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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Towards Higher Algebra: A Journey from Homological Algebra to the Derived ∞-Category of R-Modules

av

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Abstract

During recent times, (higher) category theory seem to have become increasingly relevant to know for anyone interested in more modern foundations of algebraic geometry, homotopy theory, and maybe other areas. One framework of interest might go under the name"Higher Algebra" or "Homotopical Algebra". Motivated by our interest in this framework, we try to cover most of the homological algebra necessary for engagement with Lurie's seminal work "Higher Algebra" ([Lur17]), by following part I of Aaron Mazel-Gee's lecture notes "Higher Algebra: Chapter 0". We start by the impetus for the move towards *derived algebraic* geometry. We then introduce important constructions such as chain complexes, chain-homotopies, quasi-isomorphisms and other concepts. We focus on how one may use homotopy co/kernels to glean information about a chain map $M \xrightarrow{f} N$, and how one may use the de/suspension-operator Σ together with homotopy co/kernels to create exact sequences, for an arbitrary chain map f. In the later part of the thesis, we say more about the dg-category of complexes of R-modules and how it can be viewed as enriched in a certain symmetric monoidal category $\mathsf{Ch}_{\Bbbk}$ of complexes of $\Bbbk$-modules with the tensor product of complexes. The last two chapters are devoted to *resolutions* and a brief introduction to $\Bbbk$-linear ∞-categories. We end by saying something about more general ∞-categories, using the framework of *quasicategories*, and defining the derived ∞-category of R-modules as a certain ∞-categorical localization of the category Ch_R of chain complexes of R-modules.

Sammanfattning

I mer modern tid verkar *högre kategori teori* bli allmter relevant att förstå för de som är intresserade av mer moderna formuleringar av algebraisk geometri, homotopi-teori och kanske fler områden. Ett ramverk för att tänka kring dessa områden kan kallas "Högre Algebra" eller "Homotopisk Algebra". Motiverad av vårt intresse inom detta områden, försökte vi täcka den större delen av den homologiska algebran som påstås krävas för att förstå Lurie's seminala arbete "Higher Algebra" ([Lur17]), genom att följa del I av Aaron Mazel-Gee's föreläsningsanteckningar "Higher Algebra: Chapter 0". Vi inleder med motivationen för att gå över till *deriverad algebraisk geometri*. Vi introducerar sedan konstruktioner såsom kedje-komplex, kedje-homotopier, kvasi-isomorfier och andra koncept. Vi fokuserar vidare på hur man kan använda homotopi ko/kärnor av en kedjekarta $M \xrightarrow{f} N$ för att kunna uttala sig om dess egenskaper, och hur de/suspensions-operatorn Σ kan användas för att skapa exakta sekvenser från en godtycklig kedjeavbildning f. I den senare delen av uppsatsen pratar vi om dg-kategorin av komplex av R-moduler, och hur man kan see den som berikad i en viss symmetrisk monoidal kategori $\mathsf{Ch}_{\Bbbk}$ av komplex med tensorprodukten av komplex av $\Bbbk$-moduler. De sista två kapitlena läggs på att diskutera *resolutioner*, samt att ge en kort introduktion till $\Bbbk$-linjära ∞-kategorier. Vi avslutar med att säga något om mer generella ∞-kategorier, genom konceptet *kvasikategorier*, och definierar sedermera den deriverade ∞-kategorin av R-moduler som en viss ∞-kategorisk lokalisering av kategorin Ch_R av kedjekomplex av R-moduler.

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Prelude

Structure of thesis

Some general comments about the structure of this thesis.

The motivation for this thesis is mainly our interest in (higher) category theory, derived algebraic geometry and "homotopical algebra". Our thesis is primarily based on "Higher Algebra: Chapter 0" by Aaron Mazel-Gee ([Maz23]). The title of the as-yet-to-be-published[1] book is, I believe, an allusion to the book "Abstract Algebra: Chapter 0" by Paolo Aluffi ([Alu09]), but in the context of *higher* algebra. Consequently, our supposition is that the book has the ambition of being a precursor before reading "Higher Algebra" by J. Lurie ([Lur17]). The material in this thesis is in this sense very conservative, in that we don't prove any new results. Our main motivation has been to understand (some) of the tools used in the aforementioned areas of mathematics. Of course, this thesis can't possibly claim to capture even a fraction of this vast area of (still ongoing) mathematical activity. What we do cover in this thesis is most (except not saying much about derived functors) of the homological algebra in [Maz23, Part I]. What is not covered at all in this thesis is the theory of *model categories*, which is the subject of part II of [Maz23] (there is also e.g. [Hov99]).

