)) = Q(\wedge V)$, $H^*(Q(\wedge W)) = Q(\wedge W)$ and $Q(f)^* = Q(f)$, so $Q(f)$ is an actual isomorphism, which after the identification $Q(\wedge V) = V$ and $Q(\wedge W) = W$ gives us an isomorphism between the underlying vector spaces. It follows that $f : \wedge V \to \wedge W$ is an isomorphism. □

We wish to define homotopies between morphisms of cdga's. We want a cdga that represents I in the definition of a topological homotopy $H : X \times I \to Y$. Let $\wedge(t, dt)$ be the free commutative graded algebra on the generators t and dt, where $|t| = 0$ and $|dt| = 1$. If $d(t) = dt$ and $d(dt) = 0$ then $\wedge(t, dt)$ is a contractible cdga $H^*(\wedge(t, dt)) = \mathbb{k}$. This cdga has two cdga morphisms $p_0, p_1 : \wedge(t, dt) \to \mathbb{k}$ defined by $p_0(t) = 0 = p_0(dt)$ and $p_1(t) = 1, p_1(dt) = 0$, both of which are quasi-isomorphisms. Intuitevely $\wedge(t, dt)$ should be viewed as the polynomial differential forms on I and p_0, p_1 should be viewed as the evaluations at the two endpoints.

Definition 1.7. Given two morphisms of cdga's $f, g : A \to B$, a homotopy from f to g is a morphism of cdga's $H : A \to B \otimes \wedge(t, dt)$ such that $(\mathrm{id} \otimes p_0) \circ H = f$ and $(\mathrm{id} \otimes p_1) \circ H = g$ (under identification of $B \otimes \mathbb{k} = B$). If there exists a homotopy from f to g we call f and g homotopic and write $f \simeq g$.

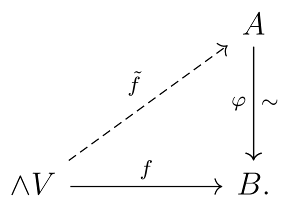
Note that $\mathrm{id} \otimes p_i$ is a quasi-isomorphism, since $\wedge(t, dt)$ is contractible. By abuse of notation we write simply p_i. We also have an inclusion $i : B \to$

$B \otimes \wedge(t, dt)$ that is a left inverse to both of these maps. Furthermore, it is a quasi-isomorphism, as the induced map is essentially the isomorphism $H^*(B) \cong H^*(B) \otimes \Bbbk$. Thus, if $f \simeq g$ then $f = g$ as morphisms in Ho(**CDGA**), as the following zig-zags are all equal

$$\xrightarrow{f} = \xrightarrow{H}\xrightarrow{p_0} = \xrightarrow{H}\xleftarrow{i}\xrightarrow{i}\xrightarrow{p_0} = \xrightarrow{H}\xleftarrow{i} = \xrightarrow{H}\xleftarrow{i}\xrightarrow{i}\xrightarrow{p_1} = \xrightarrow{H}\xrightarrow{p_1} = \xrightarrow{g} .$$

Furthermore, the converse is true if the domain is a minimal cdga. This is a consequence of the following important theorem

Theorem 1.8 (Lifting Theorem [DGMS75]). *Let $\wedge V$ be a minimal cdga. Given a quasi-isomorphism $\varphi : A \to B$ of cdga's and a cdga morphism $f : \wedge V \to B$ there exists a homotopy lift $\tilde{f} : \wedge V \to A$ such that $\varphi\tilde{f} \simeq f$. In other words, the dotted arrow exists and the diagram commutes up to homotopy*



For a proof see Lemma 12.4 in [FHT01]. We get the following two corollaries

Corollary 1.9 (Corollary 1.3 [DGMS75]). If $\wedge V$ is a minimal cdga then homotopy is an equivalence relation on the set of cdga morphisms $\wedge V \to A$ for any cdga A.

The set of homotopy equivalence classes of maps $\wedge V \to A$ is denoted by $[\wedge V, A]$.

Corollary 1.10 (Corollary 1.4 [DGMS75]). Given a quasi-isomorphism $\psi : A \to B$ the induced function of sets $\psi_* : [\wedge V, A] \to [\wedge V, B]$ is a bijection.

Definition 1.11. Given a cdga A, we say that a minimal cdga $\wedge V$ is a minimal model of A if there is a quasi-isomorphism $\wedge V \xrightarrow{\sim} A$.

Theorem 1.12 (Theorem 14.12 [FHT01]). *Let $\Bbbk$ be a field of characteristic 0 and let A be a cdga with $H^0(A) = \Bbbk$, then there exists a, unique up to isomorphism, minimal model $\wedge V \xrightarrow{\sim} A$.*

By the Lifting Theorem 1.8 if $f : A \to B$ then we get a induced map $\wedge f : \wedge V \to \wedge W$ between the minimal models of A and B respectively.

We now wish to associate to each space a cdga. For manifolds we have the de Rham complex mentioned as our first Example 1.2. Recall that the cochain complex $C^*(X;\mathbb{k})$ of singular cochains with coefficients in $\mathbb{k}$ has multiplicative structure coming from the cup product $\cup : C^k(X;\mathbb{k}) \otimes_{\mathbb{k}} C^\ell(X;\mathbb{k}) \to C^{k+\ell}(X;\mathbb{k})$ which only is graded commutative when passing to cohomology. So, $C^*(X;\mathbb{k})$ is a $\mathbb{k}$-dga but rarely a cdga. Therefore, we wish to associate to each topological space X a cdga with the same cohomology as $C^*(X;\mathbb{k})$.

Recall that a simplicial object in a category $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ where Δ is the category with objects are the ordered sets $[n] = \{0, \dots, n\}$ and the morphisms are order-preserving functions $\varphi : [m] \to [n]$. We introduce the short-hand notation $X_n = X([n])$ and $X(\varphi) = \varphi^* : X_n \to X_m$. The morphisms in the category Δ is generated by a special class of morphisms, namely, $\partial^i : [n-1] \to [n]$, the unique order preserving injection such that $i \notin \operatorname{im} \partial^i$ and $s^i : [n+1] \to [n]$, the unique order preserving surjection such that $s^i(i) = s^i(i+1)$. If X is a simplicial object, then $(\partial^i)^*$ is denoted ∂_i and called the face maps of X, while $(s^i)^* = s_i$ are called the degeneracy maps. A morphism $f : X \to Y$ of simplicial objects is a natural transformation between functors. More concretely a collection of morphisms $f_n : X_n \to Y_n$ in $\mathcal{C}$ such that $(\partial_i)_Y \circ f_n = f_{n-1} \circ (\partial_i)_X$ and $(s_i)_Y \circ f_n = f_{n+1} \circ (s_i)_X$ for all i and n. The category of simplicial objects in $\mathcal{C}$ is denoted $s\mathcal{C}$.

We define a simplicial cdga $A_{PL} : \Delta \to \mathbf{CDGA}$ by

$$(A_{PL})_n = \wedge(t_0, \dots, t_n, dt_0, \dots, dt_n) / \left(\textstyle\sum_{i=1}^n t_i - 1, \sum_{i=1}^n dt_i\right)$$

where $|t_i| = 0$ and $|dt_i| = 1$ and the differential is defined by $d(t_i) = dt_i$ and $d(dt_i) = 0$ for all $i \in [n]$. The order-preserving functions $\varphi : [m] \to [n]$ in Δ are mapped to $\varphi^* : (A_{PL})_n \to (A_{PL})_m$ defined by

$$\varphi^*(t_i) = \sum_{j \in \varphi^{-1}(i)} t_j.$$

In particular, the face and degeneracy maps are the cdga morphisms satisfying:

$$\partial_i : (A_{PL})_n \to (A_{PL})_{n-1}, \quad \partial_i(t_k) = \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}$$

$$s_j : (A_{PL})_n \to (A_{PL})_{n+1}, \quad s_j(t_k) = \begin{cases} t_k, & k < j \\ t_j + t_{j+1}, & k = j \\ t_{k+1}, & k > j. \end{cases}$$

Note that the generators dt_k are determined by the fact that φ^* is a cdga morphism so $\varphi^*(dt_k) = \varphi^*(d(t_k)) = d\varphi^*(t_k)$. To prove that it is indeed a simplicial cdga, it is enough to check that it is functorial, or that the face and degeneracy maps satisfy the simplicial identities. Note that

$$\wedge(t_0, \ldots, t_n, dt_0, \ldots, dt_n) \cong \Bbbk[t_0, \ldots, t_n] \otimes \operatorname{Ext}(dt_0, \ldots, dt_n) \tag{1}$$

as algebras, due to the degree of the generators. Thus, one should intuitively view $(A_{PL})_n$ as $\Bbbk$-polynomial differential forms in n variables.

Now A_{PL} defines a functor $A_{PL}(-) : \mathrm{s}\mathbf{Set}^{\mathrm{op}} \to \mathbf{CDGA}$ by $X \mapsto A_{PL}(X)$ where $A_{PL}(X)$ is graded by

$$A_{PL}(X)^q = \hom_{\mathrm{s}\mathbf{Set}}(X, A_{PL}^q).$$

In other words, a q-form $\omega \in A_{PL}(X)^q$ is a function such that for each n-simplex $\sigma \in X_n$ we have a "polynomial form" $\omega_\sigma \in (A_{PL})_n^q$ such that $\varphi^*(\omega_\sigma) = \omega_{\varphi^*(\sigma)}$. This inherits the structure of a cdga from A_{PL} pointwise. In particular we have

$$(d\omega)_\sigma = d(\omega_\sigma).$$

A simplicial map $\Phi : X \to Y$ is mapped to the pullback $\Phi^* : A_{PL}(Y) \to A_{PL}(X)$. If we wish to make the underlying field $\Bbbk$ explicit, we write $A_{PL}(-;\Bbbk)$.

Similarly to $A_{PL}(-)$ we define a functor $C^*(-;\Bbbk) : \mathrm{s}\mathbf{Set}^{\mathrm{op}} \to \mathbf{DGA}$ by $X \mapsto C^*(X,\Bbbk)$ graded by

$$C^p(X;\Bbbk) = \{f : X_p \to \Bbbk \mid f(s_i(\sigma)) = 0 \text{ for all } 0 \leq i \leq p-1\}.$$

The differential $d : C^p(X;\Bbbk) \to C^{p+1}(X;\Bbbk)$ is defined by

$$df(\sigma) = \sum_{k=0}^{p} (-1)^k f(d_i(\sigma)) \quad \sigma \in X_{p+1}.$$

The multiplication is the cup product $C^p(X;\Bbbk) \otimes C^q(X;\Bbbk) \to C^{p+q}(X;\Bbbk)$, defined by

$$(f \cup g)(\tau) = f(\varphi_p(\tau))g(\varphi_q(\tau))$$

where $\varphi_p = (\varphi^p)^*$ is the image of $\varphi^p : [p] \to [p+q]$, defined by $\varphi(i) = i$, and $\varphi_q = (\varphi^q)^*$, where $\varphi^q : [q] \to [p+q]$ is defined by $\varphi^q(i) = p+i$. Unlike the multiplication of $A_{PL}(X)$ this is rarely (graded) commutative, but passing to cohomology $H^*(C^*(X;\Bbbk))$ the induced product is graded commutative.

Recall that for each topological space X we associate a simplicial set $S_*(X)$ with $S_n(X)$ being the set of continuous functions from the standard (topological) n-simplex $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^n t_i = 1\}$ to X, usually called the set of singular n-simplices. An order-preserving function $\varphi : [m] \to [n]$ is mapped to the function $\varphi^* : S_n(X) \to S_m(X)$ by

$$\varphi^*(\sigma) = \sigma \circ \varphi_* \quad \text{where} \quad \varphi_* : \Delta^m \to \Delta^n,$$
$$\varphi_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n) \quad \text{for } s_i = \sum_{j \in \varphi^{-1}(i)} t_j.$$

Thus we define, for a topological space X, $A_{PL}(X) = A_{PL}(S_*(X))$. This is also functorial, so a continuous map $X \to Y$ is mapped to a morphism of simplicial sets $S_*(X) \to S_*(Y)$ by post-composing. We abbrieviate $A_{PL}(S_*(X))$ and $C^*(S_*(X); \Bbbk)$ by $A_{PL}(X)$ and $C^*(X; \Bbbk)$ respectively, and note that the latter is just the standard cochain complex of singular cochains on a topological space.

Lastly, we introduce another simplicial dga. Consider the simplicial set

$$\Delta[n] = \hom_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

and note that each order preserving function $\varphi : [m] \to [n]$ defines a morphism of simplicial sets $\varphi_* : \Delta[m] \to \Delta[n]$. We define a simplicial dga C_{PL} by setting $(C_{PL})_n = C^*(\Delta[n]; \Bbbk)$ and for each function $\varphi : [m] \to [n]$ we get a morphism $\varphi^* : (C_{PL})_n \to (C_{PL})_m$, by $(\varphi_*)^* : C^*(\Delta[n]; \Bbbk) \to C^*(\Delta[m]; \Bbbk)$, for φ_* as above. Just like A_{PL}, the simplicial dga C_{PL} defines a functor $C_{PL}(-) : \mathrm{s}\mathbf{Set} \to \mathbf{DGA}$ by

$$C_{PL}(X) = \hom_{\mathrm{s}\mathbf{Set}}(X, C_{PL})$$

with dga structure pointwise just like for $A_{PL}(X)$.

Now let $\Bbbk$ be of characteristic 0, i.e., an extension of $\mathbb{Q}$.

Theorem 1.13 (Theorem 10.9 [FHT01]). *Let X be a simplicial set. There is a natural isomorphism $C_{PL}(X) \xrightarrow{\cong} C^*(X; \Bbbk)$ of dga's and natural quasi-isomorphisms of dga's*

$$C_{PL}(X) \xrightarrow{\sim} (C_{PL} \otimes A_{PL})(X) \xleftarrow{\sim} A_{PL}(X).$$

In particular, for a topological space X, we have the same chain of quasi-isomorphism as above, and $H^*(X; \Bbbk) \cong H^*(A_{PL}(X))$.

The more standard statement is a direct natural quasi-isomorphism of cochain complexes given by integration

$$\int_X : A_{PL}(X) \to C^*(X; \Bbbk)$$

similar to the proof of de Rham's Theorem. The integration map is given by $\omega \in A^p_{PL}(X)$ is mapped to $\int \omega \in C^p(X; \Bbbk)$ defined by

$$X_p \ni \sigma \mapsto \int_\sigma \omega = \int_{\Delta^p} \omega_\sigma$$

We note that this makes sense as $\omega_\sigma \in (A_{PL})^p_p \cong \Bbbk[t_1, \ldots, t_n]dt_1 \cdots dt_p$ where the isomorphism comes from looking at p degree elements of (1) and eliminating t_0 and dt_0 using the relation $\sum t_i - 1 = 0$ and $\sum dt_i = 0$. Here it is clear why we need to restrict to $\Bbbk$ being of characteristic 0, for example if $p = 1$ we just get the integral

$$\int_{\Delta^1} t_1^k dt_1 = \int_0^1 t^k dt = \frac{1}{k+1}$$

and in general one can calculate that

$$\int_{\Delta^p} t_1^{k_1} \cdots t_p^{k_p} dt_1 \cdots dt_p = \frac{k_1! \cdots k_p!}{(k_1 + \cdots + k_p + p)!}.$$

Note that Stokes' theorem proves that it is a morphism of cochain complexes:

$$\begin{aligned}
\int_\sigma d\omega = \int_{\Delta^p} d(\omega_\sigma) &= \int_{\partial \Delta^p} \omega_\sigma \\
&= \sum_{k=0}^{p} (-1)^k \int_{\Delta^{p-1}} d_k(\omega_\sigma) \\
&= \sum_{k=0}^{p} (-1)^k \int_{\Delta^{p-1}} \omega_{d_k \sigma} \\
&= d\left(\int_\sigma \omega \right).
\end{aligned}$$

One can also check that the integration map is natural.

Theorem 1.14 (Theorem 10.15 [FHT01]). *The natural map*

$$\int_X : A_{PL}(X) \to C^*(X; \Bbbk)$$

is a quasi-isomorphism of cochain complexes.

If X is a path connected then $H^0(X; \Bbbk) = \Bbbk$ and thus $H^0(A_{PL}(X); \Bbbk) = \Bbbk$. Hence, by Theorem 1.12 we get a minimal model of $A_{PL}(X; \Bbbk)$ over $\Bbbk$. This is called the $\Bbbk$-minimal model of X.

Definition 1.15.

- A $\Bbbk$-cdga A is called $\Bbbk$-formal if it connected by a zig-zag of quasi-isomorphism of cdga's to its cohomology $H^*(A)$. In other words, A is isomorphic to $H^*(A)$ in $\mathrm{Ho}(\mathbf{CDGA}_{\Bbbk})$.
- A topological space X is called formal if $A_{PL}(X)$ is $\mathbb{Q}$-formal.
- A topological space X is called $\Bbbk$-formal if $A_{PL}(X;\Bbbk)$ is $\Bbbk$-formal.

Proposition 1.16. A $\Bbbk$-cdga (A,d) is $\Bbbk$-formal if and only if it is connected via a zig-zag of quasi-isomorphisms to a cdga with the zero-differential $(B,0)$.

Proof. A cdga with $d=0$ is its own cohomology, as $\ker d = B$ and $\operatorname{im} d = 0$. Thus the zig-zag

$$(A,d) \xleftarrow{\sim} \cdots \xrightarrow{\sim} (B,0)$$

gives an isomorphism $H(A,d) \cong H(B,0) = (B,0)$, so simply add this to the end to get a zig-zag from (A,d) to $H(A,d)$

$$(A,d) \xleftarrow{\sim} \cdots \xrightarrow{\sim} (B,0) \cong H(A,d)$$

□

One also sees that formality of A when $H^0(A)=\Bbbk$ is equivalent to existence of a quasi-isomorphism from the minimal model $\wedge V \to H^*(A)$, as the zig-zag can be lifted to the level of minimal models and quasi-isomorphisms between minimal models are isomorphisms.

Proposition 1.17. Let A be a cdga, if $H^*(A)$ is a free commutative graded algebra then A is formal.

Proof. Suppose $H^*(A) = \wedge W$. We define a map $\varphi : \wedge W \to A$ by $\varphi(w) = a_w$ so that $[a_w] = w$. This defines a map of graded algebras $\wedge W \to A$. It is a cdga morphism since $0 = \varphi(dw) = da_w = 0$. Lastly it is a quasi-isomorphism as the induced map on cohomology is the identity. □

We claim that if X is formal it is $\Bbbk$-formal for all $\Bbbk$. It is enough to prove that $A_{PL}(X;\Bbbk) \simeq A_{PL}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk$. To see this, note that since the cdga's $(A_{PL,\Bbbk})_n$ are defined as free algebras we get directly that $(A_{PL,\Bbbk})_n = (A_{PL,\mathbb{Q}})_n \otimes_{\mathbb{Q}} \Bbbk$. Thus, there is an inclusion $A_{PL}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk \to A_{PL}(X;\Bbbk)$ given by mapping each simple tensor $\omega_\sigma \otimes \lambda \in (A_{PL,\mathbb{Q}})_n \otimes_{\mathbb{Q}} \Bbbk$ to $\lambda\omega_\sigma \in (A_{PL,\Bbbk})_n$. This inclusion is a quasi-isomorphism since $H^*(A_{PL}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk) \cong H^*(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk \cong H^*(X;\Bbbk) \cong H^*(A_{PL}(X;\Bbbk))$. Hence, if X is formal, then by tensoring that zig-zag with $\Bbbk$, we see that X is $\Bbbk$-formal. Moreover, the converse is true:

Theorem 1.18 (Corollary 6.9 [HS79])**.** *If X is $\mathbb{k}$-formal for some extension $\mathbb{k}/\mathbb{Q}$ then X is formal.*

Let M be a smooth manifold. Let us consider the relation between the real cdga of de Rham forms $A_{dR}(M)$ and the rational cdga of polynomial forms $A_{PL}(M)$ is. The geometric simplicies Δ^n are manifolds with corners and thus we can define $S_*^\infty(M)$ to be the sub-simplicial set of smooth functions $\Delta^n \to M$. We can also take the de Rham complex of Δ^n and the collection $A_{dR}(\Delta^n)$ becomes a simplicial $\mathbb{R}$-cdga, let us shorten it to simply A_{dR}. Notice that this gives us a functor $A_{dR}(-) : \mathrm{s}\mathbf{Set} \to \mathbf{CDGA}_{\mathbb{R}}$, in the same way as $A_{PL}(-)$ and $C^*(-;\mathbb{k})$. Furthermore, note that $\sigma \in S_n^\infty(M)$ is a smooth map $\sigma : \Delta^n \to M$ and thus defines a real cdga morphism $A_{dR}(\sigma) : A_{dR}(M) \to A_{dR}(\Delta^n)$. Hence, we define a natural morphism $\alpha : A_{dR}(M) \to A_{dR}(S_*^\infty(M))$ by mapping $\omega \in A_{dR}(M)$ to the simplicial map taking smooth n-simplices $\sigma \in S_n^\infty(M)$ to $A_{dR}(\sigma)(\omega) \in A_{dR}(\Delta^n)$.