Below are some comments on the chapters in this thesis.

- Chapter 1 recalls some constructions (and theorems) from (classical) algebraic geometry. We then give some examples that touch upon *intersections* in algebraic geometry. The last given example gives a motivation for where the *scheme-theoretic* intersection fails to encode data about the way in which two varieties can intersect. That is, the ordinary tensor product of coordinate rings, when the tensor factors are viewed as complexes *concentrated in degree zero*, can only see the degree-zero part of the homology of the derived tensor product, but can't see anything with respect to *higher* homology.

- In chapter 2, we start by recalling the definition of the (relative) tensor product of modules, we talk about how modules of a commutative ring R relates to a $\Bbbk$-algebra structure on it. We prove a tensor-hom adjunction and introduce *homology*. We then talk briefly about *localization of categories* both with and without a so called "calculus of fractions" available. Lastly, we introduce the *tensor product of complexes*, and show among other things, that it defines a symmetric monoidal structure on the category of chain complexes.

- In chapter 3, after briefly introducing the concept of chain complexes, chain ho-

[1] As we understand it, contracted to be published by Cambridge University Press (see https://etale.site/book.html).

motopies and quasi-isomorphisms, the main focus is on *homotopy cokernels* and *homotopy kernels*. We show how one can use properties of the homotopy co/kernel of a chain map to gain information about the chain map itself. We also try to illuminate the analogy between "ordinary" co/kernels and homotopy co/kernels. In the last part of chapter 3, we talk about *exact sequences*. In particular, we show how one may, starting from a chain map, produce a long exact sequence in homology, using homotopy co/kernels and something called de/*suspension*..

- In chapter 4, we introduce hom-complexes, which besides containing all chain maps between two chain complexes also contain "higher homotopies". We then introduce the concept of *enrichment* from enriched category theory, and show that the dg-category of complexes of R-modules $\mathbf{K}_R$ with the given definition of enrichment, is enriched in the symmetric monoidal category $\left(\mathsf{Ch}_{\Bbbk}, \otimes_{\Bbbk}, \underset{\sim}{\Bbbk}\right)$, for R a (central) $\Bbbk$ algebra. Lastly, we prove a tensor-hom adjunction that relates the two dg-categories $\mathbf{K}_R$ and $\mathbf{K}_{\Bbbk}$, and we briefly mention how certain functors induced from these categories behave better with respect to homotopy co/kernels than the corresponding functors in the category of modules behave with respect to ordinary co/kernels.
- In chapter 5, we focus mostly on projective and injective resolutions. We show that the category of modules Mod_R for a commutative ring R has enough injectives. We have a brief interlude about derived functors. We then focus on bounded below projective resolutions and bounded above injective resolutions. Lastly, we talk about cell complexes and lifting criteria. We show, via something called a *small object argument*, how any given chain map $M \xrightarrow{f} N$ gives rize to a factorization $f = f^{\infty} \circ c^{\infty}$ where the chain maps f^{∞} and c^{∞} satisfies some properties. In particular we show that one may use this procedure on the zero morphism $0 \to N$ for any complex N to create an associated projective resolution $N^{(\infty)} \xrightarrow{\approx} N$.
- In chapter 6, we first talk a little about the homotopy theory for dg-categories of complexes, we then briefly introduce $\Bbbk$-linear ∞-categories. In the last chapter, we say something more about general ∞-categories. We use the framework of *quasi-categories* to define ∞-categories. We talk about a general procedure for producing ∞-categories from underlying 1-categories (i.e. "ordinary categories"). Lastly, we then say how one may construct the derived ∞-categories of R-modules as a certain "∞-categorical localization", called a *Dwyer-Kan localization.*

Conventions

We will denote by R a commutative ring unless specified otherwise, and almost everywhere R is a $\Bbbk$-algebra for some commutative ring $\Bbbk$. We aim to ignore any set-theoretic issues as long as they don't cause any obvious problems. We will try to provide full proofs for most of the theorems, and whenever this is for various reasons not possible, we might give a sketch of a proof, or refer to auxiliary sources. We will (try) to follow the convention that $\cong$ is "just" an isomorphism in the appropriate category, *without any claim about this isomorphism being natural*, while we use $\approx$ or sometimes $\simeq$ to denote *natural isomorphism* [2]. Observe that we also use $\simeq$ to denote *homotopy equivalence* at times.