Furthermore, we have the inclusion $\beta : S_*^\infty(M) \longrightarrow S_*(M)$ and a morphism $\gamma : A_{PL}^{\mathbb{R}} \longrightarrow A_{dR}$, the latter being the identification of elements of $(A_{PL}^{\mathbb{R}})_n$ as $\mathbb{R}$-polynomial forms in n variables on Δ^n. This gives

$$A_{PL}(M;\mathbb{R}) = A_{PL}(S_*(M);\mathbb{R}) \xrightarrow{\gamma_*} A_{dR}(S_*(M)) \xrightarrow{\beta^*} A_{dR}(S_*^\infty(M))$$

Proposition 1.19 (Theorem 11.4 [FHT01])**.** The maps α, β^* and γ^* are all quasi-isomorphisms. In particular, we have a weak equivalence between $A_{PL}(M;\mathbb{R})$ and $A_{dR}(M)$.

Note that this implies that if $\wedge V$ is a $\mathbb{R}$ minimal model of $A_{dR}(M)$ then it is a $\mathbb{R}$ minimal model of M.

Lastly, the reason why minimal models are of interest in rational homotopy theory is that there is a connection between the rational homotopy groups and the $\mathbb{Q}$-minimal model. Recall that a group G is nilpotent if the sequence of subgroups G_i, defined recursively by $G_{i+1} = [G_i, G] = \{ghg^{-1}h^{-1} : g \in G_i, h \in G\}$ and $G_0 = G$, terminates in 0. Similarly G acts nilpotently on a group H if the sequence of subgroups H_i defined by $H_{i+1} = [H_i, G] = \langle g(h)h^{-1} : h \in H_i, g \in G\rangle$ terminates in 0. We say that a path connected space X is nilpotent if the fundamental group $\pi_1(X)$ is nilpotent and acts nilpotently on the higher homotopy $\pi_n(X)$ for $n \geq 2$. The action of the fundamental group on higher homotopy groups is explained in section 4.1 of [Hat01].

Theorem 1.20 (Theorem 11.3 [BG76])**.** *Let X be a nilpotent space with finite Betti numbers and with rational minimal model $\wedge V$, for $n \geq 2$ there are natural isomorphisms*

$$V^n \xrightarrow{\cong} (\pi_n(X) \otimes \mathbb{Q})^*$$

in particular $\operatorname{rank} \pi_n(X) = \dim V^n$.

There is quite a lot more to say about cdga, minimal models and their applications. For more information and proofs of the Theorems presented see [DGMS75], [FHT01] and [FOT08].

2 Lie groups and nilmanifolds

Recall that a Lie group G is a smooth manifold with a group structure such that the multiplication $G \times G \to G$ is smooth. A real Lie algebra $\mathfrak{g}$ is a real vector space equipped with an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket and denoted $[x, y]$, that satisfies the Jacobi identity:

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

For any $g \in G$, the map $L_g : G \to G$, defined by $L_g(h) = gh$, is a diffeomorphism of G with inverse $L_{g^{-1}}$. The map $R_g(h) = hg$ is similarly a diffeomorphism of G. A vector field $X \in \mathfrak{X}(G)$ is left invariant if $dL_g(X) = X$, where $dL_g : TG \to TG$ is the differential of L_g. Similarly, a vector field X is right invariant if $dR_g(X) = X$. Let $\mathfrak{g}$ be the vector space of left invariant vector fields. Recall that on any manifold M there is a bracket of vector fields $[-,-] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. The bracket satisfies the following relation on smooth functions f,

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

The bracket of vector fields on M is an alternating bilinear map that satisfies the Jacobi identity. If $F : M \to N$ is a diffeomorphism, then $dF[X, Y] = [dF(X), dF(Y)]$. Hence, if X, Y are left invariant vector fields, then so is $[X, Y]$. Therefore, $\mathfrak{g}$ has the structure of a Lie algebra. Furthermore, note that if X is left invariant, then

$$X_g = (dL_g)(X_e)$$

for all $g \in G$. In particular, a left invariant vector field is uniquely determined by it's value at the identity. Conversely, every $v \in T_eG$ defines a left invariant vector field by $X_g = (dL_g)(v)$. Thus, we may think of $\mathfrak{g}$ as the tangent space at the identity, in particular, it is finite dimensional.

Similarly to vector fields, we can define left (and right) invariant differential forms by noting that left multiplication by g, L_g, induces a map of differential forms $L_g^* : A_{dR}(G) \to A_{dR}(G)$ by

$$L_g^*\omega(X_1, \ldots, X_n)(x) = \omega_{gx}(dL_gX_1, \ldots, dL_gX_n).$$

Hence, we say that ω is left invariant if $L_g^*\omega = \omega$ for all $g \in G$. Since L_g^* is a map of chain complexes, the set of left invariant forms is closed under differentiation

$$L_g^*d\omega = d(L_g^*\omega) = d\omega.$$

As it is also closed under multiplication by the Leibniz rule, we see that the space of left invariant differential forms is a sub-cdga of $A_{dR}(G)$, we denote it by $A_{dR}^L(G)$. Just like for vector fields, left invariant forms are uniquely determined by their value at the identity, so we get $A_{dR}^L(G) \cong \Lambda^*T_e^*G = \wedge\mathfrak{g}^*$.

Theorem 2.1 (Theorem 1.28 [FOT08]). *If G be a compact connected Lie group, then the inclusion $A^L_{dR}(G) \to A_{dR}(G)$ is a quasi-isomorphism.*

We equip $\wedge\mathfrak{g}^*$ with the Chevalley–Eilenberg differential, defined on 1-forms ω by

$$d\omega(X,Y) = -\omega([X,Y]),$$

and extended to arbitrary forms using the Leibniz rule. We claim that the isomorphism $A^L_{dR}(G) \cong \wedge\mathfrak{g}^*$ is an isomorphism of cdga's. It suffices to prove that it commutes with the differential. If we identify $\mathfrak{g}$ with the tangent space at the identity, then we express the isomorphism as taking an element $\omega \in \wedge\mathfrak{g}^*$ to the left invariant form ω^L, defined by

$$\omega^L_g(v_1,\ldots,v_n) = \omega(dL_g^{-1}v_1,\ldots,dL_g^{-1}v_n)$$

for tangent vectors $v_i \in T_gG$. Recall the following formula for the differential of 1-forms:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

If α, X, Y are left invariant, then this simplifies to $d\omega(X,Y) = -\omega([X,Y])$ as $\omega(X)$ and $\omega(Y)$ are constant functions

$$\omega(X)(g) = \omega_g(X_g) = \omega_g(dL_gX_e) = \omega_e(X_e) = \omega(X)(e),$$

so $X(\omega(Y)) = 0$ and $Y(\omega(X)) = 0$. Thus, we calculate that

$$\begin{aligned} d\omega^L_g(v_1,v_2) = d\omega^L_g(X_g,Y_g) &= -\omega^L_g([X,Y]_g) \\ &= -\omega(dL_g^{-1}[X,Y]_g) \\ &= -\omega([dL_g^{-1}X, dL_g^{-1}Y]_g) \\ &= d\omega(dL_g^{-1}X_g, dL_g^{-1}Y_g) \\ &= (d\omega)^L_g(X_g,Y_g) = (d\omega)^L_g(v_1,v_2), \end{aligned}$$

where X,Y are the unique left invariant vector fields with $X_g = v_1$ and $Y_g = v_2$. It follows from Theorem 2.1 that for compact connected Lie groups G with Lie algebra $\mathfrak{g}$ we have

$$H(\wedge\mathfrak{g}^*, d) \cong H^*(G;\mathbb{R}).$$

Example 2.2. Let $G = S^3 = \mathrm{Sp}(1)$ be the unit quaternions. The tangent space at the identity is given by the imaginary quaternions $\mathfrak{g} = \mathrm{im}(\mathbb{H})$ and thus spanned by i, j, k. The bracket is given by the commutator and thus $[i,j] = 2k$, $[k,i] = 2j$ and $[j,k] = 2i$. We get that

$$(\wedge\mathfrak{g}^*, d) = (\wedge(x,y,z), dx = -\tfrac{1}{2}yz, dy = -\tfrac{1}{2}zx, dz = -\tfrac{1}{2}xy).$$

We calculate that

$$\begin{aligned}
H^0(\wedge\mathfrak{g}^*, d) &= \mathbb{R} \\
H^1(\wedge\mathfrak{g}^*, d) &= \frac{\ker(d : \text{span}(x, y, z) \to \text{span}(xy, xz, yz))}{\text{im}(d : \mathbb{R} \to \text{span}(x, y, z))} = 0 \\
H^2(\wedge\mathfrak{g}^*, d) &= \frac{\ker(d : \text{span}(xy, xz, yz) \to \text{span}(xyz))}{\text{im}(d : \text{span}(x, y, z) \to \text{span}(xy, xz, yz))} = \frac{\text{span}(xy, xz, yz)}{\text{span}(xy, xz, yz)} = 0 \\
H^3(\wedge\mathfrak{g}^*, d) &= \frac{\ker(d : \text{span}(xyz) \to 0)}{\text{im}(d : \text{span}(xy, xz, yz) \to \text{span}(xyz))} = \text{span}(xyz) \cong \mathbb{R}
\end{aligned}$$

aligning with $H^*(S^3; \mathbb{R})$. Let us note however that $(\wedge\mathfrak{g}^*, d)$ is not a $\mathbb{R}$-minimal model of S^3, as there is no ordering we can put on $\{x, y, z\}$ so that $\wedge\mathfrak{g}^*$ is a Sullivan cdga.

Minimal models of compact connected Lie groups are in fact much simpler. Hopf's Theorem (see Theorem 1.34 [FOT08]) tells us that $H^*(G; \mathbb{R})$ is an exterior algebra on a finite set of variables in odd degrees $x_1, \ldots, x_n$. So we may pick differential forms α_i representing x_i and create the cdga morphism

$$\varphi : (\wedge(x_i), d = 0) \to A_{dR}(G), \quad \varphi(x_i) = \alpha_i.$$

As the elements α_i are closed, this is a chain map, and the induced map on cohomology is the isomorphism from Hopf's Theorem. Note that this implies that compact connected Lie groups are formal.

Recall that a connected Lie group N is called nilpotent if the corresponding Lie alegbra $\mathfrak{n}$ is nilpotent. That is, the sequence of Lie subalgebras $\mathfrak{n}_i$, defined recursively by $\mathfrak{n}_{i+1} = [\mathfrak{n}_i, \mathfrak{n}]$ and $\mathfrak{n}_0 = \mathfrak{n}$, terminates to 0.

Definition 2.3. A nilmanifold $M = N/\Gamma$ is a quotient of a simply connected nilpotent Lie group N by a co-compact discrete subgroup Γ.

Usually the quotient is taken to be the right cosets, so that M still has a left action by N even if Γ isn't a normal subgroup. This action is transitive, so nilmanifolds are examples of homogeneous spaces. Note that co-compact means exactly that N/Γ is compact.

Let us remark that a simply connected nilpotent Lie group N is diffeomorphic to its Lie algebra $\mathfrak{n}$, which is just diffeomorphic to $\mathbb{R}^n$, see Theorem 1.104 in [Kna96]. Thus, the covering map $N \to N/\Gamma$ is the universal covering by a contractible space, so that $\pi_n(N/\Gamma) = 0$ for $n \geq 2$ and $\pi_1(N/\Gamma) \cong \Gamma$. In particular, nilmanifold are Eilenberg Maclane spaces $K(\Gamma, 1)$.

Example 2.4. Consider $N = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, then $N/\Gamma = \mathbb{R}^n/\mathbb{Z}^n = T^n$, so the n-torus is a nilmanifold.

Example 2.5. Let N be the Lie group of real $n \times n$ upper triangular matrices with 1's on the diagonal, this group is nilpotent as the Lie algebra is comprised of upper triangular matrices with zeroes on the diagonal. Let Γ be the subgroup of such matrices with integer coefficients, then the quotient $M = N/\Gamma$ is a nilmanifold. When $n = 3$ the Lie group N_3 is called the Heisenberg group and the nilmanifold N_3/Γ is called the Heisenberg manifold.

To construct more examples one might wonder when we can find a subgroup Γ such that there is a nilmanifold of the form N/Γ. A. Maltsev proved the following:

Theorem 2.6 (Theorem 7 [Mal49])**.** *Let N be a simply connected, nilpotent Lie group. There is a discrete co-compact subgroup Γ if and only if there is a basis e_i of the Lie algebra $\mathfrak{n}$ such that the structure constants c_{ij}^k, coming from the bracket*

$$[e_i, e_j] = \sum_k c_{ij}^k e_k,$$

are rational.

A proof in English can be found in [Rag72]. Note that having rational structure constants for the Lie bracket is equivalent to having rational structure constants for the Chevalley-Eilenberg differential d on $\wedge\mathfrak{n}^*$.

Using Lie's third theorem, we can thus find nilmanifold by starting with a nilpotent Lie algebra $\mathfrak{n}$ with rational structure constants, letting N be the corresponding simply connected nilpotent Lie group and Γ be the co-compact discrete subgroup given by Maltsev's Theorem.

For a Nilpotent Lie algebra, the cdga $(\wedge\mathfrak{n}^*, d)$ is a minimal cdga. The ordering can be defined by using the sequence of Lie subalgebras $\mathfrak{n}_i$. If $\dim \mathfrak{n} = d$ and the last non-zero algebra in the sequence is $\mathfrak{n}_k$, then choose a basis for $\mathfrak{n}_k$ and set the basis elements to be the largest. As $\mathfrak{n}_{k-1} \neq \mathfrak{n}_k$ we have to expand the basis by at least one element, we let these basis elements be smaller than the ones for $\mathfrak{n}_k$. We continue in this fashion until we arrive at a basis for $\mathfrak{n}$. If $\{X_1, \ldots, X_d\}$ is this basis, then we claim that in the decomposition $[X_i, X_j] = \sum_k c_{ij}^k X_k$ one has $c_{ij}^k = 0$ for $k \leq \max\{i, j\}$. Since, if $X_i \in \mathfrak{n}_{\ell_i}$ and $X_j \in \mathfrak{n}_{\ell_j}$, then $[X_i, X_j] = -[X_j, X_i] \in \mathfrak{n}_{\ell_{\max\{i,j\}}+1}$, so must be able to be expressed in only basis elements strictly larger than $\max\{i, j\}$. Hence, in $\wedge\mathfrak{n}^*$, the decomposition $dx_k = \sum -c_{ij}^k x_i x_j$ must have $i < k$ and $j < k$, and we have a well-ordering on the elements such that $dx_\alpha \in \wedge x_{\beta<\alpha}$. It is also immediate that the cdga is decomposable. We can however not conclude that $(\wedge\mathfrak{n}^*, d)$ is a $\mathbb{R}$ minimal model of N, as N need not be compact. On the other hand, we have the following Theorem:

Theorem 2.7 (Nomizu's Theorem [Nom54])**.** *The inclusion $A^L_{dR}(N) \to A_{dR}(N/\Gamma)$ is a quasi-isomorphism.*

Since $(\wedge\mathfrak{n}^*, d) \cong A^L_{dR}(N)$, we get that $(\wedge\mathfrak{n}^*, d)$ is the $\mathbb{R}$-minimal model of the nilmanifold N/Γ.

Let us suppose that the nilmanifold N/Γ is formal. This means that there is a quasi-isomorphism $(\wedge\mathfrak{n}^*, d) \xrightarrow{\sim} (H^*(N/\Gamma;\mathbb{R}), 0)$. By definition, $\wedge\mathfrak{n}^*$ is generated by elements of degree 1, and if the dimension of the nilmanifold is n, then there are n such generators. Furthermore, as the generators are in degree 1, the algebra $\wedge\mathfrak{n}^*$ is an exterior algebra, and, in particular, the top dimension $(\wedge\mathfrak{n}^*)^n$ is the span of the single element $x_1x_2\cdots x_n$. Denote the quasi-isomorphism by φ. The element $x_1\cdots x_n = z$ is closed as it's in the top dimension. Since N/Γ is a compact oriented manifold it has non-zero top dimension, and as φ^* is an isomorphism this means that z is not exact and that $\varphi^*([z]) \neq 0$. In particular $\varphi(z) \neq 0$, and since $\varphi(z) = \varphi(x_1)\cdots\varphi(x_n)$ all of these are non-zero. This makes φ injective, if $y = ax_I + \cdots$ for some indexing set $I \subset [n]$ we can find its complement I^C and get $x_{I^C}y = \pm az$. So applying φ on both sides leads to $\varphi(y) \neq 0$. This in turns makes the differential d trivial on $\wedge\mathfrak{n}^*$, since

$$0 = d\varphi(y) = \varphi(dy)$$

and so $dy = 0$ for all y. In particular, $(\wedge\mathfrak{n}^*, d) = (\wedge(x_1, \ldots, x_n), 0)$ which is the minimal model of an n-torus by Proposition 1.17. In fact this result can be strengthened to say that N/Γ must be diffeomorphic to a torus using Mostow's classification of nilmanifolds by their fundamental group [Mos54]. We thus have

Proposition 2.8. If a nilmanifold is formal, it is diffeomorphic to a torus.

For more on Lie groups see [Kna96], for nilmanifolds see section 3.2 in [FOT08] or chapter 2 of [TO97].

3 Complex manifolds

The main class of topological spaces we will study in the rest of this thesis are complex manifolds. There are two notions of complex manifolds that turn out to be equivalent.

Definition 3.1. A complex manifold M of (complex) dimension n is a $2n$ dimensional smooth manifold with atlas $\{\varphi_U\}_{U\in\mathcal{U}}$, such that, after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, the transition maps $\varphi_U \circ \varphi_V^{-1} = \varphi_{UV}$ are holomorphic functions between open subsets of $\mathbb{C}^n$ (when defined). We call such an atlas a holomorphic structure on M.

Note that there are two canonical identifications of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$

$$(x_1, y_1, \dots, x_n, y_n) \longleftrightarrow (x_1 + iy_1, \dots, x_n + iy_n) \quad \text{and}$$
$$(x_1, \dots, x_n, y_1, \dots, y_n) \longleftrightarrow (x_1 + iy_1, \dots, x_n + iy_n),$$

we shall prefer the latter in this thesis.

The second definition comes from the more general definition of an almost complex manifold

Definition 3.2. An almost complex manifold is a pair (M, J) where M is a smooth manifold and J is a (1,1)-tensor field, which we identify with a linear map $J_p : T_pM \to T_pM$ over each point $p \in M$, such that $J^2 = -\operatorname{id}$ (so $J_p^2(v_p) = -v_p$, for $v_p \in T_pM$). The tensor field J is called an almost complex structure and (M, J), or just M, is called an almost complex manifold.

Note that the condition $J^2 = -\operatorname{id}$ forces M to be of even dimension. Over each point p we have $J_p^2 = -\operatorname{id}_p$, so taking determinants we get $\det(J_p)^2 = (-1)^{\dim M}$, which only has a real solution when $\dim M$ is even.

Note that the almost complex structure makes the tangent spaces T_pM into complex vector spaces, by defining

$$(a + bi)X_p = aX_p + bJ_pX_p.$$

Hence, the bundle TM becomes a complex vector bundle.

Let us show that complex manifolds are almost complex. For each n define a linear map $j_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by the block matrix

$$j_n = \begin{pmatrix} 0 & -\operatorname{id}_n \\ \operatorname{id}_n & 0 \end{pmatrix}$$

and note that $j_n^2 = -\operatorname{id}_{2n}$. This will be the almost complex structure of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ under the identification from before. Now let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a smooth function, and write $F = G + iH$ for real functions $G, H : \mathbb{R}^{2n} \to \mathbb{R}^m$ using the identification. Then $F = (G, H) : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ and the differential dF can be written as

$$dF = \begin{pmatrix} \left(\frac{\partial G_j}{\partial x_k}\right) & \left(\frac{\partial G_j}{\partial y_k}\right) \\ \left(\frac{\partial H_j}{\partial x_k}\right) & \left(\frac{\partial H_j}{\partial y_k}\right) \end{pmatrix}.$$

Calculate that

$$(dF)j_n = \begin{pmatrix} \left(\frac{\partial G_j}{\partial x_k}\right) & \left(\frac{\partial G_j}{\partial y_k}\right) \\ \left(\frac{\partial H_j}{\partial x_k}\right) & \left(\frac{\partial H_j}{\partial y_k}\right) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial G_j}{\partial y_k}\right) & -\left(\frac{\partial G_j}{\partial x_k}\right) \\ \left(\frac{\partial H_j}{\partial y_k}\right) & -\left(\frac{\partial H_j}{\partial x_k}\right) \end{pmatrix}$$

$$j_m(dF) = \begin{pmatrix} 0 & -\mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial G_j}{\partial x_k}\right) & \left(\frac{\partial G_j}{\partial y_k}\right) \\ \left(\frac{\partial H_j}{\partial x_k}\right) & \left(\frac{\partial H_j}{\partial y_k}\right) \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial H_j}{\partial x_k}\right) & -\left(\frac{\partial H_j}{\partial y_k}\right) \\ \left(\frac{\partial G_j}{\partial x_k}\right) & \left(\frac{\partial G_j}{\partial y_k}\right) \end{pmatrix}$$

and note that these matrices are equal if and only if

$$\frac{\partial G_j}{\partial y_k} = -\frac{\partial H_j}{\partial x_k} \quad \text{and} \quad \frac{\partial G_j}{\partial x_k} = \frac{\partial H_j}{\partial y_k}$$

for all $1 \leq j \leq m$ and $1 \leq k \leq n$. We see that this is exactly the Cauchy-Riemann equations for each component G_j, H_j and each pair of variables x_k, y_k. Hence, we conclude that F is holomorphic if and only if $(dF)j_n = j_m(dF)$, or informally, dF commutes with the complex structure j. The reason is that, under the identification, j_n is exactly multiplication by i, so a smooth map is holomorphic if and only if its total differential commutes with i.