[2]Note however that we also use $\approx$ for quasi-isomorphisms. Hopefully it is clear from the context which of these we mean, since a (naive) natural bijection between two complexes (viewed as complexes) is an ill-defined notion anyways, there is however an added complication in that one could use $\approx$ to mean that two complexes are *canonically* isomorphic, for example whenever we say have two different *models* of the

We will often omit specifying which type of map we mean, when this is (we hope) clear from the context. For example, we might say "map" when we mean *chain* map or R-module homomorphism (cf. how [Hat01] uses "map" for continuous maps).

Observe that our convention for the universal map u with respect to the *cone* model of the homotopy cokernel (definition 3.2.3) is with degree n-component as $b \mapsto (0, -b)$, while perhaps it is more common to define this as $b \mapsto (0, b)$. We realized this quite late in our writing of this thesis, and so have kept this convention. If one wants to use the more conventional definition of the universal map then (as far as we can tell) most proofs go through up to changing signs whenever appropriate.

homotopy cokernel. All in all, one might say that *context matters*!

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Chapter 1

Towards derived algebraic geometry

1.1 Recollection

Below, we will recall some basic definitions from algebraic geometry, and giv an example for why going to the (classical) *derived setting* is motivated. For more background on (classical) algebraic geometry, we refer the reader to the (relatively) short but clear [Gat24], or [Har77] for a more exhaustive (and perhaps advanced) treatment.

Definition 1.1.1 (Algebraic subsets (affine varieties))**.** Fixing some base-field k, and subset $S \subset k[x_1, \dots, x_n]$ of the polynomial ring over k in n variables, we let

$$V(S) := \{a \in \mathbb{A}^n_k \; : \; f(a) = 0, \forall f \in S\},$$

and call subsets of **affine n-space** $\mathbb{A}^n_k$ of this form **algebraic subsets**, and whenever we have a set $Z \subseteq \mathbb{A}^n_k$ on the form $Z = V(S)$ we say that Z is the **vanishing locus** of S. We call sets Z on the form $V(S)$ **affine varieties**.

Remark 1.1.2*.* Observe that some authors will reserve the name affine varieties for subsets $Z = V(S)$ that can *not* be written on the form $Z = Z_1 \cup Z_2$ for closed (in the Zariski-topology) subsets $Z_1, Z_2 \subsetneq \mathbb{A}^n_k$, i.e. so called **irreducible** subsets.

Definition 1.1.3 (Vanishing ideal)**.** For a subset $Z \subset \mathbb{A}^n_k$, we define the **vanishing ideal**

$$I(Z) := \{f \in k[x_1, \dots, x_n] \; : \; f(z) = 0, \forall z \in Z\}.$$

Definition 1.1.4 (Coordinate ring $\mathscr{O}(Z)$ of a (reduced) affine variety Z)**.** With the above constructions in mind, we let

$$\mathscr{O}(Z) := k[x_1, \dots, x_n]/I(Z)$$

for a *reduced* affine variety Z, and call this the **coordinate ring** of Z.

Remark 1.1.5*.* By $\mathscr{O}$ we really $\mathscr{O}_X$ for fixed scheme X whenever we are working from "the modern point of view" (i.e. with schemes), so that we get back $\mathscr{O}_X(X) = \Gamma(X, \mathscr{O}_X)$, the ring of *global sections* with respect to the *structure sheaf* $\mathscr{O}_X$ of the scheme X. However, if $\mathscr{O}$ takes the input of something defined classically, we mean *taking coordinate rings.*