Now for a complex manifold M with holomorphic structure $\{\varphi_U\}_{U\in\mathcal{U}}$ we define a local almost complex structure J_U on $U \subset M$ by

$$J_U = d\varphi_U^{-1} j_n d\varphi_U : TU \to TU.$$

Now if (V, φ_V) is another chart, overlapping with U, we show that they agree on the overlap. This is due to the fact that the transition functions are holomorphic, so their differentials commute with j_n, and we have

$$\begin{aligned} J_V &= d\varphi_V^{-1} j_n d\varphi_V \\ &= d\varphi_U^{-1} d\varphi_U d\varphi_V^{-1} j_n d\varphi_V \\ &= d\varphi_U^{-1} d(\varphi_U \circ \varphi_V^{-1}) j_n d\varphi_V \\ &= d\varphi_U^{-1} j_n d(\varphi_U \circ \varphi_V^{-1}) d\varphi_V \\ &= d\varphi_U^{-1} j_n d\varphi_U \\ &= J_U. \end{aligned}$$

Thus, the collection of local almost complex structures J_U for $U \in \mathcal{U}$ gives rise to a global almost complex structure J on M.

Now, as you might have guessed, not all almost complex manifolds have holomorphic structure. We call an almost complex structure coming from a holomorphic structure a complex structure. The next theorem, due to L. Newlander and A. Nirenberg, gives us a condition for when an almost complex structure comes from a holomorphic structure. Before we can state it we need to introduce the complexified tangent bundle.

Definition 3.3. The complexified tangent bundle is the extension of scalars

$$TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$$

over each point, $T_pM^{\mathbb{C}} = T_pM \otimes_{\mathbb{R}} \mathbb{C}$.

All real tensors on TM can be extended to complex tensors on $TM^{\mathbb{C}}$ by $\mathbb{C}$-linearity. In particular, for an almost complex manifold, the almost complex structure J extends to $TM^{\mathbb{C}}$ and can be diagonalized. As $J^2 = -\operatorname{id}$ the only possible eigenvalues are i and $-i$. Thus we denote the two eigenbundles corresponding to i and $-i$ by $T^{1,0}M$ and $T^{0,1}M$ respectively. The bundle $T^{1,0}M$ is called the holomorphic bundle of M, while $T^{0,1}M$ is called the anti-holomorphic bundle of M. Note that $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ and, via some algebra,

$$T^{(1,0)}M = \{X - iJX : X \in TM\} \quad T^{(0,1)}M = \{X + iJX : X \in TM\}. \tag{2}$$

The Newlander-Nirenberg theorem can now be stated:

Theorem 3.4 (Newlander-Nirenberg)**.** *Given an almost complex manifold (M, J), the almost complex structure comes from a holomorphic structure on M if and only if for all antiholomorphic vector fields $X, Y \in \Gamma(T^{0,1}M)$ their Lie bracket is also antiholomorphic $[X, Y] \in \Gamma(T^{0,1}M)$.*

For a proof see Section 11 in [Dem12]. We can now view complex manifolds either as manifolds with holomorphic transition functions or as almost complex manifolds (M, J) such that the Lie bracket is an operation on $\Gamma(T^{0,1}M)$.

We now complexify the exterior bundle $\Lambda^*_{\mathbb{C}}M = \Lambda^*M \otimes_{\mathbb{R}} \mathbb{C}$, so that complex valued differential forms are sections of $\Lambda^*_{\mathbb{C}}M$. Similarly as before, we decompose $\Lambda^1_{\mathbb{C}}M$ into a sum of subbundles $\Lambda^1_{\mathbb{C}}M = \Lambda^{(1,0)}M \oplus \Lambda^{(0,1)}M$ where

$$\Lambda^{(1,0)}M = \left\{\xi \in \Lambda^1_{\mathbb{C}}M : \xi(Z) = 0 \text{ for all } Z \in T^{(0,1)}M\right\}$$

and

$$\Lambda^{(0,1)}M = \left\{\xi \in \Lambda^1_{\mathbb{C}}M : \xi(Z) = 0 \text{ for all } Z \in T^{(1,0)}M\right\}.$$

The sections of these subbundles are called forms of type (1,0) and forms of type (0,1) respectively, or just $(1,0)$ forms and $(0,1)$ forms. It is straightforward to check that

$$\Lambda^{(1,0)}M = \{\omega - i\omega \circ J : \omega \in \Lambda^1 M\} \quad \Lambda^{(0,1)}M = \{\omega + i\omega \circ J : \omega \in \Lambda^1 M\} \quad (3)$$

and that

$$\Lambda^1_{\mathbb{C}}M = \Lambda^{(1,0)}M \oplus \Lambda^{(0,1)}M. \quad (4)$$

Denote the k:th exterior power of $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$ by $\Lambda^{(k,0)}$ and $\Lambda^{(0,k)}$ respectively and denote the tensor product $\Lambda^{(p,0)} \otimes \Lambda^{(0,q)}$ by $\Lambda^{(p,q)}$. Since

$$\Lambda^k(E \oplus F) \cong \bigoplus_{i=0}^{k} \Lambda^i E \otimes \Lambda^{k-i} F$$

we get by (4) that

$$\Lambda^k_{\mathbb{C}}M \cong \bigoplus_{p+q=k} \Lambda^{(p,q)}M$$

using our established notation. Sections of $\Lambda^{(p,q)}M$ are called (p,q) forms and the space of such is denoted by $\Omega^{(p,q)}M$.

Definition 3.5. Let J be an almost complex structure on M. The Nijenhuis tensor of J is the $(2,1)$-tensor, denoted N^J, defined by

$$N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

Note that, unlike the bracket, the Nijenhuis tensor is in fact a tensor, i.e. it only depends on the pointwise value of X and Y. This can be verified in a local chart.

Proposition 3.6 (Proposition 8.3 [Mor07])**.** Let J be an almost complex structure on M. The following are equivalent

(i) J is complex

(ii) $[Z,W] \in \Gamma(T^{0,1}M)$ for all $Z, W \in \Gamma(T^{0,1}M)$

(iii) $d(\Omega^{1,0}M) \subset \Omega^{2,0}M \oplus \Omega^{1,1}M$

(iv) $d(\Omega^{p,q}M) \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$

(v) $N^J = 0$.

In particular (iv) means that we can decompose d into $\partial+\bar{\partial}$ with $\partial : \Omega^{p,q}M \to \Omega^{p+1,q}M$ and $\bar{\partial} : \Omega^{p,q+1}M$, called the holomorphic and antiholomorphic differential. The equation $d^2 = 0$ gives

$$0 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

and as $\partial^2, (\partial\bar{\partial} + \bar{\partial}\partial)$ and $\bar{\partial}^2$ take values in $\Omega^{p+2,q}M$, $\Omega^{p+1,q+1}M$ and $\Omega^{p,q+2}M$ respectively, we get the important equations

$$\partial^2 = 0 \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0 \qquad \bar{\partial}^2 = 0.$$

Therefore, for each fixed q we get a cochain complex $\Omega^{*,q}M$ with differential ∂ and for each fixed p we get a cochain complex $\Omega^{p,*}M$ with differential $\bar{\partial}$.

Furthermore, we note that if $\omega \in \Omega^{p,q}M$ and $\tau \in \Omega^{k,\ell}M$, then their product $\omega \wedge \tau \in \Omega^{p+k,q+\ell}M$, so the cochain complexes $\Omega^{*,0}M$ and $\Omega^{0,*}M$ also have multiplicative structure and the equation $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{|\omega|}\omega \wedge d\tau$ gives the Leibniz rule,

$$\partial(\omega \wedge \tau) = \partial\omega \wedge \tau + (-1)^{|\omega|}\omega \wedge \partial\tau$$

for $\omega \in \Omega^{*,0}M$

Let us also note that for complex manifolds we have a local basis of $\Omega^{p,q}M$, namely if U is a chart with local holomorphic coordinates $z_i = x_i + iy_i$ and anti-holomorphic $\bar{z}_i = x_i - iy_i$, we get that

$$\Omega^{p,q}U = \operatorname{span}_{\mathbb{C}} \left\{ dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \mid i_1 < \cdots < i_p, j_1 < \cdots < j_q \right\}$$

just like in ordinary complex variables. From this we see that locally we have a conjugation map, $\Omega^{p,q}U \to \Omega^{q,p}U$ that is conjugate linear. One can check that it agrees on overlap and thus defines a global conjugation map $\Omega^{p,q}M \to \Omega^{q,p}M$.

Recall that a Riemannian metric g on a smooth manifold M is a (0,2)-tensor field such that $g_p : T_pM \times T_pM \to \mathbb{R}$ is a symmetric, non-degenerate, positive definite bilinear form.

Definition 3.7. A Hermitian metric h on an almost complex manifold (M, J) is a Riemmanian metric on M such that $h(X, Y) = h(JX, JY)$ for all $X, Y \in TM$. The triple (M, J, h) is called a Hermitian manifold. The fundamental 2-form Ω of a Hermitian metric is defined by $\Omega(X, Y) = h(JX, Y)$.

Extending this to $TM^{\mathbb{C}}$ we get additional relations

$$\begin{aligned} h(\overline{Z}, \overline{W}) &= \overline{h(Z, W)}, \quad \text{for all } Z, W \in TM^{\mathbb{C}} \\ h(Z, \overline{Z}) &> 0, \quad \text{for all } Z \neq 0 \\ h(Z, W) &= 0, \quad \text{for all } Z, W \in T^{1,0}M \text{ and } Z, W \in T^{0,1}M. \end{aligned}$$

Our chosen notion of Hermitian metric is closely related to the notion of a Hermitian structure on the tangent bundle.

Definition 3.8. Given a complex vector bundle $E \to M$, a Hermitian structure on E is a smooth field of Hermitian products, i.e. over each $p \in M$ the map $H_p : E_p \times E_p \to \mathbb{C}$ satisfies

1. H_p is $\mathbb{C}$-linear in the first coordinate.
2. H_p is conjugate symmetric, $H_p(u, v) = \overline{H_p(v, u)}$
3. $H_p(u, u) > 0$ for $u \neq 0$

A complex vector bundle with Hermitian structure is called a Hermitian bundle.

Since the tangent bundle TM is a complex vector bundle under the action $(a+bi)X = aX+bJX$ discussed earlier we may wonder if a Hermitian structure on the tangent bundle TM, called a Hermitian form on M, is connected to a Hermitian metric. In fact, they are equivalent in the following way.

Proposition 3.9.

1. Given a Hermitian metric h, $H(X, Y) = h(X, Y) - ih(JX, Y)$ defines a Hermitian form.
2. Given a Hermitian form H, the real part $\operatorname{Re} H$ defines a Hermitian metric.
3. These constructions are inverses to each other

Proof.

1. We have that

$$\begin{aligned} H((a+bi)X, Y) &= h(aX + bJX, Y) - ih(J(aX + bJX), Y) \\ &= a(h(X, Y) - ih(JX, Y)) + bi(h(X, Y) - ih(JX, Y)) \\ &= (a + bi)H(X, Y) \end{aligned}$$

and

$$\begin{aligned} H(X, Y) &= h(X, Y) - ih(JX, Y) \\ &= h(Y, X) - ih(JY, J^2X) \\ &= h(Y, X) + ih(JY, X) = \overline{H(Y, X)}. \end{aligned}$$

Now, since $h(JX, X) = h(J^2X, JX) = -h(JX, X)$, we see that $h(JX, X) = 0$ and thus

$$H(X, X) = h(X, X) - ih(JX, X) = h(X, X) > 0.$$

Lastly H is smooth as h and J are.

2. It is immediate that $\operatorname{Re} H$ is $\mathbb{R}$-linear in both coordinate and symmetric as $\operatorname{Re} H(X,Y) = \operatorname{Re} \overline{H(Y,X)} = \operatorname{Re} H(Y,X)$. Positivity is built into H so it holds for $\operatorname{Re} H$ and thus $\operatorname{Re} H$ is a Riemannian metric. Lastly as $H(JX,JY) = H(iX,iY) = H(X,Y)$ we get that $\operatorname{Re} H(X,Y) = \operatorname{Re} H(JX,JY)$.

3. Given a hermitian form H, applying both constructions yields a Hermitian form $\widetilde{H}(X,Y) = \operatorname{Re} H(X,Y) - i \operatorname{Re} H(JX,Y)$, we calculate that

$$\operatorname{Re} H(JX,Y) = \operatorname{Re}(iH(X,Y)) = -\operatorname{Im} H(X,Y)$$

and thus, $\widetilde{H}(X,Y) = \operatorname{Re} H(X,Y) + i \operatorname{Im} H(X,Y) = H(X,Y)$.

Conversely given a hermitian metric h, we have

$$\tilde{h}(X,Y) = \operatorname{Re}(h(X,Y) - ih(JX,Y)) = h(X,Y).$$

□

The equation connecting the Hermitian metric h, Hermitian form H and the fundamental form Ω is

$$h(JX,Y) = \Omega(X,Y) = -\operatorname{Im} H(X,Y).$$

Lastly we note that every almost complex manifold admits a Hermitian metric. Choose any Riemmanian metric g and define $h(X,Y) = g(X,Y) + g(JX,JY)$, it is an easy check that this defines a Hermitian metric. The fundamental form Ω is non-degenerate and thus Ω^n is a volume form of type (n,n), in particular, all almost complex manifolds are orientable. Let us also note that a complex hermitian manifold has two natural connections on its tangent bundle TM, the Levi-Civita connection that every Riemannian manifold admits, but also the Chern connection ∇_C which is the unique connection on TM such that the Hermitian structure H is parallel to ∇_C and, when extended to $\Omega^{p,q}M$, the antiholomorphic piece $\nabla_C^{(0,1)} : \Lambda^{p,q}M \to \Lambda^{p,q+1}M$ is equal to $\bar{\partial}$.

We are now ready to define Kähler manifolds.

Definition 3.10 (Kähler manifolds)**.** A complex Hermitian manifold (M,J,h) is called a Kähler manifold if the fundamental form is closed, $d\Omega = 0$. The Hermitian metric h is then called a Kähler metric.

Since h is non-degenerate, if M is Kähler then the fundamental form Ω is a symplectic form, and thus Kähler manifolds are particular cases of symplectic manifolds.

There are many equivalent ways to express that a Hermitian manifold (M, J, h) is a Kähler manifold. Being complex can be described in all the equivalent conditions present in Proposition 3.6. So for example, a Hermitian manifold (M, J, h) is Kähler if $N^J = 0$ and $d\Omega = 0$. But there are many more

Theorem 3.11 (Theorem 11.5 [Mor07])**.** *A Hermitian manifold is Kähler if and only if J is parallel with respect to the Levi-Civita connection of h.*

Theorem 3.12 (Proposition 11.8 [Mor07])**.** *A complex Hermitian manifold is Kähler if and only if the Levi-Civita connection is equal to the Chern connection.*

Theorem 3.13 (p. 107 [GH94])**.** *A complex Hermitian manifold (M, J, h) is Kähler if and only if for each point $p \in M$ there is a neighbourhood of p with local coordinates z_i such that*

$$h = (\delta_{i,j} + \mathcal{O}(|z|^2))dz_i \wedge d\bar{z}_j$$

where $\delta_{i,j}$ is the Kronecker delta function. In other words, there exists local holomorphic coordinates such that h is equal to the standard hermitian metric up to order 2.

The last theorem can be remember as a holomorphic version of the classic theorem in Riemmanian geometry that there are local coordinates where the Riemmanian metric is equal to the standard metric up to order 2.

To end this section let us give a brief introduction to Hodge Theory.

Let (M^n, g) be an oriented Riemannian manifold with volume form dv. Let $E \to M$ and $F \to M$ be two Hermitian bundles with associated Hermitian structures $\langle -, - \rangle_E$ and $\langle -, - \rangle_F$ respectively.

Definition 3.14. Suppose $P : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(E)$ are linear differential operators. We say that Q is a formal adjoint of P if

$$\int_M \langle P\alpha, \beta \rangle_F \, dv = \int_M \langle \alpha, Q\beta \rangle_E \, dv$$

for all compactly supported smooth sections $\alpha \in \Gamma_c(E), \beta \in \Gamma_c(F)$.

One can prove that if P has a formal adjoint, then it is unique. It is also clear that if Q is a formal adjoint of P, then P is a formal adjoint of Q.

Lemma 3.15. *Let $P : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(E)$ be linear differential operators. If there exists a section $\omega \in \Gamma(E^* \otimes F^* \otimes \Lambda^{n-1}M)$ such that*

$$(\langle P\alpha, \beta \rangle_F - \langle \alpha, Q\beta \rangle_E) \, dv = d(\omega(\alpha, \beta))$$

for all $\alpha \in \Gamma(E), \beta \in \Gamma(F)$ then Q is a formal adjoint of P.

Proof. If α and β are compactly supported, then so is $\omega(\alpha, \beta)$. Integrating both sides and noting that the right hand side is zero due to Stokes' Theorem gives

$$\int_M \langle P\alpha, \beta \rangle_F \, dv = \int_M \langle \alpha, Q\beta \rangle dv.$$

$\square$

Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame of M parallel at a point. Using the musical isomorphisms, we identity vectors and 1-forms and write $dv = e_1 \wedge \cdots \wedge e_n$. Let φ be the embedding $\Lambda^k M \to (T^*M)^{\otimes k}$ defined by

$$\varphi(\omega)(X_1, \dots, X_k) = \omega(X_1, \dots, X_k).$$

In particular,

$$\varphi(e_1 \wedge \cdots \wedge e_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}.$$

We define the Riemannian metric g on tensors by

$$g(v_1 \otimes v_2, w_1 \otimes w_2) = g(v_1, w_1) g(v_2, w_2)$$

and similarly for k tensors. Which gives a scalar product on $\Lambda^k M$, defined by

$$\langle \omega, \tau \rangle = \frac{1}{k!} g(\varphi(\omega), \varphi(\tau)).$$

The scalar product can be characterized by the fact that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is an orthonormal basis of $\Lambda^k M$. With this scalar product, the interior and exterior products are adjoint operators,

$$\langle X \lrcorner\, \omega, \tau \rangle = \langle \omega, X \wedge \tau \rangle$$

for $X \in TM, \omega \in \Lambda^k M, \tau \in \Lambda^{k-1} M$. Extending J to act on $\Lambda^k M$ under this identification we get $JJ = (-1)^k$. Thus we write J^{-1} and remember that J^{-1} is either $-J$ or J depending on the degree of the form it acts upon.

Definition 3.16. The Hodge star-operator, $* : \Lambda^k M \to \Lambda^{n-k} M$, is defined by the equation

$$\omega \wedge *\tau = \langle \omega, \tau \rangle dv$$

for all $\omega, \tau \in \Lambda^k M$.

Note that $*1 = dv$, as $1 \wedge *1 = dv$, and $*dv = 1$, as $dv \wedge *dv = dv$. If $\omega = \sum_I \omega_I e_I$ then $*\omega = \sum_{I^c} \omega_{I^c} \operatorname{sgn}(I, I^c) e_{I^c}$ where I^c is the complementary multi-index, i.e. if $I = (i_1, \ldots, i_k)$ then $I^c = (j_1, \ldots, j_{n-k})$, and $\operatorname{sgn}(I, I^c)$ is the sign of the permutation $(I, I^C) = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$. Using this, we calculate that

$$\langle *\omega, *\tau \rangle = \langle \omega, \tau \rangle$$

and

$$*^2 = (-1)^{k(n-k)}$$

on $\Lambda^k M$.

Definition 3.17. The codifferential $d^* : \Omega^{k+1} M \to \Omega^k M$, also denoted δ, is the formal adjoint of $d : \Omega^k M \to \Omega^{k+1} M$ given by $d^* = -(-1)^{nk} * d *$. If $d^*\omega = 0$ we call ω coclosed.

To see that this in fact is the formal adjoint we calculate

$$\begin{aligned} \langle d\alpha, \beta \rangle dv &= d\alpha \wedge *\beta \\ &= d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d * \beta \\ &= d(\alpha \wedge *\beta) - (-1)^{k+k(n-k)} \alpha \wedge * * d * \beta \\ &= d(\alpha \wedge *\beta) - (-1)^{nk} \langle \alpha, * d * \beta \rangle dv. \end{aligned}$$

Hence, setting ω to be defined by $\omega(\alpha, \beta) = \alpha \wedge *\beta$ in Lemma 3.15 makes d^* the formal adjoint of d.

Definition 3.18. The Laplacian of d, $\Delta_d : \Omega^k M \to \Omega^k M$, is defined by

$$\Delta_d = dd^* + d^* d.$$

A differential form ω is called harmonic if $\Delta_d \omega = 0$.

By direct calculations one gets that the Laplacian is its own formal adjoint.

Lemma 3.19. *On a compact Kähler manifold a form α is harmonic if and only if it is both closed and coclosed.*

Proof. The if direction is clear. Let ω be harmonic, then

$$0 = \int_M \langle \Delta_d \omega, \omega \rangle dv = \int_M \langle dd^*\omega, \omega \rangle + \langle d^* d\omega, \omega \rangle dv = \int_M |d^*\omega|^2 + |d\omega|^2 dv,$$

so $d^*\omega = d\omega = 0$. □

Definition 3.20. For a Hermitian manifold (M^{2n}, h, J) we define

$$L : \Lambda^k M \to \Lambda^{k+2} M \quad \text{by} \quad L(\omega) = \Omega \wedge \omega = \frac{1}{2} \sum_{i=1}^{2n} e_i \wedge Je_i \wedge \omega \quad \text{and}$$

$$\Lambda : \Lambda^{k+2} M \to \Lambda^k M \quad \text{by} \quad \Lambda(\omega) = \frac{1}{2} \sum_{i=1}^{2n} Je_i \lrcorner\, e_i \lrcorner\, \omega.$$

We also define $[P, Q] = PQ - QP$, the usual commutator for differential operators. Note that by definition of d^*, if the manifold has even dimension, $d^* = - * d *$ in all degrees.

Lemma 3.21.

1. *The Hodge star-operator $*$ maps (p, q)–forms to $(n-q, n-p)$–forms.*
2. $[X \lrcorner\,, \Lambda] = 0$ *and* $[X \lrcorner\,, L] = JX \wedge -$.

Proof. 1. Follows from the fact that the volume form dv is a (n, n)-form and $\omega \wedge *\omega = \langle \omega, \omega \rangle dv$. While 2. is a direct calculation. We have

$$[X \lrcorner\,, \Lambda](\omega) = \frac{1}{2} \sum_{i=1}^{2n} X \lrcorner\, Je_i \lrcorner\, e_i \lrcorner\, \omega - \frac{1}{2} \sum_{i=1}^{2n} Je_i \lrcorner\, e_i \lrcorner\, X \lrcorner\, \omega = 0,$$

since $X \lrcorner\, Y \lrcorner\, \omega = -Y \lrcorner\, X \lrcorner\, \omega$. Also,

$$\begin{aligned} [X \lrcorner\,, L](\omega) &= X \lrcorner\, (\Omega \wedge \omega) - \Omega \wedge (X \lrcorner\, \omega) \\ &= (X \lrcorner\, \Omega) \wedge \omega + \Omega \wedge (X \lrcorner\, \omega) - \Omega \wedge (X \lrcorner\, \omega) \\ &= h(JX, -) \wedge \omega \\ &= JX \wedge \omega, \end{aligned}$$

by the identification of vector fields and 1-forms. □

Definition 3.22. Let M be a Kähler manifold. Define the twisted differential $d^c : \Omega^k M \to \Omega^{k+1} M$ by

$$d^c(\omega) = \sum_{i=1}^{2m} Je_i \wedge \nabla_{e_i} \omega$$

and its formal adjoint $(d^c)^* = \delta^c : \Omega^{k+1} M \to \Omega^k M$ by

$$\delta^c = - * d^c * = - \sum_{i=1}^{2m} Je_i \lrcorner\, \nabla_{e_i}(\omega).$$

The Laplacian of d^c is defined as $\Delta_{d^c} = d^c\delta^c + \delta^c d^c$.

Lemma 3.23. *$d^c = J^{-1}dJ = (-1)^k JdJ$.*

Proof. Let ω be a k-form. Since $d\omega = \sum_{i=1}^{2m} e_i \wedge \nabla_{e_i}\omega$ we have

$$\begin{aligned} J^{-1}dJ\omega &= (-1)^k J\left(\sum_{i=1}^{2m} e_i \wedge \nabla_{e_i}(J\omega)\right) \\ &= (-1)^k J\left(\sum_{i=1}^{2m} e_i \wedge J\nabla_{e_i}\omega\right) \qquad \text{by Theorem 3.11} \\ &= (-1)^k \sum_{i=1}^{2m} Je_i \wedge (-1)^k \nabla_{e_i}\omega \\ &= \sum_{i=1}^{2m} Je_i \wedge \nabla_{e_i}\omega. \end{aligned}$$

□

Furthermore, via some tedious calculations, $J \circ * = (-1)^k * \circ J$, and it follows that $\delta^c = J^{-1}d^*J = (-1)^k Jd^*J$.

Lemma 3.24 (Lemma 14.5 [Mor07])**.** *On a Kähler manifold M the following holds*

$$[L, d^*] = d^c, \quad [L, d] = 0 \quad \textit{and} \quad [\Lambda, d] = -\delta^c, \quad [\Lambda, d^*] = 0.$$

These are called the Kähler identities.

By the decomposition $d = \partial + \bar{\partial}$, we have a corresponding decomposition $d^* = \partial^* + \bar{\partial}^*$, which satisfies

$$\partial^* : \Omega^{p,q}M \to \Omega^{p-1,q}M, \qquad \partial^* = - * \bar{\partial} *$$

and

$$\bar{\partial}^* : \Omega^{p,q}M \to \Omega^{p,q-1}M, \qquad \bar{\partial}^* = - * \partial * .$$

Furthermore, $\bar{\partial}^*$ is the formal adjoint of ∂ and ∂^* is the formal adjoint of $\bar{\partial}$ with respect to the Hermitian product $H(\omega, \tau) = \langle \omega, \overline{\tau} \rangle$. Hence, we have two additional Laplacians

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \quad \text{and} \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Lemma 3.19 holds for $\Delta_\partial, \Delta_{\bar{\partial}}$ and Δ_{d^c} with closed and coclosed swapped to ∂-closed ∂-coclosed and similarly for the others. The proof is identical.

Theorem 3.25. *If M is a Kähler manifold, then* $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial} = \Delta_{d^c}$.

Proof. The identification of TM with T^*M via the musical isomorphisms extends to $TM^{\mathbb{C}}$ and maps $(1,0)$-vectors to $(0,1)$-forms. Therefore, the following holds

$$\partial = \sum_{j}^{2n} \frac{1}{2}(e_j + iJe_j) \wedge \nabla_{e_j} \quad \text{and} \quad \bar\partial = \sum_{j=1}^{2n} \frac{1}{2}(e_j - iJe_j) \wedge \nabla_{e_j}.$$

It follows that

$$d^c = i(\bar\partial - \partial).$$

Post and precomposing with the Hodge star operator yields

$$\delta^c = i(\partial^* - \bar\partial^*).$$

It follows, by studying the bidegrees of the Kähler identities in Lemma 3.24, that

$$[L, \partial^*] = i\bar\partial, \quad [L, \bar\partial^*] = -i\partial, \quad [L, \partial] = 0, \quad [L, \partial] = 0$$

and

$$[\Lambda, \partial] = i\bar\partial^*, \quad [\Lambda, \bar\partial] = -i\partial^*, \quad [\Lambda, \partial^*] = 0, \quad [\Lambda, \bar\partial^*] = 0.$$

Using these identities, we calculate

$$\begin{aligned} -i(\bar\partial\partial^* + \partial^*\bar\partial) &= \bar\partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\bar\partial \\ &= \bar\partial\Lambda\bar\partial - \bar\partial\bar\partial\Lambda + \Lambda\bar\partial\bar\partial - \bar\partial\Lambda\bar\partial \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} i(\partial\bar\partial^* + \bar\partial^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial\Lambda\partial - \partial\partial\Lambda + \Lambda\partial\partial - \partial\Lambda\partial \\ &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta_d &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= \partial\partial^* + \partial\bar\partial^* + \bar\partial\partial^* + \bar\partial\bar\partial^* + \partial^*\partial + \partial^*\bar\partial + \bar\partial^*\partial + \bar\partial^*\bar\partial \\ &= (\partial\partial^* + \partial^*\partial) + (\partial\bar\partial^* + \bar\partial^*\partial) + (\bar\partial\partial^* + \partial^*\bar\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) \\ &= \Delta_\partial + 0 + 0 + \Delta_{\bar\partial}. \end{aligned}$$

We have $\Delta_\partial = \Delta_{\bar\partial}$, since

$$\begin{aligned} -i\Delta_\partial &= -i(\partial\partial^* + \partial^*\partial) = \partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\partial \\ &= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial \\ &= \partial\Lambda\bar\partial + \bar\partial\partial\Lambda - \Lambda\partial\bar\partial - \bar\partial\Lambda\partial \\ &= [\partial, \Lambda]\bar\partial + \bar\partial[\partial, \Lambda] \\ &= -i\bar\partial^*\bar\partial - i\bar\partial\bar\partial^* = -i\Delta_{\bar\partial}. \end{aligned}$$

Lastly, since $d^c = J^{-1}dJ$ and $\delta^c = J^{-1}\delta J$ we have

$$\begin{aligned} \Delta_{d^c} &= d^c\delta^c + \delta^c d^c \\ &= J^{-1}dJJ^{-1}\delta J + J^{-1}\delta JJ^{-1}dJ \\ &= J^{-1}(d\delta + \delta d)J \\ &= J^{-1}\Delta_d J \end{aligned}$$

(there is some subtlety with the signs here as d and δ changes the degree, feel free to check). Now we claim that $J^{-1}\Delta_\partial J = \Delta_\partial$, but this is clear as on (p,q)–forms J just acts as i^{q-p} and Δ_∂ preserves bidegree. Similarly $J^{-1}\Delta_{\bar\partial}J = \Delta_{\bar\partial}$, and thus $\Delta_{d^c} = \Delta_d$. $\square$

Note that the theorem concerns complex forms, but $\Delta_d = \Delta_{d^c}$ also holds for real forms. The theorem also tells us that Δ_d and Δ_{d^c} preserve bidegrees, i.e. induces a map $\Delta_d : \Omega^{p,q}M \to \Omega^{p,q}M$.

Denote the subspace of harmonic forms, with respect to the Laplacian of d, by

$$\mathcal{H}_d^k M = \Omega^k M \cap \ker \Delta_d \quad \text{and} \quad \mathcal{H}_d^{p,q} M = \Omega^{p,q} M \cap \ker \Delta_d$$

and similarly for d^c, as well as the second one for ∂ and $\bar\partial$. For a Kähler manifold we now know that these are all equal.

Theorem 3.26 (Existence of Green's function, [GMV95])**.** *Let M be a compact oriented Riemmanian manifold, then $\dim \mathcal{H}_d^k < \infty$, so there exists a unique well-defined orthogonal projection $\mathcal{H} : \Omega^k M \to \mathcal{H}_d^k M$. Furthermore, there exists a unique operator, called Green's operator,*

$$G_d : \Omega^k M \to \Omega^k M$$

for each k, that satisfies

$$G_d(\mathcal{H}_d^k M) = 0, \quad dG_d = G_d d, \quad d^* G_d = G_d d^*$$

and

$$\mathrm{id} = \mathcal{H} + \Delta_d G_d.$$

For a compact Kähler manifold we have more.

Theorem 3.27. *Let M be a compact Kähler manifold. The Green's operator G_d preserves bidegrees*

$$G_d : \Omega^{p,q}M \to \Omega^{p,q}M.$$

Furthermore, if D is any of the operators d, d^c, ∂ or $\bar{\partial}$, then the same equations hold if we replace d by D:

$$G_d(\mathcal{H}_D^{p,q}M) = 0, \quad DG_d = G_dD, \quad D^*G_d = G_dD^*.$$

Proof. By Theorem 3.25 $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}} = \Delta_{d^c}$ and each subspace $\mathcal{H}_D^{p,q}$ is equal. So the first equation is clear. Let $\mathcal{H}^{p,q}$ be the unique orthogonal projection $\Omega^{p,q}M \to \mathcal{H}_d^{p,q}M$. Then, $\bigoplus_{p+q=k} \mathcal{H}^{p,q} = \mathcal{H}$ as the subspaces $\Omega^{p,q}M$ are orthogonal, by the fact that h is Hermitian and that $\mathcal{H}$ is unique. Thus, $\mathcal{H}$ preserves bidegrees and looking at (p,q) forms in the equation $\omega = \mathcal{H}\omega + \Delta_d G_d \omega$ shows that G_d must preserve bidegrees as well, since Δ_d does. The last part follows from looking at bidegrees in the equation

$$(\partial + \bar{\partial})G_d = dG_d = G_d d = G_d(\partial + \bar{\partial}),$$

and similarly for the adjoints D^*. □

We can thus simply call the Green's operator G.

There is much more to say about Hodge theory and Kähler manifold, but we leave it here and refer the reader to [Mor07], [Voi02] and [GH94].

4 Commutative bidifferential bigraded algebras

We now generalize the notion of cdga's to bigraded algebras, having the complex de Rham forms $\Omega^{*,*}M$ as motivation. We only consider the underlying field $\Bbbk$ to be equal to $\mathbb{C}$ in this section. A $\mathbb{C}$-vector space V is called bigraded if $V = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$. If $x \in V^{p,q}$ we say that x is of bidegree (p,q) and total degree $p+q$. We write $|x|$ for its total degree or bidegree, depending on context. A linear map $\varphi : V \to W$ of bigraded $\mathbb{C}$-vector spaces is said to have bidegree (k,ℓ) if $\varphi(V^{p,q}) \subset W^{p+k,q+\ell}$. The tensor product of two bigraded $\mathbb{C}$-vector spaces is also bigraded by

$$(V \otimes W)^{p,q} = \bigoplus_{\substack{s+u=p \\ t+v=q}} V^{s,t} \otimes W^{u,v}.$$

A bigraded $\mathbb{C}$-algebra A is an algebra such that the multiplication $A \otimes A \to A$ is a linear map of bidegree $(0,0)$. Note that this necessitates that $1 \in A^{0,0}$. A bigraded $\mathbb{C}$-algebra is called commutative if it is graded-commutative with respect to total degree

$$ab = (-1)^{|a||b|}ba.$$

If V is a bigraded vector space then the free commuative graded algebra $\wedge V$, graded by total degree, is a bigraded algebra in the obvious way.

Definition 4.1. A bicomplex $A = (A^{*,*}, \partial, \bar\partial)$ is a bigraded $\mathbb{C}$-vector space $A^{*,*}$ with endomorphisms ∂ and $\bar\partial$ of bidegree $(1,0)$ and $(0,1)$ such that

$$\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$$

or equivalently, $d^2 = 0$ for $d = \partial + \bar\partial$.

We visualize this as a grid

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p,q+2} & \longrightarrow & A^{p+1,q+2} & \longrightarrow & A^{p+2,q+2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p,q+1} & \longrightarrow & A^{p+1,q+1} & \longrightarrow & A^{p+2,q+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p,q} & \longrightarrow & A^{p+1,q} & \longrightarrow & A^{p+2,q} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where each row and column is a cochain complex and each 1-by-1 square is anti-commutative. In particular, composing more than 2 arrows in any direction gives the zero map.

A bicomplex A is said to have real structure if there is an anti-linear involution $\sigma : A \to A$, such that $\sigma A^{p,q} = A^{q,p}$ and $\sigma d \sigma = d$.

Definition 4.2. A morphism of bicomplexes $f : A \to B$ is a linear map $f : A^{*,*} \to B^{*,*}$ of bidegree $(0,0)$ that satisfies $fd = df$, or equivalently $f\partial = \partial f$ and $f\bar{\partial} = \bar{\partial} f$. If A and B have real structures σ_A and σ_B, then f is called real if $f\sigma_A = \sigma_B f$.

We denote the category of bicomplexes by **BiCo** and the category of bicomplexes with real structure by $\mathbb{R}$-**BiCo**.

Definition 4.3. A commutative bidifferential bigraded algebra (cbba) A is a bicomplex $A = (A^{*,*}, \partial, \bar{\partial})$ such that $A^{*,*}$ is a commutative bigraded $\mathbb{C}$-algebra and the differentials ∂ and $\bar{\partial}$ satisfy Leibniz rule

$$\partial(ab) = (\partial a)b + (-1)^{|a|} a(\partial b).$$

A is a $\mathbb{R}$-cbba if the bicomplex has real structure and the multiplication is real, i.e., if $\sigma(ab) = \sigma(a)\sigma(b)$.

A morphism of ($\mathbb{R}$-)cbba is a (real) map of bicomplexes that is also an algebra homomorphism. We denote the category of cbba's by **CBBA** and the category of $\mathbb{R}$-cbba's by $\mathbb{R}$-**CBBA**.

The main example we keep in mind, as noted before, is the $\mathbb{R}$-cbba $\Omega^{*,*}M$ of complex de Rham forms on a complex manifold M, which in this context we denote by $A(M)$. The real structure is given by conjugation, i.e., locally it is given by

$$A^{p,q}(U) \ni dz_{\alpha_1} \cdots dz_{\alpha_p} d\bar{z}_{\beta_1} \cdots d\bar{z}_{\beta_q} \overset{\sigma}{\mapsto} d\bar{z}_{\alpha_1} \cdots d\bar{z}_{\alpha_p} dz_{\beta_1} \cdots dz_{\beta_q} \in A^{q,p}(U).$$

A holomorphic map $f : N \to M$ of complex manifolds induces a morphism of $\mathbb{R}$-cbba's $f^* : A(M) \to A(N)$.

For bicomplexes there are various cohomologies that we can take. The simplest ones are

$$H_d(A) = \frac{\ker d}{\operatorname{im} d}, \quad H_\partial(A) = \frac{\ker \partial}{\operatorname{im} \partial} \quad \text{and} \quad H_{\bar{\partial}}(A) = \frac{\ker \bar{\partial}}{\operatorname{im} \bar{\partial}}$$

but also Bott-Chern and Aeppli cohomology

$$H_{BC}(A) = \frac{\ker \partial \cap \ker \bar{\partial}}{\operatorname{im} \partial\bar{\partial}} \quad \text{and} \quad H_A(A) = \frac{\ker \partial\bar{\partial}}{\operatorname{im} \partial + \operatorname{im} \bar{\partial}},$$

all of which are functors $\mathbf{BiCo} \to \mathbf{BiVect}_{\mathbb{C}}$ from the category of bicomplexes to bigraded vector spaces over $\mathbb{C}$. These are all related by what we will call the cohomology diamond:

$$\begin{array}{ccccc} & & H_{BC}(A) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\partial}(A) & & H_d(A) & & H_{\bar{\partial}}(A) \\ & \searrow & \downarrow & \swarrow & \\ & & H_A(A) & & \end{array}$$

where all the maps are induced by the inclusion of the kernels, so that it commutes. To see that the maps make sense one just needs to verify that

$$\ker \partial \cap \ker \bar{\partial} \subset \ker \partial, \ker d, \ker \bar{\partial} \subset \ker \partial\bar{\partial}$$

and

$$\operatorname{im} \partial\bar{\partial} \subset \operatorname{im} \partial, \operatorname{im} d, \operatorname{im} \bar{\partial} \subset \operatorname{im} \partial + \operatorname{im} \bar{\partial}.$$

Definition 4.4. A map of bicomplex $f : A \to B$ is called a pluripotential quasi-isomorphism if $H_{BC}(f) : H_{BC}(A) \to H_{BC}(B)$ and $H_A(f) : H_A(A) \to H_A(B)$ are isomorphisms.

Just like for cdga's, this gives us homotopy categories by formally inverting pluripotential quasi-isomorphism. We denote the categories by Ho(**BiCo**), Ho($\mathbb{R}$-**BiCo**), Ho(**CBBA**) and Ho($\mathbb{R}$-**CBBA**). We call an isomorphism in Ho(**BiCo**) and Ho(**CBBA**) a pluripotential weak equivalence and an isomorphism in Ho($\mathbb{R}$-**BiCo**) and Ho($\mathbb{R}$-**CBBA**) a real pluripotential weak equivalence. Just like before, such an isomorphism is explicitly a zig-zag of (real) pluripotential quasi-isomorphisms.

Similarly to cdga's, we say that an augmented cbba is a cbba A and an augmentation $\epsilon : A \to \mathbb{C}$, a morphism of cbba's where $\mathbb{C}$ is situated in bidegree $(0, 0)$. A morphism of augemented cbba's is one that preserves the augmentation and the kernel $\ker \epsilon$ is called the augmentation ideal and denoted A^+. We will always consider $A(M)$ to be augmented by evaluation at some point $x_0 \in M$, and $\wedge V$ to be augmented by its coefficients.

Definition 4.5. A cbba M is called minimal if it can be written as $M = \wedge V$ for a bigraded vector space V that admits a well-ordered basis v_α such that $dv_\alpha \in \wedge(v_{\beta<\alpha})$ and $\operatorname{im} \partial\bar{\partial} \subset \wedge^{\geq 2} V$. A minimal cbba M is called a minimal model of an augmented cbba A if there is a weak equivalence $M \to A$ of augmented cbba's.

Existence and uniqueness of minimal models of cbba's is proved in [Ste25]. For cbba's we have two new notions of formality.

Definition 4.6. A cbba A is called strongly formal (over $\mathbb{R}$) if there is a (real) pluripotential weak equivalence $A \simeq H_{BC}(A)$ where we view $H_{BC}(A)$ as a (real) cbba with trivial differentials.