A consequence of *Hilbert's basis theorem* is that any vanishing ideal $I(Z)$ is *finitely generated*, i.e., we can always write $I(Z)$ as $\langle f_1, \dots, f_n \rangle$ for a finite set of polynomials $f_i \in k[x_1, \dots, x_n]$. For a general (not necessarily algebraically closed) field k, the constructions $I(\,\cdot\,)$ and $V(\,\cdot\,)$ determine functions as illustrated below:

$$\{\text{subsets of } \mathbb{A}^n_k\} \underset{V(\,\cdot\,)}{\overset{I(\,\cdot\,)}{\rightleftarrows}} \{\text{ideals of } k[x_1, \dots, x_n]\} \tag{1.1.1}$$

By Hilbert's nullstellensatz (see e.g. [Gat24, Prop. 1.10]), if we promote k to an algebraic closure $\overline{k}$ of k, then we have a bijective correspondence

$$\left\{\text{affine varieties of } \mathbb{A}^n_{\overline{k}}\right\} \underset{V(\,\cdot\,)}{\overset{I(\,\cdot\,)}{\rightleftarrows}} \{\text{radical ideals of } \overline{k}[x_1, \dots, x_n]\}\ . \tag{1.1.2}$$

Definition 1.1.6 (Mutual right adjoints)**.** Let $\mathcal{C}^{\mathsf{op}} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D}^{\mathsf{op}} \xrightarrow{G} C$ be (covariant) functors. If we have natural isomorphisms

$$\mathsf{hom}_{\mathcal{D}}(T, F(S)) \approx \mathsf{hom}_{\mathcal{C}}(S, G(T))$$

for all objects S in $\mathcal{C}$ and all objects T in $\mathcal{D}$, then we say that F and G are **mutual right adjoints**.

Going back to 1.1.1 one checks that $I(\,\cdot\,)$ and $V(\,\cdot\,)$ define covariant *functors*

$$P\left(k[x_1, \dots, x_n]\right)^{\mathsf{op}} \xrightarrow{V(\,\cdot\,)} P\left(\mathbb{A}^n_k\right) \qquad \text{and} \qquad P\left(\mathbb{A}^n_k\right)^{\mathsf{op}} \xrightarrow{I(\,\cdot\,)} P\left(k[x_1, \dots, x_n]\right),$$

between their respective (opposite) *poset*-categories ordered by *inclusion*. Since poset-categories have hom-sets that are either empty or consists of a single element, and since

$$S \subseteq I(T) \Leftrightarrow T \subseteq V(S),$$

it follows that $I(\,\cdot\,)$ and $V(\,\cdot\,)$ participate in natural bijections such as in definition 1.1.6. In the case when the categories involved are preorders (since all posets are preorders, this applies here), we call this an **antitone Galois connection**. If we promote k to an algebraic closure $\overline{k}$ of k, one may use [Rie16, Lemma 4.2.11][1] together with Hillbert's nullstellensatz to see that (observe that it is relatively straightforward to find the *unit* and *counit* with respect to an adjunction between poset categories due to the simplicity of the hom-sets) that we get an equivalence of categories between the poset category of affine varieties in $\mathbb{A}^n_{\overline{k}}$ and the opposite category of the poset-category of radical ideals of $\overline{k}[x_1, \dots, x_n]$. Any equivalence of categories between poset categories defines an isomorphism, so that the equivalence just mentioned gives an isomorphism between the two categories just mentioned (observe that the opposite of a poset-category is a poset category).

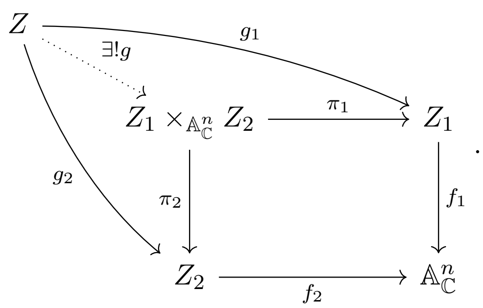
[1]Use that $\overline{V} := V^{\mathsf{op}}(\,\cdot\,) \dashv I(\,\cdot\,)$ defines an "ordinary" adjunction, with $P(\overline{k}[x_1, \dots, x_n]) \xrightarrow{\overline{V}} P\left(\mathbb{A}^n\right)^{\mathsf{op}}$ its opposite functor.