Similarly to ordinary formality, it is enough that there is a pluripotential weak equivalence $A \simeq H$ to any cbba H with trivial differentials $\partial = 0, \bar{\partial} = 0$.

Definition 4.7. A cbba A is called weakly formal (over $\mathbb{R}$) if there is a (real) pluripotential weak equivalence $A \simeq H$ where H is a (real) cbba with $\partial\bar{\partial} = 0$.

A complex manifold M is strongly/weakly formal if $A(M)$ is strongly/weakly formal. Let us say that a cbba $A = (A, \partial, \bar{\partial})$ is formal if the total cdga (A, d) is $\mathbb{C}$-formal. Thus, we have 3 notions of formality for cbba's.

Let us return to nilmanifolds and Lie groups. Let G be a real Lie group equipped with a left invariant almost complex structure J. Then J defines a linear map $J : \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -\operatorname{id}$. Conversely, given such a linear map $J : \mathfrak{g} \to \mathfrak{g}$, we get a left invariant almost complex structure on G. Recall that if J is a complex structure, then the Nijenhuis tensor N^J is identically zero, Proposition 3.6. If we view N^J as a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, then J is a complex structure if this disappears as well, due to the fact that it is a tensor, so can be checked pointwise. Complexifying $\mathfrak{g}$, we have $\mathfrak{g}_{\mathbb{C}}$ with subspaces $\mathfrak{g}_{\mathbb{C}}^{1,0}$ and $\mathfrak{g}_{\mathbb{C}}^{0,1}$ of holomorphic and antiholomorphic parts, just like for complex manifolds. Considering $\wedge\mathfrak{g}_{\mathbb{C}}^*$ with the (complexified) Chevalley-Eilenberg differential again we get a real cbba $(\wedge\mathfrak{g}_{\mathbb{C}}^*, \partial + \bar{\partial} = d)$.

Now suppose that $\mathfrak{n}$ is a nilpotent Lie algebra with associated Lie group N and rational structure coefficients, so that it admits a nilmanifold N/Γ. If N has a left invariant complex structure J then so does N/Γ (recall that we consider right cosets N/Γ). We say that such a left invariant complex structure J is nilpotent if there is a basis $\{\omega_1, \ldots, \omega_n\}$ of $(\mathfrak{g}_{\mathbb{C}}^*)^{1,0}$ such that

$$d\omega_i \in \wedge(\omega_{j<i}, \bar{\omega}_{j<i}).$$

See [CFGU00] for a more detailed treatment. We have the following analogue to Nomizu's Theorem:

Theorem 4.8 (Theorem 3.7 [Ste25])**.** *If N/Γ is a nilmanifold with nilpotent complex structure, then $(\wedge\mathfrak{g}_{\mathbb{C}}^*, d)$ is a pluripotential minimal model of $A(N/\Gamma)$.*

Example 4.9. Consider the Lie algebra

$$\mathfrak{n} = \mathrm{span}_{\mathbb{R}}(X_1, \ldots, X_n, Y_1, \ldots, Y_n, W, U)$$

with bracket $[X_i, Y_i] = -W$ for all i. This Lie algebra is nilpotent and has rational structure constants, so we get a $2n+2$ dimensional nilmanifold N/Γ that has minimal model

$$(\wedge \mathfrak{n}^*, d) = \left(\wedge(x_1, \ldots, x_n, y_1, \ldots, y_n, w, u), dw = \sum_{i=1}^{n} x_i y_i \right).$$

Since d is not zero this nilmanifold is not formal by Proposition 2.8. This Lie algebra has a left-invariant almost complex structure given by

$$J(X_i) = Y_i, J(Y_i) = -X_i \quad \text{and} \quad J(W) = U, J(U) = -W.$$

It is a simple check that the Nijenhuis tensor N^J, in terms of $\mathfrak{n}$, is zero. Thus, N and N/Γ are complex manifolds. Setting $Z_i = X_i - iY_i$ and $\Xi = 2i(W - iU)$, we get

$$A = (\wedge \mathfrak{n}^*_{\mathbb{C}}, d) = (\wedge(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, \xi, \bar{\xi}), d)$$

with differential satisfying

$$d(z_i) = 0, \quad d(\bar{z}_i) = 0, \quad d(\xi) = 2id(w) = 2i \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} z_i \bar{z}_i, \quad d(\bar{\xi}) = \sum_{i=1}^{n} z_i \bar{z}_i.$$

We see that the complex structure J is nilpotent with this ordering. Note that

$$\partial\bar{\partial}(\xi\bar{\xi}) = \sum_{i=1}^{n} \partial(z_i \bar{z}_i \bar{\xi}) = \sum_{i=1}^{n} z_i \bar{z}_i \sum_{j=1}^{n} z_j \bar{z}_j = 2 \sum_{i<j} z_i \bar{z}_i z_j \bar{z}_j, \tag{5}$$

so if $n = 1$, then $\partial\bar{\partial}(\xi\bar{\xi}) = 0$, and in fact if $n = 1$, then $\partial\bar{\partial} \equiv 0$. By Theorem 4.8 $A \simeq A(N/\Gamma)$.

When $n = 1$ the nilmanifold N/Γ is the Kodaira-Thurston surface, denoted KT. The nilpotent Lie group can be realized as real matrices of the form

$$\begin{pmatrix} 1 & u_1 & u_3 & 0 & 0 \\ 0 & 1 & u_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and Γ as the subgroup of such matrices with integer coefficients. By this matrix representation it is not hard to see that KT is homeomorphic to $H_3 \times S^1$ where H_3 is the 3-dimensional Heisenberg nilmanifold, discussed in Example 2.5. The Lie algebra $\mathfrak{n}$ is then given by such matrices but with 0's on the diagonals. As discussed, $\partial\bar{\partial} \equiv 0$ on A so by the pluripotential quasi-isomorphism $A \simeq A(KT)$ we get an example of weakly formal complex manifold that isn't formal. Another reason why this example is of importance is that KT is a symplectic manifold, a closed non-degenerate 2-form is given by $\omega = wy + xu$. So, symplectic manifolds need not be formal.

For more theory on bicomplexes and cbba's see [Ste25].

5 Formality of compact Kähler manifolds

Definition 5.1. A cbba $(A, \partial, \bar\partial)$ satisfies the $\partial\bar\partial$-property if for all $a \in A$, such that $\partial a = \bar\partial a = 0$, the following implications are true:

$$a = \partial c \implies a = \partial\bar\partial b \quad \text{and} \quad a = \bar\partial c \implies a = \bar\partial\partial b.$$

In other words the morphisms $H_{BC}(A) \to H_\partial(A)$ and $H_{BC} \to H_{\bar\partial}(A)$ in the cohomology diamond are injective.

The $\partial\bar\partial$-property can also be expressed as

$$\ker\partial \cap \ker\bar\partial \cap \operatorname{im}\partial = \operatorname{im}\partial\bar\partial \quad \text{and} \quad \ker\partial \cap \ker\bar\partial \cap \operatorname{im}\bar\partial = \operatorname{im}\partial\bar\partial. \tag{6}$$

where the $\subset$ is the property, as $\supset$ is always true. Technically, this way of writing is redundant, as $\ker\partial \cap \operatorname{im}\partial = \operatorname{im}\partial$, but it indicates better how it is related to the morphisms in the cohomology diamond.

Lemma 5.2. *The $\partial\bar\partial$-property is equivalent to*

$$\ker\partial \cap \ker\bar\partial \cap (\operatorname{im}\partial + \operatorname{im}\bar\partial) = \operatorname{im}\partial\bar\partial. \tag{7}$$

In other words, the morphism $H_{BC}(A) \to H_A(A)$ is injective.

Proof. Note that just like in (6), it is the $\subset$ part of (7) that needs proving. Let $a \in \ker\partial \cap \ker\bar\partial$ and $a = \partial c + \bar\partial c'$. Then, $a - \partial c = \bar\partial c' \in \ker\partial \cap \operatorname{im}\bar\partial$, so by (6) $a - \partial c = \partial\bar\partial b$. Similarly, $a - \bar\partial c = \partial c \in \ker\bar\partial \cap \operatorname{im}\partial$, so $a - \bar\partial c' = \partial\bar\partial b'$. Adding these we get

$$2a - \partial c - \bar\partial c' = \partial\bar\partial(b + b').$$

Since $a = \partial c + \bar\partial c'$, we have $a = \partial\bar\partial(b + b') \in \operatorname{im}\partial\bar\partial$. Conversely, if (7) holds, both inclusions in (6) follows from $\operatorname{im}\partial, \operatorname{im}\bar\partial \subset (\operatorname{im}\partial + \operatorname{im}\bar\partial)$. $\square$

A direct consequence of this lemma is that the $\partial\bar\partial$-property is invariant under pluripotential quasi-isomorphisms. Since we get a commutative diagram

$$\begin{array}{ccc} H_{BC}(A) & \xrightarrow{\cong} & H_{BC}(B) \\ \downarrow & & \downarrow \\ H_A(A) & \xrightarrow[\cong]{} & H_A(B) \end{array}$$

Hence, if the left arrow is injective, so must the right arrow be. In particular, if A is strongly formal, A satisfies the $\partial\bar\partial$-property.

Additionally, if A satisfies the $\partial\bar{\partial}$-property and is weakly formal, it follows that it is strongly formal. The weak equivalence $A \simeq H$ implies that H also has the $\partial\bar{\partial}$-property and since $\partial\bar{\partial} = 0$ we get that $\operatorname{im}\partial + \operatorname{im}\bar{\partial} \subset \ker\partial \cap \ker\bar{\partial}$ and must be equal to $\operatorname{im}\partial\bar{\partial} = 0$ by Lemma 5.2 and hence $\partial = \bar{\partial} = 0$.

But something much stronger is true

Theorem 5.3. *If A satisfies the $\partial\bar{\partial}$-property, then all the maps in the cohomology diamond are isomorphisms.*

Proof. Recall the cohomology diamond

$$\begin{array}{ccccc} & & H_{BC}(A) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\partial}(A) & & H_{d}(A) & & H_{\bar{\partial}}(A) \\ & \searrow & \downarrow & \swarrow & \\ & & H_{A}(A) & & \end{array}$$

We prove that all the maps are surjective and injective from the assumption that the two upper outer maps are injective, expressed by (6).

- $H_{BC}(A) \to H_A(A)$ is injective: This is Lemma 5.2.
- $H_{BC}(A) \to H_d(A)$ is injective: Injectivity of this map is expressed as

$$\ker\partial \cap \ker\bar{\partial} \cap \operatorname{im} d \subset \operatorname{im}\partial\bar{\partial},$$

  but since $\operatorname{im} d \subset \operatorname{im}\partial + \operatorname{im}\bar{\partial}$, this is immediate from the previous point.
- $H_{\partial}(A) \to H_A(A)$ and $H_{\bar{\partial}}(A) \to H_A(A)$ are injective: Injectivity of the first is

$$\ker\partial \cap (\operatorname{im}\partial + \operatorname{im}\bar{\partial}) \subset \operatorname{im}\partial.$$

  If $\partial a = 0$ and $a = \partial c + \bar{\partial}c'$, then $a - \partial c = \bar{\partial}c' \in \ker\partial \cap \operatorname{im}\bar{\partial}$, so $a - \partial c = \partial\bar{\partial}b$, and thus $a = \partial(c + \bar{\partial}b)$. The proof of $H_{\bar{\partial}}(A) \to H_A(A)$ is essentially the same.
- $H_{\partial}(A) \to H_A(A)$ and $H_{\bar{\partial}}(A) \to H_A(A)$ are surjective: Surjectivity of the first one means that

$$\ker\partial\bar{\partial} \subset \ker\partial + (\operatorname{im}\partial + \operatorname{im}\bar{\partial})$$

since the map is surjective if for each $a \in \ker \partial\bar{\partial}$, we can find some $b \in \ker \partial$ so that $[b]_A = [a]_A$, i.e. $b + \partial c + \bar{\partial} c' = a$. If $\partial\bar{\partial} a = 0$, then we have $\partial a \in \ker \bar{\partial} \cap \ker \partial \cap \operatorname{im} \partial$, so $\partial a = \partial\bar{\partial} b$ and

$$a = (a - \bar{\partial} b) + \bar{\partial} b.$$

Now note that $\bar{\partial} b \in \operatorname{im} \partial + \operatorname{im} \bar{\partial}$ and $a - \bar{\partial} b \in \ker \partial$, since

$$\partial(a - \bar{\partial} b) = \partial\bar{\partial} b - \partial\bar{\partial} b = 0.$$

Hence, $a \in \ker \partial + (\operatorname{im} \partial + \operatorname{im} \bar{\partial})$. Surjectivity of $H_{\bar{\partial}}(A) \to H_A(A)$ is proved in essentially the same way, swapping the roles of ∂ and $\bar{\partial}$.

- $H_{BC}(A) \to H_{\partial}(A)$ and $H_{BC}(A) \to H_{\bar{\partial}}(A)$ are surjective: The first being surjective is equivalent to

  $$\ker \partial \subset \ker \partial \cap \ker \bar{\partial} + \operatorname{im} \partial.$$

  Since $\ker \partial \subset \ker \partial\bar{\partial}$, we use the previous point to write $a = b + \partial c + \bar{\partial} c'$ where $b \in \ker \bar{\partial}$. Applying ∂, gives us $\partial b + \partial\bar{\partial} c' = 0$, and $\bar{\partial}(b + \bar{\partial} c') = 0$ directly. Thus, $a \in \ker \partial \cap \ker \bar{\partial} + \operatorname{im} \partial$ by writing $a = b + \bar{\partial} c' + \partial c$. The other map is proved analogously.

- $H_{BC}(A) \to H_d(A)$ is surjective: This can be expressed as

  $$\ker d \subset \ker \partial \cap \ker \bar{\partial} + \operatorname{im} d.$$

  Note that by the previous two points, the composite $H_{BC}(A) \to H_A(A)$ is surjective, so

  $$\ker \partial\bar{\partial} \subset \ker \partial \cap \ker \bar{\partial} + \operatorname{im} \partial + \operatorname{im} \bar{\partial}.$$

  Since $\ker d \subset \ker \partial\bar{\partial}$, if $a \in \ker d$, then we can write $a = b + \partial c + \bar{\partial} c'$ where $b \in \ker \partial \cap \ker \bar{\partial}$. Thus, to prove the statement we need to write $\partial c + \bar{\partial} c' = d\tilde{c}$. Write $b = a - (\partial c + \bar{\partial} c')$, so that

  $$0 = \partial b = \partial a - \partial\bar{\partial} c' \quad \text{and} \quad 0 = \bar{\partial} b = \bar{\partial} a - \bar{\partial}\partial c.$$

  Since $0 = da = \bar{\partial}\partial c + \partial\bar{\partial} c' = \partial\bar{\partial}(c' - c)$, by applying d to $a = b + \partial c + \bar{\partial} c'$, we see that $c' - c \in \ker \partial\bar{\partial}$ and we again write

  $$c' - c = w + \partial u + \bar{\partial} v,$$

where $w \in \ker \partial \cap \ker \bar{\partial}$. Applying ∂ and $\bar{\partial}$ to $c' - c$ we get

$$\partial c' - \partial c = \partial\bar{\partial} v \quad \text{and} \quad \bar{\partial} c' - \bar{\partial} c = \bar{\partial}\partial u.$$

It follows that

$$\partial c + \bar{\partial} c' = \partial c' - \partial\bar{\partial} v + \bar{\partial}\partial u + \bar{\partial} c = \partial c' + \bar{\partial} c - \partial\bar{\partial}(v + u),$$

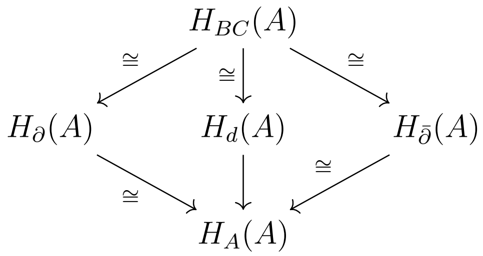
and therefore

$$d(c + c') = \partial c + \bar{\partial} c + \partial c' + \bar{\partial} c' = 2(\partial c' + \bar{\partial} c) - \partial\bar{\partial}(v + u).$$

We define $\tilde{c} = \frac{1}{2}(c + c' - \bar{\partial}(v + u))$ and calculate that

$$d\tilde{c} = \frac{1}{2} d(c + c') - \frac{1}{2} d\bar{\partial}(v + u) = \partial c' + \bar{\partial} c - \partial\bar{\partial}(v + u) = \partial c + \bar{\partial} c'.$$

We have now proved that



and the last morphism is an isomorphism by commutativity. □

Hence, if A satisfies the $\partial\bar{\partial}$-property, then we may talk about the cohomology of A, $H(A)$, and mean any of the cohomologies in the cohomology diamond.

Proposition 5.4. If A satisfies the $\partial\bar{\partial}$-property and $H_{BC}(A)$ (or any of the isomorphic cohomologies in the cohomology diamond) is a free bigraded algebra, then A is strongly formal.

Proof. The proof is essentially the same as the proof of Proposition 1.17. Let $H_{BC}(A) = \wedge W$ and define a map $\varphi : \wedge W \to A$ by $w \mapsto a_w$ with $[a_w]_{BC} = w$. Since $\partial a_w = \bar{\partial} a_w = 0$ this is a morphism of cbba's. The induced map on Bott chern cohomology is the identity and, by Theorem 5.3, so is the induced map on Aeppli cohomology, thus φ is a pluripotential quasi-isomorphism. □

Definition 5.5. Let $(A, \partial, \bar{\partial})$ be a cbba, define $d^c = i(\partial - \bar{\partial})$. A cbba $(A, \partial, \bar{\partial})$ satisfies the dd^c-property if for all $a \in A$ such that $da = d^c a = 0$ we have

$$a = dc \implies a = dd^c b \quad \text{and} \quad a = d^c c \implies d^c db.$$

Just like the $\partial\bar{\partial}$-property we can summarize this as

$$\ker d \cap \ker d^c \cap \operatorname{im} d = \operatorname{im} dd^c \quad \text{and} \quad \ker d \cap \ker d^c \cap \operatorname{im} d^c = \operatorname{im} dd^c$$

from the fact that $dd^c = -d^c d$ and $d^c d^c = 0$. The proof of Lemma 5.2 goes through word for word by swapping ∂ and $\bar{\partial}$ to d and d^c. Thus, the dd^c-property is equivalent to

$$\ker d \cap \ker d^c \cap (\operatorname{im} d + \operatorname{im} d^c) = \operatorname{im} dd^c. \tag{8}$$

Proposition 5.6. The $\partial\bar{\partial}$-property is equivalent to the dd^c-property.

Proof. We just need to prove that equations (7) and (8) are equivalent. We first claim that $\ker \partial \cap \ker \bar{\partial} = \ker d \cap \ker d^c$. The left inclusion is immediate by definition of d and d^c. For the right inclusion, we have $0 = da = \partial a + \bar{\partial} a$ and $0 = id^c a = \bar{\partial} a - \partial a$. It follows that $\partial a = \bar{\partial} a = 0$ by adding and subtracting the previous equations.

Secondly, $\operatorname{im} \partial + \operatorname{im} \bar{\partial} = \operatorname{im} d + \operatorname{im} d^c$, here the right inclusion is immediate. For the other, note that

$$\partial = \frac{1}{2}(d - id^c) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(d + id^c),$$

so

$$\partial a + \bar{\partial} b = \frac{1}{2}\left(d(a+b) - id^c(a-b)\right).$$

Lastly, $\operatorname{im} \partial\bar{\partial} = \operatorname{im} dd^c$, but this is immediate by calculating that

$$dd^c = i(\partial + \bar{\partial})(\partial - \bar{\partial}) = -2i\partial\bar{\partial}.$$

□

Lemma 5.7 ($\partial\bar{\partial}$-Lemma)**.** *If M is a compact Kähler manifold, then $A(M)$ satisfies the $\partial\bar{\partial}$-property.*

Proof. Let $\alpha \in A(M)$ satisfy $\partial\alpha = \bar{\partial}\alpha = 0$ and $\alpha = \partial\gamma$. Recall Theorem 3.27, we can write $\alpha = \mathcal{H}\alpha + 2\Delta_\partial G\alpha$ as $\Delta_d = 2\Delta_\partial$. If h is harmonic, then

$$\langle \alpha, h \rangle = \langle \partial\gamma, h \rangle = \langle \gamma, \partial^* h \rangle = 0$$

by Lemma 3.19 (the lemma is still valid if we swap d to ∂, and all Laplacians are equal up to constants by Lemma 3.25), hence $\mathcal{H}\alpha = 0$. Furthermore,

$$\Delta_\partial G\alpha = \partial\partial^* G\alpha + \partial^*\partial G\alpha = \partial\partial^* G\alpha + \partial^* G(\partial\alpha) = \partial\partial^* G\alpha$$

and thus $\alpha = 2\partial\partial^* G\alpha$. Doing the exact same trick with $\bar{\partial}$ we get that $\alpha = 2\bar{\partial}\bar{\partial}^* G\alpha$. Combining this we have

$$\begin{aligned}\alpha &= 2\partial\partial^* G\alpha \\ &= 2\partial\partial^* G(2\bar{\partial}\bar{\partial}^* G\alpha) \\ &= 4\partial\partial^*\bar{\partial} G(\bar{\partial}^* G\alpha) \\ &= -4\partial\bar{\partial}\partial^* G(\bar{\partial}^* G\alpha)\end{aligned}$$

and hence in the image of $\partial\bar{\partial}$. The last equality comes from

$$\begin{aligned}\bar{\partial}\partial^* + \partial^*\bar{\partial} &= i\bar{\partial}[\Lambda, \bar{\partial}] + i[\Lambda, \bar{\partial}]\bar{\partial} \\ &= i\bar{\partial}\Lambda\bar{\partial} - i\bar{\partial}\Lambda\bar{\partial} \\ &= 0\end{aligned}$$

using the identities established in the proof of Theorem 3.25. It is clear that the same argument works for when $\alpha = \bar{\partial}\gamma$. □

The reason for why we even mentioned the dd^c-property is that in [DGMS75] they instead prove the dd^c-lemma, the de Rham differential forms on a compact Kähler manifold M satisfy the dd^c-property. Recall that d^c is a real operator on differential forms 3.22, that when extended to the complex differential forms $A(M)$ is equal to the d^c we defined above. The authors then prove the following in terms of d and d^c instead. However, we choose to continue with the language of cbba's.

Theorem 5.8. *Let $A = (A^{*,*}, \partial, \bar{\partial})$ be a cbba, if A satisfies the $\partial\bar{\partial}$-Property then the total $\mathbb{C}$-cdga (A^*, d) is formal.*

Proof. Consider the three $\mathbb{C}$-cdga's $(H_\partial A, \bar{\partial})$, $(Z_\partial A, \bar{\partial})$ and (A, d), all graded by total degree, where $Z_\partial A$ denotes the ∂-closed forms of A. There are natural morphisms, the first given by the inclusion $\iota : (Z_\partial A, \bar{\partial}) \hookrightarrow (A, d)$ and the second given by the projection $\rho : (Z_\partial A, \bar{\partial}) \to (H_\partial A, \bar{\partial})$ projecting a ∂-closed form α to its homology class $[\alpha]_\partial \in H_\partial A$. Note that ι is a morphisms of cdga's because $d\alpha = \partial\alpha + \bar{\partial}\alpha = \bar{\partial}\alpha$ if $\alpha \in Z_\partial A$. Thus we have the zig-zag

$$(H_\partial A, \bar{\partial}) \xleftarrow{\rho} (Z_\partial A, \bar{\partial}) \xrightarrow{\iota} (A, d).$$

We start by proving that ι is a quasi-isomorphism using the $\partial\bar{\partial}$-property. We continue by proving that $\bar{\partial} = 0$ on $H_\partial A$ and that ρ is a quasi-isomorphism, with repeated use of the $\partial\bar{\partial}$-property. We are then done as we have connected (A, d) via a zig-zag to a cdga with zero differential.

- ι^* is injective: Let $\alpha \in (Z_\partial A, \bar\partial)$ be $\bar\partial$-closed element such that $[\alpha]_d = 0$, i.e. $\alpha = d\beta$ for some $\beta \in A$. Then, as $\partial(\partial\beta) = 0$ and

$$\bar\partial(\partial\beta) = -\partial\bar\partial\beta = -\partial(d\beta - \partial\beta) = -\partial\alpha = 0,$$

we apply the $\partial\bar\partial$-property and find $\gamma \in A$ such that $\partial\beta = \partial\bar\partial\gamma = -\bar\partial\partial\gamma$. Note that

$$\alpha = \partial\beta + \bar\partial\beta = -\bar\partial\partial\gamma + \bar\partial\beta = \bar\partial(\beta - \partial\gamma).$$

Thus, α is a coboundary and $[\alpha]_{\bar\partial} = 0$, proving injectivity.

- ι^* is surjective: Let $[\alpha]_d \in H(A, d)$. We have $\partial(\partial\alpha) = 0$ and

$$\bar\partial\partial\alpha = \bar\partial(d\alpha - \bar\partial\alpha) = 0,$$

so by the $\partial\bar\partial$-property, we find β such that $\partial\alpha = \partial\bar\partial\beta$. Now let $\gamma = \alpha - d\beta$, then

$$\partial\gamma = \partial(\alpha - d\beta) = \partial\alpha - \partial\bar\partial\beta = 0$$

and

$$\bar\partial\gamma = \bar\partial(\alpha - d\beta) = -\partial\alpha - \bar\partial\partial\beta = 0$$

using that since $d\alpha = 0$, one has $\partial\alpha = -\bar\partial\alpha$. Therefore, γ defines a cohomology class $[\gamma]_{\bar\partial} \in H(Z_\partial A, \bar\partial)$ and $[\gamma]_d$ is, by construction, equal to $[\alpha]_d$.

- $\bar\partial = 0$ on $H_\partial A$: Let $[\alpha]_\partial \in H_\partial A$. Then $\bar\partial(\bar\partial\alpha) = 0$ and $\partial(\bar\partial\alpha) = -\bar\partial\partial\alpha = 0$, so by the $\partial\bar\partial$-property we find β such that $\bar\partial\alpha = -\partial\bar\partial\beta$. It follows that

$$\bar\partial[\alpha]_\partial = [\bar\partial\alpha]_\partial = [-\partial\bar\partial\beta]_\partial = 0.$$

In particular, $H(H_\partial A, \bar\partial) = (H_\partial A, \bar\partial)$ and we consider

$$\rho^* : H(Z_\partial A, \bar\partial) \to (H_\partial A, \bar\partial).$$

- ρ^* is surjective: Let $[\alpha]_\partial \in H_\partial A$, then in the same way as we argued above, there is some β such that $\bar\partial\alpha = \bar\partial\partial\beta$. Set $\gamma = \alpha - \partial\beta$. Then, $\partial\gamma = 0$ and therefore $\gamma \in Z_\partial A$. Furthermore, $\bar\partial\gamma = \bar\partial\alpha - \bar\partial\partial\beta = 0$, so $[\gamma]_{\bar\partial} \in H(Z_\partial A, \bar\partial)$ and

$$\rho^*[\gamma]_{\bar\partial} = [\gamma]_\partial = [\alpha]_\partial.$$

- ρ^* is injective: Assume $\partial\alpha = \bar\partial\alpha = 0$, so that $[\alpha]_{\bar\partial} \in H(Z_\partial A, \bar\partial)$. Furthermore, assume that $\rho^*[\alpha]_{\bar\partial} = [\alpha]_\partial = 0$, i.e. $\alpha = \partial\beta$. Then, by the $\partial\bar\partial$-property, there exists some β such that $\alpha = \partial\bar\partial\gamma = -\bar\partial\partial\gamma$, and hence $[\alpha]_{\bar\partial} = 0$.

$\square$

Note that this also implies that strong formality of $(A, \partial, \bar{\partial})$ implies $\mathbb{C}$-formality of (A, d).

Corollary 5.9. Compact Kähler Manifolds are formal.

Proof. Let M be a compact Kähler manifold. The cbba $A(M)$ satisfies the $\partial\bar{\partial}$-property by Lemma 5.7, hence the cdga $(A(M), d)$ is formal. But note that this is just the complex de Rham forms with differential d, i.e $A_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C}$. By Theorem 1.19 $A_{dR}(M) \simeq A_{PL}(M; \mathbb{R})$, and since quasi-isomorphisms are preserved under tensoring $A_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C} \simeq A_{PL}(M; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Since $A_{PL}(M; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong A_{PL}(M; \mathbb{C})$, M is $\mathbb{C}$-formal, and by Theorem 1.18 formal. $\square$

The natural next question is if all compact Kähler manifolds are weakly or strongly formal. Since they satisfy the $\partial\bar{\partial}$-property weak formality and strong formality are equivalent, as remarked previously. In [PSZ24] the question is answered negatively, if M is a closed Riemann surface of genus at least 2, then $A(M)$ carry non-trivial triple ABC-Massey products, discussed in section 7, which are obstruction to weak, and thus also strong, formality.

6 Formality of spaces with certain cohomology

Recall that a sequence of elements $r_1, \ldots, r_d \in R$ is called a regular sequence if r_i is a non-zero-divisor in the quotient ring $R/(r_1, \ldots, r_{i-1})$ for all i. A $\Bbbk$-algebra of the form $A/(a_1, \ldots, a_d)$ for a regular sequence $a_1, \ldots, a_d$ is said to be of complete intersection type.

Definition 6.1. Let A be a commutative (cohomologically) graded algebra over $\Bbbk$ and $a_1, \ldots, a_r \in A$. The Koszul complex of $a_1, \ldots, a_r$ in A is the cdga

$$K(a_1, \ldots, a_r) = A \otimes \wedge(y_1, \ldots, y_r)$$

where $|y_i| = |a_i| - 1$, and with differential given by

$$d|_A = 0 \quad d(y_i) = a_i$$

and Leibniz rule.

We introduce a chain complex structure on $K(a_1, \ldots, a_r)$ by suppressing the degree of A all to zero, letting $|y_i| = 1$ and letting d act the same. As d is the same map, it follows that $H^*(K(a_1, \ldots, a_r)) = H_*(K(a_1, \ldots, a_r))$.

Theorem 6.2 (Theorem 6.2.3 [Smi95])**.** *If $a_1, \ldots, a_r$ is a regular sequence, then*

$$H^*(K(a_1, \ldots a_r)) = A/(a_1, \ldots a_r).$$

Proof. Let us fix a Koszul complex $K(a_1, \ldots, a_r)$ and denote it simply by K. By the remark above, it is enough to prove that $H_*(K) = A/(a_1, \ldots, a_r)$ with the homological grading, i.e.

$$H_n(K) = \begin{cases} A/(a_1, \ldots, a_r) & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Let us start by showing that $H_0(K) = A/(a_1, \ldots, a_r)$. Note that $K_0 = A$, that each K_i is a A-module and that $d_i : K_i \to K_{i-1}$ is a A-module homomorphism by the Leibniz rule. In particular, $\mathrm{im}(d_1 : K_1 \to K_0 = A)$ is an ideal $I \subset A$. By definition

$$H_0(K) = K_0/\,\mathrm{im}(d_1 : K_1 \to K_0) = A/I,$$

so we just need to show that $I = (a_1, \ldots, a_r)$. It is clear that $a_i \in I$ as $d(y_i) = a_i$, so $I \supset (a_1, \ldots, a_r)$. Conversely, as $x \in K_1$ if and only if $x = \sum \alpha_i y_i$, we have

$$a = d(x) = \sum \alpha_i d(y_i) = \sum \alpha_i a_i \in (a_1, \ldots, a_r).$$

Thus, $H_0(K) = A/(a_1, \ldots, a_r)$.

For the rest of the degrees, let us do induction of the subcomplexes $K(i) = K(a_1, \ldots, a_i)$ ($K(0) = A$). We have inclusions $K(i-1) \hookrightarrow K(i)$, and an isomorphism $sK(i-1) \xrightarrow{\cong} K(i)/K(i-1)$, given by $x \mapsto xy_i$, where s is the suspension, so that the degrees line up. The map is injective, since $xy_i \notin K(i-1)$ for any x, and surjective, since $K(i)/K(i-1)$ consists of exactly elements of the form xy_i, as $y_i^2 = 0$ (it is of odd degree). We thus have an exact sequence of chain complexes

$$K(i-1) \to K(i) \to sK(i-1),$$

giving us a long exact sequence in homology

$$\cdots \to H_j(K(i-1)) \to H_j(K(i)) \to H_j(sK(i-1)) \xrightarrow{\partial} H_{j-1}(K(i-1)) \to \cdots .$$

Using the induction hypothesis, that $H_j(sK(i-1)) = H_{j-1}(K(i-1))$ and that we have already proved the statement in degree zero, the sequence simplifies to

$$0 \to H_1(K(i)) \to A/(a_1, \ldots, a_{i-1}) \xrightarrow{\partial} A/(a_1, \ldots, a_{i-1}) \to A/(a_1, \ldots, a_i) \to 0.$$

We just have to check what the connecting homomorphism ∂ is. Note that the map $K(i) \to sK(i-1) = K(i-1)$ is given by the projection $K(i) \to K(i)/K(i-1)$, followed by the inverse of the isomorphism $sK(i-1) \to K(i)/K(i-1)$, which is given by multiplying with y_i. It follows that the connecting homomorphism $\partial : H_1(sK(i-1)) \to H_0(K(i-1))$ is given by

$$A/(a_1, \ldots, a_{i-1}) \ni [x] \mapsto [d(xy_i)] = [xa_i] \in A/(a_1, \ldots, a_{i-1}),$$

as $xy_i \in K(i)_1$ is an element that is mapped to $x \in K(i-1)_0$. In other words, ∂ is given by multiplying by a_i, and thus as a_i is a non-zero divisor in $A/(a_1, \ldots, a_{i-1})$ by regularity of the sequence, it follows that ∂ is injective and that $H_1(K(i)) = 0$. □

Corollary 6.3. Let A be a $\Bbbk$-cdga for a field $\Bbbk$ of characteristic zero. If

$$H^*(A) = \Bbbk[x_1, \ldots, x_n]/(f_1, \ldots, f_r) = \wedge(x_1, \ldots, x_n)/(f_1, \ldots, f_r)$$

with x_i of even positive degree and $f_1, \ldots, f_r$ a regular sequence, then A is formal.

Proof. We want to show that there is a zig-zag of quasi-isomorphism

$$H^*(A) \xleftarrow{\sim} K \xrightarrow{\sim} A$$

where $K = K(f_1, \dots, f_r)$ is the Koszul complex of the regular sequence $f_1, \dots, f_r$ over the free commutative graded algebra $\Bbbk[x_1, \dots, x_n] = \wedge(x_1, \dots, x_n)$. We can write K as $\wedge(x_1, \dots, x_n, y_1, \dots, y_r)$, with $dx_i = 0$ and $dy_i = f_i$. The first map, call it ψ, is defined by $\psi(x_i) = x_i$ and $\psi(y_i) = 0$. Since $d(x_i) = 0$ and $d(y_i) = f_i$, ψ is a morphism pf cdga's. By the previous theorem, we have $H^*(K) = \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_d)$, and since $\psi(x_i) = x_i$, it follows that it is a quasi-isomorphism.

The second map, φ, is defined by $\varphi(x_i) = a_i$ where a_i is such that $[a_i] = x_i$, and $\varphi(y_i) = b_i$ such that $db_i = f_i(a_k)$. It is clear that φ is a morphism of cdga's and a quasi-isomorphism. □

Therefore if the rational cohomology of a topological space X is of the form $\mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_d)$ then the space is formal. Examples of such spaces are elliptic spaces with non-zero Euler characteristic, see Proposition 32.10 in [FHT01]. A space is called elliptic if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$. In fact a space X with $\chi(X) \neq 0$ is elliptic if and only if $H^*(X; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$.

Let us take this to the bigraded world, see [PSZ25]. We shall use the following lemma

Lemma 6.4 (Part of Theorem 1.21 [Ste25])**.** *Let A and B be first quadrant bicomplexes, i.e. $A^{p,q} = 0$ for all $(p, q) \notin \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, then a map of bicomplexes $f : A \to B$ is a pluripotential quasi-isomorphism if and only if*

$$f^* : H_\partial(A) \to H_\partial(B) \quad \text{and} \quad f^* : H_{\bar\partial}(A) \to H_{\bar\partial}(B)$$

are isomorphisms.

The lemma holds more generally by considering locally bounded bicomplexes, but as we haven't define this notion we stick to this version, as it is all we need.

Definition 6.5. Consider the free bigraded algebra $\mathbb{C}[x_1, \dots, x_n]$, where $|x_i| = (p_i, p_i)$ with $p_i \geq 1$, and a regular sequence $r_1, \dots, r_m$ in $\mathbb{C}[x_1, \dots, x_n]$ of elements of pure bidegree, which must also be diagonal as x_i are. The bigraded Koszul complex of such a regular sequence is the cbba

$$K(r_1, \dots, r_m) = \mathbb{C}[x_1, \dots, x_n] \otimes \wedge(y_1, \dots, y_m, \partial y_1, \dots, \partial y_m, \bar\partial y_1, \dots, \bar\partial y_m),$$

where $|y_i| = |r_i| - (1, 1)$, and with differentials given by

$$dx_i = 0, \quad dy_i = \partial y_i + \bar\partial y_i, \quad i\partial\bar\partial y_i = r_i.$$

Theorem 6.6 (Lemma 4.1.3 [PSZ25]). *For such a regular sequence, there is a pluripotenial quasi-isomorphism*

$$\varphi : K(r_1, \dots, r_m) \to (\mathbb{C}[x_1, \dots, x_n]/(r_1, \dots, r_m), d = 0)$$

given by $x_i \mapsto x_i$ *and* $y_i \mapsto 0$.

Proof. Denote $K(r_1, \dots, r_m)$ by K and $\mathbb{C}[x_1, \dots, x_n]/(r_1, \dots, r_m)$ by H. As the total degree of all the elements are non-negative, except possibly if r_i are scalars but then it is a degenerate case, we can use Lemma 6.4 to conclude that it is a quasi-isomorphism by checking it on ∂ and $\bar{\partial}$ cohomology instead. Let $U_\partial : \mathbf{CBBA} \to \mathbf{CDGA}$ be the functor that forgets the $\bar{\partial}$ differential, and similarly $U_{\bar{\partial}}$ be the functor that forgets the ∂ differential. Then the induced map $\varphi^* : H_\partial(K) \to H_\partial(H)$ is equal to $U(\varphi)^* : H(U_\partial(K)) \to H(U_\partial(H))$. We can decompose $U_\partial(K)$ into

$$\mathbb{C}[x_1, \dots, x_n] \otimes \wedge(\bar{\partial} y_1, \dots, \bar{\partial} y_m) \otimes \wedge(y_1, \dots, y_m, \partial y_1, \dots, \partial y_m).$$

As $\partial(\bar{\partial} y_i) = -ir_i$, we get that

$$\mathbb{C}[x_1, \dots, x_n] \otimes \wedge(\bar{\partial} y_1, \dots, \bar{\partial} y_m)$$

is the Koszul complex of $r_1, \dots, r_m$ in $\mathbb{C}[x_1, \dots, x_n]$ seen as a cdga with ∂ as differential. Thus, by Theorem 6.2, its cohomology is $\mathbb{C}[x_1, \dots, x_n]/(r_1, \dots, r_m)$. On the other hand,

$$\wedge(y_1, \dots, y_m, \partial y_1, \dots, \partial y_m)$$

has no cohomology. To see this note that we can divide it up into the tensor product

$$\bigotimes_{i=1}^{m} \wedge(y_i, \partial y_i)$$

and $H(\wedge(y_i, \partial y_i), \partial) = \mathbb{C}$. Therefore, we have

$$H^*(U_\partial(K)) = \mathbb{C}[x_1, \dots, x_n]/(r_1, \dots, r_m).$$

Similarly, to the proof of Corollary 6.3 we see that $U_\partial(\varphi) : U_\partial(K) \to U_\partial(H) = H$ is a quasi-isomorphism, as it sends $x_i \in K$ to $x_i \in H$. By symmetry, the same goes for $U_{\bar{\partial}}(\varphi)$, so $\varphi^* : H_\partial(K) \to H_\partial(H)$ and $\varphi^* : H_{\bar{\partial}}(K) \to H_{\bar{\partial}}(H)$ are isomorphisms and we are done. $\square$

Recall that if a cbba A satisfies the $\partial\bar{\partial}$-property then all the cohomologies in the cohomology diamond are caonically isomorphic and we can thus just think of $H(A)$.

Corollary 6.7. Suppose A is a cbba satisfying the $\partial\bar{\partial}$-property. If

$$H(A) = \mathbb{C}[x_1, \ldots, x_n]/(r_1, \ldots, r_m)$$

for x_i and r_i as above, then A is strongly formal.

Proof. We create a zig-zag

$$H(A) \xleftarrow{\sim} K(r_1, \ldots, r_m) \xrightarrow{\sim} A,$$

where the left morphism is a quasi-isomorphism by the previous theorem. The second morphism, call it ψ, is constructed by mapping $x_i \in K$ to a representative $a_i \in A$ of $x_i \in H(A)$. If we take $H(A)$ to be $H_{BC}(A)$, then $r_i(x_1, \ldots, x_n) \in K$ is mapped to $r_i(a_1, \ldots, a_n)$, which in turn is 0 in $H_{BC}(A)$. Thus there is some $\xi_i \in A$, such that $i\partial\bar{\partial}\xi_i = r_i$, and we map y_i to such a ξ_i. This makes ψ a well-defined morphism of cbba's and by the same argument as above ψ is a quasi-isomorphism. □

For a geometric application, recall that a homogeneous space G/H is the quotient of Lie group by a closed Lie subgroup.

Corollary 6.8. Compact Homogeneous Kähler manifolds are strongly formal.

Proof. Suppose X is a compact homogeneous Kähler manifold. By [BR62] we may write $X = T \times F$, where T is a complex torus and F is a homogeneous projective rational variety, i.e. a generalized flag manifold. Strong formality can be proven on each factor separately by Proposition 4.21 in [MS24]. Tori are Kähler and $H_d(T) = \wedge(dz_1, d\bar{z}_1, \ldots, dz_n, d\bar{z}_n)$, so by Proposition 5.4 T is strongly formal. By [RWY09], the de Rham cohomology of F is of the form $\mathbb{C}[x_1, \ldots, x_n]/(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n$ is a regular sequence. Furthermore it is generated by Schubert classes, fundamental classes of subvarieties, which by Proposition 11.20 in [Voi02] have diagonal bidegree. □

The classic example of such spaces are complex Grassmannians. Let us remark that in [PSZ25] this theorem is a bit stronger, Placini, Stelzig and Zoller prove that compact homogeneous Kähler manifolds are strongly formal over $\mathbb{Q}$. We have decided to not include the "over $\mathbb{Q}$" part for simplicity. Strong formality does not imply strong formality over $\mathbb{Q}$, as is proved in the same article, so there is a point in remarking that the theorem can be strengthened.

7 Massey Products

Let (A, d) be a cdga with cohomology $H^*(A)$. For $a \in A$, define $\bar{a} = (-1)^{|a|+1}a$, and note that

$$\overline{ab} = -\bar{a}\bar{b}, \quad \overline{da} = -d\bar{a}, \quad d(ab) = (da)b - \bar{a}db, \quad \bar{\bar{a}} = a. \tag{9}$$

Consider classes $x, y, z \in H^*(A)$, such that $xy = yz = 0$. Let $a, b, c \in A$ represent x, y, z respectively. Since xy is zero in cohomology, $\bar{a}b = ds$ for some element $s \in A$. Similarly, $\bar{b}c = dt$ for some $t \in A$. Define the triple Massey product, $\langle x, y, z\rangle \in H^*(A)$, as the set of cohomology classes of $\bar{s}c + \bar{a}t$ for a, b, c, s, t defined as above. In other words,

$$\langle [a], [b], [c]\rangle = \left\{[\bar{a}t + \bar{s}c] \mid \bar{a}b = ds, \bar{b}c = dt\right\}.$$

We say that the triple Massey product of x, y, z is zero if $\langle x, y, z\rangle$ contains 0. In general this is a set, but it becomes a unique element in the quotient $H^*(A)/(xH^*(A) + H^*(A)z)$. To see this, suppose we pick some other element, s', such that $\bar{a}b = ds'$. Then, $d(s - s') = 0$ and thus, $s - s'$ is a cocycle. We calculate that

$$[\bar{a}t + \bar{s}c] - [\bar{a}t + \bar{s'}c] = [(-1)^{|s|+1}(s - s')c] = [(-1)^{|s|+1}(s - s')]z \in H^*(A)z.$$

Hence, their difference is zero in the quotient. The same argument works for t.

Note that we define the triple Massey product $\langle x, y, z\rangle$ when the ordinary products $xy = yz = 0$. If the triple Massey product is zero, i.e. contains zero, we define another product one step higher. The quadruple Massey product $\langle x, y, z, w\rangle$ is defined when the Triple Massey products $\langle x, y, z\rangle$ and $\langle y, z, w\rangle$ are zero (and defined). The quintuple Massey product when the quadruple are zero (and defined), and so on. In other words, the n-fold Massey product is defined when all the lower products are defined and are zero. Let us now define all these higher Massey products.

For a cdga (A, d) with cohomology $H^*(A)$, a defining system for the n-fold Massey product $\langle x_1, \ldots, x_n\rangle$, where $x_1, \ldots, x_n \in H^*(A)$ and $n \geq 2$, is a family of cochains, $a_{i,j} \in A$, for $0 \leq i < j \leq n$ with $(i, j) \neq (0, n)$, such that $a_{i-1,i}$ is a representative of x_i for all $0 \leq i \leq n$, and

$$da_{i,j} = \sum_{i<k<j} \bar{a}_{i,k}a_{k,j}.$$

The associated cocycle of such a defining system is

$$\sum_{0<k<n} \bar{a}_{0,k}a_{k,n}.$$

It is indeed a cocycle, since

$$\begin{aligned}
d\left(\sum_{0<k<n} \bar{a}_{0,k}a_{k,n}\right) &= \sum_{0<k<n} (d(\bar{a}_{0,k})a_{k,n} - a_{0,k}d(a_{k,n})) \\
&= -\sum_{0<k<n}\left(\overline{\sum_{0<j<k} \bar{a}_{0,j}a_{j,k}}a_{k,n} - a_{0,k}\sum_{k<j<n}\bar{a}_{k,j}a_{j,n}\right) \\
&= \sum_{0<k<n}\sum_{0<j<k} a_{0,j}\bar{a}_{j,k}a_{k,n} - \sum_{0<k<n}\sum_{k<j<n} a_{0,k}\bar{a}_{k,j}a_{j,n},
\end{aligned}$$

which is zero by noting that the first sum is just the sum of all pairs (j,k) with $0<j<k<n$, while the second sum is the sum over all pairs (k,j) with $0<k<j<n$.

Proposition 7.1 ([Kra66]). A defining system for the n-fold Massey product $\langle x_1,\ldots,x_n\rangle$ exists if and only if a defining system for $\langle x_i,\ldots,x_j\rangle$ exists for all $1\le i<j\le n$ and $(i,j)\neq(1,n)$ and the set of associated cocycles contain 0.

When a defining system exists, we define the n-fold Massey product to be the set of cohomology classes of the associated cocycles of the possible defining systems. We say that the product is zero, or that it is trivial, if it contains zero.

If $n=2$, a defining system for $\langle x_1,x_2\rangle$ consists of just two cocycles, $a_{0,1}$ and $a_{1,2}$, such that, $[a_{0,1}]=x_1$ and $[a_{1,2}]=x_2$. The product is thus the singleton set

$$\langle x_1,x_2\rangle = \{[\bar{a}_{0,1}a_{1,2}] \mid [a_{0,1}]=x_1,[a_{1,2}]=x_2\} = \left\{(-1)^{|x_1|+1}x_1x_2\right\}.$$

If $n=3$, a defining system consists of cocycles, $a_{0,1}$, $a_{1,2}$ and $a_{1,3}$, representing x_1,x_2 and x_3, as well as $a_{0,2},a_{1,3}$, such that

$$da_{0,2}=\bar{a}_{0,1}a_{1,2} \quad\text{and}\quad da_{1,3}=\bar{a}_{1,2}a_{2,3}.$$

Translating to what we wrote above about the triple Massey products, we have $a_{0,1}=a, a_{1,2}=b$, $a_{2,3}=c$, $a_{0,2}=s$ and $a_{1,3}=t$.

In general one can view $a_{i,j}$ as a witness that $\langle x_i,\ldots,x_j\rangle$ contains zero. It is of course natural to code this into a $(n+1)\times(n+1)$ matrix, indexed from 0, by setting the entries of the matrix to be $a_{i,j}$ for $1\le i<j\le n$, $(i,j)\neq(0,n)$, and the rest to zero.

Lemma 7.2. *The subset of cohomology classes of associated cocycles of defining systems for $\langle[a_1],\ldots,[a_n]\rangle$ that satisfy $a_{i-1,i}=a_i$, for all $1\le i\le n$, is equal to the whole set. In other words, every element in the Massey product $\langle[a_1],\ldots,[a_n]\rangle$ can be realized from a defining system $(a_{i,j})$ with $a_{i-1,i}=a_i$.*

Proof. Denote the set of cohomology classes of associated cocycles of defining systems for $\langle [a_1], \ldots, [a_n] \rangle$ such that $a_{i-1,i} = a_i$ for all $1 \leq i \leq n$ by $(a_1, \ldots, a_n)$, i.e.,

$$(a_1, \ldots, a_n) = \left\{ \left[\sum_{0<k<n} \bar{a}_{0,k} a_{k,n} \right] \in H(A) \;\middle|\; \begin{array}{c} (a_{i,j}) \text{ defining system for} \\ \langle [a_1], \ldots, [a_n] \rangle \text{ with } a_{i-1,i} = a_i \end{array} \right\}.$$

Note that, $\langle x_1, \ldots, x_n \rangle = \bigcup (a_1, \ldots, a_n)$, where the union is taken over all tuples $a_1, \ldots, a_n$, such that $[a_i] = x_i$. In particular, for two such tuples, their difference, for each i, is a coboundary. We want to show that, for every t, we have $(a_1, \ldots, a_t, \ldots, a_n) = (a_1, \ldots, a_t + db, \ldots, a_n)$, as this will prove the statement, by applying it to each component where they differ. By symmetry, it is enough to show $\subset$, by setting $a_t = a_t - db$. Let $x \in (a_1, \ldots, a_n)$, with defining system $(a_{i,j})$. We wish to find a defining system, $(a'_{i,j})$ for $(a_1, \ldots, a_t + db, \ldots, a_n)$, such that the cohomology class of its associated cocycle is equal to x. Let

$$\begin{aligned} a'_{i,j} &= a_{i,j} && \text{for } i \neq t-1 \text{ and } j \neq t, \\ a'_{t-1,t} &= a_{t-1,t} + db, && \\ a'_{i,t} &= a_{i,t} - a_{i,t-1} b && \text{for } i < t-1, \\ a'_{t-1,j} &= a_{t-1,j} - \bar{b} a_{t,j} && \text{for } j > t, \end{aligned}$$

and check that it is a defining system for the product

$$\langle [a_1], \ldots, [a_t + db], \ldots, [a_n] \rangle = \langle [a_1], \ldots, [a_t], \ldots, [a_n] \rangle.$$

Clearly, it holds that $[a'_{i-1,i}] = [a_{i-1,i}] = [a_i]$, for $i \neq t$. Furthermore, $[a'_{t-1,t}] = [a_{t-1,t} + db] = [a_t]$, by assumption. We verify that

$$da'_{i,j} = \sum_{i<r<j} \bar{a}'_{i,r} a'_{r,j}.$$

If $i < j < t$ or $t-1 < i < j$, then it is immediate. Suppose that $i < t-1 < t < j$, then

$$\begin{aligned} &\sum_{i<r<j} \bar{a}'_{i,r} a'_{r,j} \\ &= \sum_{i<r<t-1} \bar{a}_{i,r} a_{r,j} + \bar{a}_{i,t-1}(a_{t-1,j} - \bar{b} a_{t,j}) + \overline{(a_{i,t} - a_{i,t-1} b)} a_{t,j} + \sum_{t<r<j} \bar{a}_{i,r} a_{r,j} \\ &= \sum_{i<r<t-1} \bar{a}_{i,r} a_{r,j} + \bar{a}_{i,t-1} a_{t-1,j} - \bar{a}_{i,t-1} \bar{b} a_{t,j} + \bar{a}_{i,t} a_{t,j} + \bar{a}_{i,t-1} \bar{b} a_{t,j} + \sum_{t<r<j} \bar{a}_{i,r} a_{r,j} \\ &= \sum_{i<r<j} \bar{a}_{i,r} a_{r,j}, \end{aligned}$$

which is equal to $da_{i,j} = da'_{i,j}$ by assumption. Now let $i = t-1$, if $j = t$, then $da'_{t-1,t} = 0$ directly. Otherwise, if $t < j$, we have

$$\begin{aligned}
\sum_{t-1<r<j} \bar{a}'_{t-1,r} a'_{r,j} &= \overline{(a_{t-1,t} + db)} a_{t,j} + \sum_{t<r<j} \overline{(a_{t-1,r} - \bar{b} a_{t,r})} a_{r,j} \\
&= \bar{a}_{t-1,t} a_{t,j} - d\bar{b} a_{t,j} + \sum_{t<r<j} \bar{a}_{t-1,r} a_{r,j} + b \sum_{t<r<j} \bar{a}_{t,r} a_{r,j} \\
&= \sum_{t-1<r<j} \bar{a}_{t-1,r} a_{r,j} - d\bar{b} a_{t,j} + b d a_{t,j} \\
&= da_{t-1,j} - d(\bar{b} a_{t,j}),
\end{aligned}$$

which is equal to the left hand side, $da'_{t-1,j} = d(a_{t-1,j} - \bar{b} a_{t,j})$. Lastly, let $i < t-1$ and $j = t$, then

$$\begin{aligned}
\sum_{i<r<t} \bar{a}'_{i,r} a'_{r,t} &= \sum_{i<r<t-1} \bar{a}_{i,r} (a_{r,t} - a_{r,t-1} b) + \bar{a}_{i,t-1} (a_{t-1,t} + db) \\
&= \sum_{i<r<t} \bar{a}_{i,r} a_{r,t} - \sum_{i<r<t-1} \bar{a}_{i,r} a_{r,t-1} b + \bar{a}_{i,t-1} db \\
&= da_{i,t} - da_{i,t-1} b + \bar{a}_{i,t-1} db \\
&= da_{i,t} - d(a_{i,t-1} b),
\end{aligned}$$

which is equal to $d(a'_{i,t}) = d(a_{i,t} - a_{i,t-1} b)$. Thus, (a'_{ij}) is such a defining system. If $1 < t < n$, we calculate that

$$\begin{aligned}
&\sum_{0<k<n} \bar{a}'_{0,k} a'_{k,n} \\
&= \sum_{0<k<t-1} \bar{a}_{0,k} a_{k,n} + \bar{a}_{0,t-1} (a_{t-1,n} - \bar{b} a_{t,n}) + \overline{(a_{0,t} - a_{0,t-1} b)} a_{t,n} + \sum_{t<k<n} \bar{a}_{0,k} a_{k,n} \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - \bar{a}_{0,t-1} \bar{b} a_{t,n} + \bar{a}_{0,t-1} \bar{b} a_{t,n} \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n}.
\end{aligned}$$

Thus, the associated cocycles are exactly equal. If $t = 1$, then

$$\begin{aligned}
\sum_{0<k<n} \bar{a}'_{0,k} a'_{k,n} &= \overline{(a_{0,1} + db)} a_{1,n} + \sum_{1<k<n} \overline{(a_{0,k} - \bar{b} a_{1,k})} a_{k,n} \\
&= \bar{a}_{0,1} a_{1,n} - d\bar{b} a_{1,n} + \sum_{1<k<n} \bar{a}_{0,k} a_{k,n} + b \sum_{1<k<n} \bar{a}_{1,k} a_{k,n} \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - d\bar{b} a_{1,n} + b d a_{1,n} \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - d(\bar{b} a_{1,n}).
\end{aligned}$$

Hence, their associated cocyles differ by a coboundary, so equal in cohomology. Lastly, for $t = n$, we get

$$\begin{aligned}
\sum_{0<k<n} \bar{a}'_{0,k} a'_{k,n} &= \sum_{0<k<n-1} \bar{a}_{0,k}(a_{k,n} - a_{k,n-1} b) + \bar{a}_{0,n-1}(a_{n-1,n} + db) \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - \sum_{0<k<n-1} \bar{a}_{0,k} a_{k,n} b + \bar{a}_{0,n-1} db \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - d a_{0,n-1} b + \bar{a}_{0,n-1} db \\
&= \sum_{0<k<n} \bar{a}_{0,k} a_{k,n} - d(a_{0,n-1} b),
\end{aligned}$$

so again, cohomologous. □

Theorem 7.3. *Let $\varphi : A \to B$ be a morphism of cdga's, then*

- $\varphi^* \langle x_1, \ldots, x_n \rangle \subset \langle \varphi^*(x_1), \ldots, \varphi^*(x_n) \rangle$, *and*
- *if φ is a quasi-isomorphism,* $\varphi^* \langle x_1, \ldots, x_n \rangle = \langle \varphi^*(x_1), \ldots, \varphi^*(x_n) \rangle$.

Proof. An element of $\varphi^* \langle x_1, \ldots, x_n \rangle$ is

$$\varphi^* \left[\sum_{0<k<n} \bar{a}_{0,k} a_{k,n} \right] = \left[\sum_{0<k<n} \overline{\varphi(a_{0,k})} \varphi(a_{k,n}) \right],$$

for some defining system $a_{i,j}$ of $\langle x_1, \ldots, x_n \rangle$. Therefore, we prove that $\varphi(a_{i,j})$ is a defining system of $\langle \varphi^*(x_1), \ldots, \varphi^*(x_n) \rangle$, as then, $\sum_{0<k<n} \overline{\varphi(a_{0,k})} \varphi(a_{k,n})$ is an associated cocycle of a defining system for $\langle \varphi^*(x_1), \ldots, \varphi^*(x_n) \rangle$.

Let $a_{i,j}$ be a defining system for $\langle x_1, \ldots, x_n\rangle$, then the system $\varphi(a_{i,j})$ satisfies $[\varphi(a_{i-1,i})] = \varphi^*[a_{i-1,i}] = \varphi^*(x_i)$. Moreover,

$$d\varphi(a_{i,j}) = \varphi(da_{i,j}) = \varphi\left(\sum_{i<k<j} \bar{a}_{i,k}a_{k,j}\right) = \sum_{i<k<j} \overline{\varphi(a_{i,k})}\varphi(a_{k,j}),$$

and thus, $\varphi(a_{i,j})$ is a defining system for $\langle \varphi^*(x_1), \ldots, \varphi^*(x_n)\rangle$.

For the second point, we only give a proof when $n = 3$. The general case can either be proven similarly but with notational nightmares or with the theory of A_∞-algebras [Kel01].

We need to prove that $\varphi^*\langle x_1, x_2, x_3\rangle \supset \langle \varphi^*x_1, \varphi^*x_2, \varphi^*x_3\rangle$. Let a_1, a_2, a_3 be cocycles such that $[a_1] = x_1, [a_2] = x_2$ and $[a_3] = x_3$. Then, $\varphi^*(x_i) = [\varphi(a_i)]$, so by Lemma 7.2, we can assume that a defining system for $\langle \varphi^*x_1, \varphi^*x_2, \varphi^*x_3\rangle$ is of the form $\{\varphi(a_1), \varphi(a_2), \varphi(a_3), x, y\}$, where $dx = \overline{\varphi(a_1)}\varphi(a_2) = \varphi(\bar{a}_1a_2)$, and $y = \overline{\varphi(a_2)}\varphi(a_3) = \varphi(\bar{a}_2a_3)$. In particular, we have $\varphi^*[\bar{a}_1a_2] = [dx] = 0$ and $\varphi^*[\bar{a}_2a_3] = [dy] = 0$, so, as φ^* is an isomorphism, we get $[\bar{a}_1a_2] = [\bar{a}_2a_3] = 0$, and we find z and w, such that $dz = \bar{a}_1a_2$ and $dw = \bar{a}_2a_3$. Note that

$$d(\varphi(z) - x) = \varphi(\bar{a}_1a_2) - \varphi(\bar{a}_1a_2) = 0 \quad \text{and} \quad d(\varphi(w) - y) = 0,$$

so, as φ^* is surjective, we find z' and w', such that

$$\varphi^*[z'] = [\varphi(z')] = [x - \varphi(z)] \quad \text{and} \quad \varphi^*[w'] = [\varphi(w')] = [y - \varphi(\omega)]. \tag{10}$$

Define $z'' = z' + z$ and $w'' = w' + w$. Then, $\{a_1, a_2, a_3, z'', w''\}$ is a defining system for $\langle x_1, x_2, x_3\rangle$, since $dz'' = dz' + dz = \bar{a}_1a_2$ and $dw'' = dw' + dw = \bar{a}_2a_3$. The associated cocycle of this defining system is $\bar{a}_1w'' + \bar{z}''a_3$, so we are just left with showing that $\varphi^*[\bar{a}_1w'' + \bar{z}''a_3] = [\overline{\varphi(a_1)}y + \bar{x}\varphi(a_3)]$. To see this, we calculate

$$\begin{aligned}\varphi^*[\bar{a}_1w'' + \bar{z}''a_3] &= [\overline{\varphi(a_1)}\varphi(w'') + \overline{\varphi(z'')}\varphi(a_3)] \\ &= [\overline{\varphi(a_1)}(\varphi(w') + \varphi(w)) + \overline{(\varphi(z') + \varphi(z))}\varphi(a_3)] \\ &= [\overline{\varphi(a_1)}y + \bar{x}\varphi(a_3)],\end{aligned}$$

by (10). □

Corollary 7.4. If A is formal every Massey product $\langle x_1, \ldots, x_n\rangle$ with $n \geq 3$ is trivial.

Proof. If $d = 0$, then to be zero in cohomology, means to be precisely zero. So we can pick 0 as every element in the defining system, except for $a_{i-1,i}$. So, as $n \geq 3$, the associated cocycle is exactly zero:

$$\sum_{0<k<n} \bar{a}_{0,k}a_{k,n} = 0.$$

In particular, $(H(A), d = 0)$ has trivial Massey products, and thus, by the weak equivalence, so must A have. $\square$

Thus, Massey products are obstructions to being formal, since, if we can find a non-trivial one, it follows that the cdga cannot be formal.

Example 7.5. Using Massey products we get a new way to show that the Kodaira-Thurston surface KT, discussed in example 4.9, cannot be formal. We show that the cdga $A = (\wedge \mathfrak{n}^*, d) = (\wedge(x, y, w, u), dw = xy)$ has a non-trivial ordinary Massey products. Consider $\langle x, y, y \rangle$, then $dw = xy$ and $yy = 0$, so $[wy] \in \langle x, y, y \rangle$. As the triple Massey product is unique up to $xH^*(A) + H^*(A)y$, we see that, as $[wy] \notin xH^*(A) + H^*(A)y$, we have a non-trivial triple Massey product. It thus follows that KT is not formal, since $A \simeq A_{dR}(KT)$. Now that we have proved that compact Kähler manifolds are formal, and KT is compact and symplectic, we conclude that KT has no Kähler structure.

There are generalizations of the triple Massey product to cbba's, due to Daniele Angella and Adriano Tomassini in [AT15].

Definition 7.6. Let A be a cbba. Let $a, b, c \in H_{BC}(A)$ be Bott-Chern classes, such that $ab = bc = 0$ in $H_{BC}(A)$. Furthermore, let a, b and c be represented by α, β and γ respectively, and let $\partial\bar{\partial}x = \alpha\beta$, $\partial\bar{\partial}y = \beta\gamma$ be witnesses of $ab = bc = 0$. Then the triple Aeppli-Bott-Chern-Massey (ABC-Massey) product is

$$\langle a, b, c \rangle = [x\gamma - \alpha y] \in H_A(A)/(aH_A(A) + H_A(A)c).$$

Let us show that this is well defined. First, we need to see that it actually defines a Aeppli class. As γ and α represents Bott-Chern classes, we have $\partial\gamma = \bar{\partial}\gamma = 0 = \bar{\partial}\alpha = \partial\alpha$. Thus,

$$\partial\bar{\partial}(x\gamma - \alpha y) = (\partial\bar{\partial}x)\gamma - (-1)^{2|\alpha|}\alpha\partial\bar{\partial}y = \alpha\beta\gamma - \alpha\beta\gamma = 0$$

by the Leibniz rule.

Secondly, let us show that it is independent of choice of x. Suppose that $\alpha\beta = \partial\bar{\partial}x = \partial\bar{\partial}x'$. Then $\partial\bar{\partial}(x - x') = 0$, so $[x - x'] \in H_A(A)$. Therefore, $[x\gamma - x'\gamma] = [x - x']c$, and hence zero in the quotient $H_A(A)/(aH_A(A) + H_A(A)c)$. A similar argument works for y.

Lastly, let us show that it is independent of choice of representatives α, β, γ. As any choice differs by an element of $\mathrm{im}(\partial\bar{\partial})$, by linearity, it is enough to show that if $a = 0$, $b = 0$ or $c = 0$ in $H_{BC}(A)$, then the Massey product is zero.

Without loss of generality, assume the first, and write $\alpha = \partial\bar{\partial}\xi$. Then, $x = \xi\beta$ satisfies $\partial\bar{\partial}(\xi\beta) = \alpha\beta$. It follows that

$$\begin{aligned}\xi\beta\gamma - \alpha y &= \xi\beta\gamma - \partial\bar{\partial}\xi\gamma \\ &= \xi\partial\bar{\partial}y - \partial\bar{\partial}\xi y \\ &= -\partial(\bar{\partial}\xi y) - (-1)^{|\xi|}\bar{\partial}(\xi\partial y) \in \mathrm{im}(\partial) + \mathrm{im}(\bar{\partial}),\end{aligned}$$

so the Massey product is zero.

Viewing the triple ABC-Massey product as subset, i.e. before passing to the quotient, we see that independence of representative of a, b, c still holds, but we can get multiple elements by the choice of x and y. Similarly to Theorem 7.3, we have the following:

Lemma 7.7. *Let $\varphi : A \to B$ be a morphism of cbba's*

- $\varphi^*\langle a, b, c\rangle \subset \langle \varphi^*(a), \varphi^*(b), \varphi^*(c)\rangle$, *and*
- *if φ is a pluripotential quasi-isomorphism, then* $\varphi^*\langle a, b, c\rangle = \langle \varphi^*(a), \varphi^*(b), \varphi^*(c)\rangle$.

Proof. We have that $[\varphi(\alpha)] = \varphi^*(a)$, $[\varphi(\beta)] = \varphi^*(b)$ and $[\varphi(\gamma)] = \varphi^*(c)$, so these are representatives. Furthermore, $\partial\bar{\partial}\varphi(x) = \varphi(\partial\bar{\partial}x) = \varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$, and similarly, $\partial\bar{\partial}\varphi(y) = \varphi(\beta)\varphi(\gamma)$. Thus,

$$\varphi^*[x\gamma - \alpha y] = [\varphi(x)\varphi(\gamma) - \varphi(\alpha)\varphi(y)] \in \langle \varphi^*(a), \varphi^*(b), \varphi^*(c)\rangle,$$

proving the first part.

The second part is proved in essentially the same way as for ordinary Massey products, by swapping d to $\partial\bar{\partial}$ and the independence of α, β, γ discussed above. □

In particular, when passing to the quotient, if φ is a pluripotential quasi-isomorphism, then the element $\langle a, b, c\rangle \in H_A(A)/(aH_A(A) + H_A(A)c)$ is zero if and only if the ABC-Massey product $\langle \varphi^*(a), \varphi^*(b), \varphi^*(c)\rangle$ is zero.

Corollary 7.8. If A is weakly formal, then all triple ABC-Massey products vanish.

Proof. If H is a cbba with $\partial\bar{\partial} \equiv 0$, then $H_{BC}(H) = \ker\partial \cap \ker\bar{\partial}$, as $\mathrm{im}\,\partial\bar{\partial} = 0$. Thus, if $ab = 0$ in $H_{BC}(H)$, then this means that the unique elements representing them multiply to zero as well, and similarly for bc. Thus, take x, y to be zero, and get $\langle a, b, c\rangle = 0$. Since vanishing of triple ABC-Massey products is invariant under pluripotential quasi-isomorphism, we get that A has no triple ABC-Massey products. □

Example 7.9. Recall Example 4.9 and let $n = 2$, then we have

$$A = (\wedge(z_1, z_2, \bar{z}_1, \bar{z}_2, \xi, \bar{\xi}), d(\xi) = d(\bar{\xi}) = z_1\bar{z}_1 + z_2\bar{z}_2).$$

We consider the triple ABC-Massey product $\langle 2z_1\bar{z}_1, z_2\bar{z}_2, z_2\rangle$. Note that,

$$2z_1\bar{z}_1z_2\bar{z}_2 = \partial\bar{\partial}(\xi\bar{\xi}) \quad \text{by (5), and} \quad z_2\bar{z}_2z_2 = 0 = \partial\bar{\partial}(0).$$

So we get the ABC-Massey product

$$[\xi\bar{\xi}z_2] \in H_A(A)/(2z_1\bar{z}_1H_A(A) + H_A(A)z_2)$$

and it is a routine check to see that this is non-zero in this quotient. This shows that for $n = 2$, the nilmanifold N/Γ is not weakly formal.

The natural question now is if there are higher ABC-Massey products. In fact there are, but some surprises arise. The higher ABC-Massey products are due to Milivojević and Stelzig. We give the definition of the (ad hoc) quadruple product and refer the reader to [MS24] for more information about the higher products.

Definition 7.10 (Quadruple ABC-Massey products [MS24])**.** Let a, b, c, d be Bott-Chern classes with representatives α, β, γ and δ. A defining system for the quadruple ABC-Massey product $\langle a, b, c, d\rangle$ is a quintuple $x, y, z, \tilde{\eta}$ and $\tilde{\xi}$, such that

$$x \in A^{|\alpha\beta|-(1,1)}, \quad y \in A^{|\beta\gamma|-(1,1)}, \quad z \in A^{|\gamma\delta|-(1,1)},$$

satisfying

$$\partial\bar{\partial}x = \alpha\beta, \quad \partial\bar{\partial}y = \beta\gamma, \quad \partial\bar{\partial}z = \gamma\delta,$$

and

$$\tilde{\eta} = \eta + \eta' \in A^{|\alpha\beta\gamma|-(2,1)} \oplus A^{|\alpha\beta\gamma|-(1,2)}$$
$$\tilde{\xi} = \xi + \xi' \in A^{|\beta\gamma\delta|-(2,1)} \oplus A^{|\beta\gamma\delta|-(1,2)},$$

satisfying

$$x\gamma - \alpha y = \partial\eta + \bar{\partial}\eta' \quad y\delta - \beta z = \partial\xi + \bar{\partial}\xi'.$$

The quadruple ABC-Massey product is the subset of the -1-degree cohomology in the Schweitzer complex $H^{-1}_{S_{|\alpha\beta\gamma\delta|}}(A)$, which is equal to

$$\frac{\ker\left(\partial \oplus \bar{\partial} : A^{|\alpha\beta\gamma\delta|-(2,1)} \oplus A^{|\alpha\beta\gamma\delta|-(1,2)} \to A^{|\alpha\beta\gamma\delta|-(1,1)}\right)}{\operatorname{im}\left(\partial \oplus d \oplus \bar{\partial} : A^{|\alpha\beta\gamma\delta|-(3,1)} \oplus A^{|\alpha\beta\gamma\delta|-(2,2)} \oplus A^{|\alpha\beta\gamma\delta|-(1,3)} \to A^{|\alpha\beta\gamma\delta|-(2,1)} \oplus A^{|\alpha\beta\gamma\delta|-(1,2)}\right)},$$

given by

$$[(-1)^{|\alpha|}\alpha\tilde{\xi} - (\partial x)z - (-1)^{|x|}x\bar{\partial}z + \tilde{\eta}\delta],$$

for all defining systems $x, y, z, \tilde{\eta}, \tilde{\xi}$.

Let us check that $(-1)^{|\alpha|}\alpha\tilde{\xi} - (\partial x)z - (-1)^{|x|}x\bar{\partial}z + \tilde{\eta}\delta$ actually defines such a cohomology class. By counting bidegrees, we see that the term splits as

$$\begin{aligned} (-1)^{|\alpha|}\alpha\xi - (-1)^{|x|}x\bar{\partial}z + \eta\delta &\in A^{|\alpha\beta\gamma\delta|-(2,1)} \\ (-1)^{|\alpha|}\alpha\xi' - (\partial x)z + \eta'\delta &\in A^{|\alpha\beta\gamma\delta|-(1,2)}, \end{aligned}$$

so we apply ∂ to the first and $\bar{\partial}$ to the second. This yields:

$$\begin{aligned} &(\partial \oplus \bar{\partial})((-1)^{|\alpha|}\alpha\tilde{\xi} - (\partial x)z - (-1)^{|x|}x\bar{\partial}z + \tilde{\eta}\delta) \\ &= \partial\big((-1)^{|\alpha|}\alpha\xi\big) + \bar{\partial}\big((-1)^{|\alpha|}\alpha\xi'\big) - \bar{\partial}((\partial x)z) - \partial\left((-1)^{|x|}x\bar{\partial}z\right) + \partial(\eta\delta) + \bar{\partial}(\eta'\delta) \\ &= \alpha\partial\xi + \alpha\bar{\partial}\xi' - \bar{\partial}\partial x z - (-1)^{|\partial x|}\partial x\bar{\partial}z - (-1)^{|x|}\partial x\bar{\partial}z - x\partial\bar{\partial}z + \partial\eta\delta + \bar{\partial}\eta'\delta \\ &= \alpha(y\delta - \beta z) + \alpha\beta z - x\gamma\delta + (x\gamma - \alpha y)\delta \\ &= 0. \end{aligned}$$

For some context of where this comes from, for a bicomplex A and a bidegree (p,q) the Schweitzer complex $S_{p,q}A$ is the cochain complex given by

$$\begin{array}{ccccc} & & & & A^{p+2,q} \oplus A^{p+1,q+1} \oplus A^{p,q+2} \longrightarrow \cdots \\ & & & & \uparrow{\scriptstyle d\oplus d} \\ & & A^{p,q} & \xrightarrow{\;d\;} & A^{p+1,q} \oplus A^{p,q+1} \\ & & \uparrow{\scriptstyle \partial\bar{\partial}} & & \\ A^{p-2,q-1} \oplus A^{p-1,q-2} & \xrightarrow{\partial\oplus\bar{\partial}} & A^{p-1,q-1} & & \\ \uparrow{\scriptstyle \partial\oplus d\oplus\bar{\partial}} & & & & \\ \cdots \longrightarrow A^{p-3,q-1} \oplus A^{p-2,q-2} \oplus A^{p-1,q-3} & & & & \end{array}$$

indexed so that $S^1_{p,q}A = A^{p,q}$. The nice property of this complex is that both Bott-chern and Aeppli cohomology turns up in a single complex, namely

$$H^0(S_{p,q}A) = \frac{\ker(\partial\bar{\partial} : A^{p-1,q-1} \to A^{p,q})}{\operatorname{im}(\partial \oplus \bar{\partial} : A^{p-2,q-1} \oplus A^{p-1,q-2} \to A^{p-1,q-1})} = H_A^{p-1,q-1}(A)$$

and

$$H^1(S_{p,q}A) = \frac{\ker(d : A^{p,q} \to A^{p+1,q} \oplus A^{p,q+1})}{\operatorname{im}(\partial\bar{\partial} : A^{p-1,q-1} \to A^{p,q})} = H_{BC}^{p,q}(A).$$

While, as mentioned earlier, the (-1)-degree cohomology, $H^{-1}(S_{p,q}A)$, is

$$\frac{\ker\left(\partial \oplus \bar{\partial} : A^{p-2,q-1} \oplus A^{p-1,q-2} \to A^{p-1,q-1}\right)}{\operatorname{im}\left(\partial \oplus d \oplus \bar{\partial} : A^{p-3,q-1} \oplus A^{p-2,q-2} \oplus A^{p-1,q-3} \to A^{p-2,q-1} \oplus A^{p-1,q-2}\right)}.$$

In [MS24] they claim that a similar argument as in Lemma 7.7 shows that quadruple ABC-Massey products are also invariant under pluripotential quasi-isomorphism. If we assume this, then a similar argument to Corollary 7.8 shows that weakly formal cbba's have trivial quadruple ABC-Massey products.

Example 7.11 (Space with non-trivial quadruple ABC-Massey product [MS24])**.** Consider the real Lie algebra $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$, with bracket determined by

$$\begin{array}{llll} [x_1, x_2] = -x_3, & [x_1, x_3] = -x_4 & [y_1, y_2] = x_3, & [y_1, y_3] = x_4 \\ [x_1, y_2] = -y_3, & [x_1, y_3] = -y_4 & [y_1, x_2] = -y_3 & [y_1, x_3] = -y_4 \end{array}$$

and setting the rest to zero. Note that this Lie algebra is nilpotent as

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}_{\mathbb{R}}(x_3, x_4, y_3, y_4) \quad \text{and thus} \quad [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0.$$

As the structure constants are integers, we get a compact nilmanifold N/Γ. Define $J : \mathfrak{g} \to \mathfrak{g}$ by $J(x_i) = y_i$ and $J(y_i) = -x_i$ to get an almost complex structure on N/Γ. It is a straightforward, but tedious, calculation that the Nijenhuis tensor N^J is zero, so N/Γ is complex. The dual $\wedge\mathfrak{g}_{\mathbb{C}}^*$ is generated by the elements $z_i = x_i^* + iy_i^*$ of type (1,0) and $\bar{z}_i = x_i^* - iy_i^*$ of type (0,1). Recall that the differential $d = \partial + \bar{\partial}$ is defined by the relation

$$da(X, Y) = -a([X, Y]).$$

From this relation, it is a direct calculation that

$$\begin{array}{lll} dx_1^* = dx_2^* = 0, & dx_3^* = x_1^*x_2^* - y_1^*y_2^*, & dx_4^* = x_1^*x_3^* - y_1^*y_3^*, \\ dy_1^* = dy_2^* = 0, & dy_3^* = y_1^*x_2^* + x_1^*y_2^*, & dy_4^* = y_1^*x_3^* + x_1^*y_3^*, \end{array}$$

so that

$$\begin{array}{lll} dz_1 = dz_2 = 0, & dz_3 = z_1z_2, & dz_4 = z_1z_3, \\ d\bar{z}_1 = d\bar{z}_2 = 0, & d\bar{z}_3 = \bar{z}_1\bar{z}_2, & d\bar{z}_4 = \bar{z}_1\bar{z}_3. \end{array}$$

Thus, we get the real cbba

$$A = (\wedge(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3, z_4, \bar{z}_4), dz_3 = z_1z_2, dz_4 = z_1z_3, \sigma(z) = \bar{z}),$$

with $|z_i| = (1, 0)$ and $|\bar{z}_i| = (0, 1)$. We see that the complex structure is nilpotent, so N/Γ is a nilmanifold with nilpotent complex structure, and we can conclude by Theorem 4.8 that $A \simeq A(N/\Gamma)$. Let us now show that A has a non-trivial quadruple ABC-Massey product, and thus as a consequence N/Γ, has a non-trivial ABC-Massey product and cannot be weakly formal.

For ease of notation, we skip the indices and let $x = z_1, y = z_2, z = z_3$ and $w = z_4$. Consider Bott-chern classes, $[x]_{BC}, [x\bar{y}]_{BC}$ and $[\bar{x}y]_{BC}$, we shall consider the quadruple product $\langle [x], [x], [x\bar{y}], [\bar{x}y] \rangle$. Note that $x^2 = 0$ as $|x| = 1$, $xx\bar{y} = 0$ and $x\bar{y}\bar{x}y = \partial\bar{\partial}(z\bar{z})$, where the last involves some anti-commuting to verify. Lastly, $-xz\bar{z} = \partial(-w\bar{z})$. So the set $\{0, 0, z\bar{z}, 0, -w\bar{z}\}$ constitutes a defining system for $\langle [x], [x], [x\bar{y}], [\bar{x}y] \rangle$, and $[x\omega\bar{z}]$ is a $H^{-1}_{S_{4,2}}$-cohomology class in the quadruple product $\langle [x], [x], [x\bar{y}], [\bar{x}y] \rangle$.

To show that the product is non-trivial we show that $0 \notin \langle [x], [x], [x\bar{y}], [\bar{x}y] \rangle$. By degree reasons, we can only pick 0 as the first element in the defining system, since it needs to be of degree $(1, -1)$. The second element in the defining system, call it b (called x in Definition 7.10), should be of degree $(1, 0)$ and satisfy $\partial\bar{\partial}b = xx\bar{y} = 0$. So we can pick any of the holomorphic generators, and $b \in \operatorname{span}\{x, y, z, w\}$. The third element, call it c, needs to be of degree $(1, 1)$ and satisfy $\partial\bar{\partial}c = x\bar{y}\bar{x}y$. By calculating $\partial\bar{\partial}$ of every generator of $A^{1,1}$, we get that c lies in the affine subspace

$$c \in z\bar{z} + \operatorname{span}\{x\bar{x}, x\bar{y}, x\bar{z}, x\bar{w}, y\bar{x}, y\bar{y}, y\bar{z}, y\bar{w}, z\bar{x}, z\bar{y}, w\bar{x}, w\bar{y}\}.$$

The fourth element, $\tilde{\eta} = \eta + \eta'$, has $|\eta| = (1, 0)$ and $|\eta'| = (2, -1)$, so $\eta' = 0$, while η must satisfy $\partial\eta = -xb$. As $b \in \operatorname{span}\{x, y, z, w\}$, one calculates that $\eta \in \operatorname{span}\{x, y, z, w\}$, but that it depends on b, explicitly

$$b = a_1x + a_2y + a_3z + a_4w \quad \text{gives} \quad \eta = b_1x + b_2y - a_2z - a_3w.$$

The last element is $\tilde{\xi} = \xi + \xi'$, with $|\xi| = (1, 1)$ and $|\xi'| = (2, 0)$, satisfying

$$b\bar{x}y - xc = \partial\xi + \bar{\partial}\xi'.$$

Since $\bar{\partial} = 0$ on $A^{2,0}$, we get that $\xi' \in \operatorname{span}\{xy, xz, xw, yz, yw, zw\}$, while we have that

$$\partial\xi \in -xz\bar{z} + \operatorname{span}\{xy\bar{x}, zy\bar{x}, wy\bar{x}, xy\bar{y}, xy\bar{z}, xy\bar{w}, xz\bar{x}, xz\bar{y}, xw\bar{x}, xw\bar{y}\}.$$

By calculating ∂ on $A^{1,1}$, we get that

$$\xi \in -w\bar{z} + \operatorname{span}\{x\bar{x}, x\bar{y}, x\bar{z}, x\bar{w}, y\bar{x}, y\bar{y}, y\bar{z}, y\bar{w}, z\bar{x}, z\bar{y}, z\bar{z}, z\bar{w}, w\bar{x}, w\bar{y}\}$$

with some dependence on b and c. From this, we get that the elements of the ABC-Massey products are $H^{-1}_{S_{4,2}}$ classes of elements

$$\begin{aligned} -x\tilde{\xi} + \tilde{\eta}\bar{x}y \in xw\bar{z} &+ \operatorname{span}\{xy\bar{x}, xy\bar{y}, xy\bar{z}, xy\bar{w}, xz\bar{x}, xz\bar{y}, xz\bar{z}, xz\bar{w}, xw\bar{y}, z\bar{x}y, w\bar{x}y\} \\ &+ \operatorname{span}\{xyz, xyw, xzw\}. \end{aligned}$$

To see that this is non-zero in $H^{-1}_{S_{4,2}}$, we note that we would have to have an element τ in $A^{1,1}$ or $A^{0,2}$ with $\partial\tau = xw\bar{z}$, or at least in the affine subspace listed above, but one can check that there are no such elements, and thus this proves that $0 \notin \langle [x], [x], [x\bar{y}], [\bar{x}y] \rangle$.

This example is mostly for computational importance, showcasing how one would compute quadruple ABC-Massey product.

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