1.2 Intersections

Perhaps a particularly important statement for what we aim to illustrate in this section is the following fact, where we now fix the base-field to $\mathbb{C}$: for closed subschemes (in the sense of [Gat24, Construction 12.32.(b)]) $Z_1 = \mathsf{Spec}(R/I_1), Z_2 = \mathsf{Spec}(R/I_2)$ and with $R := \mathbb{C}[x_1, \dots, x_n]$, we have that

$$Z_1 \cap Z_2 = Z_1 \times_{\mathbb{A}^n_{\mathbb{C}}} Z_2,$$

where $Z_1 \times_{\mathbb{A}^n_{\mathbb{C}}} Z_2$ is the **fiber product** of Z_1 and Z_2 over $\mathbb{A}^n_{\mathbb{C}}$ together with associated morphisms $\pi_i : Z_1 \times_{\mathbb{A}^n_{\mathbb{C}}} Z_2 \to Z_i$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$, where $f_i : Z_i \to \mathbb{A}^n_{\mathbb{C}}$ is induced from the canonical ring-homomorphisms $R \to R/I_i$, which is defined by the following universal property: For any two morphisms $g_1 : Z \to Z_1$ and $g_2 : Z \to Z_2$ from another scheme Z such that $f_1 \circ g_1 = f_2 \circ g_2$, then there is a unique dashed map $g : Z \to Z_1 \times_{\mathbb{A}^n_{\mathbb{C}}} Z_2$ as indicated in the diagram below, so that the diagram commutes,



In the case that Z_1, Z_2 are affine, then we claim that also the fiber product is affine, and is given by $\mathsf{Spec}(R/I_1 \otimes_R R/I_2)$ (see e.g. [Gat24, Def. 12.38.(a)]). Using the isomorphism theorems, we have an isomorphism of rings $R/I_1 \otimes_R R/I_2 \cong R/(I_1 + I_2)$. On applying the functor $\mathsf{Spec}(\,\cdot\,)$, we get an isomorphism $\mathsf{Spec}(R/I_1 \otimes_R R/I_2) \cong \mathsf{Spec}\,(R/(I_1 + I_2))$ of affine schemes. Then we find that

$$\begin{aligned} \mathscr{O}(Z_1 \cap Z_2) &= \mathscr{O}_{\mathsf{Spec}(R/I_1 \otimes_R R/I_2)}(\mathsf{Spec}(R/I_1 \otimes_R R/I_2)) \\ &\cong R/I_1 \otimes_R R/I_2 \\ &\cong R/(I_1, I_2), \end{aligned} \tag{1.2.1}$$

where we have used that $\mathscr{O}_{\mathsf{Spec}(R)}(\mathsf{Spec}(R)) \cong R$ for an affine scheme $\mathsf{Spec}(R)$, with $\mathscr{O}_{\mathsf{Spec}(R)}$ the **structure sheaf** of the scheme $\big(\mathsf{Spec}(R), \mathscr{O}_{\mathsf{Spec}(R)}\big)$.

Example 1.2.1. Consider the affine varieties $Z_1 = V(y - x^2)$ and $Z_2 = V(y - x)$ in the complex plane $\mathbb{C}^2$. Taking the (non-scheme theoretic) intersection

$$Z_1 \cap_{\mathsf{cl}} Z_2 = V(y - x^2, y - x),$$

we find that

$$\begin{aligned} \mathscr{O}(Z_1 \cap_{\mathsf{cl}} Z_2) &= \mathbb{C}[x, y]/I(V(y - x^2, y - x)) \\ &= \mathbb{C}[x, y]/\sqrt{(y - x^2, y - x)} \\ &= \mathbb{C}[x, y]/(y - x^2, y - x), \qquad \text{since } (y - x^2, y - x) \text{ is a radical ideal} \\ &\cong \mathbb{C}[x]/(x - x^2) \end{aligned}$$

where the last isomorphism is defined as $\overline{y} \mapsto \overline{x}$ and $\overline{x} \mapsto \overline{x}$ on generators. Since the ideals (x) and $(1-x)$ are such that $x+(1-x) = 1$, they are *coprime* in $\mathbb{C}[x]$. It follows that their intersection equals the ideal $(x - x^2)$. By the (generalized) chinese remainder theorem, we then have

$$\begin{aligned}\mathbb{C}[x]/(x - x^2) &= \mathbb{C}[x]/(x \cdot (1 - x)) \\ &\cong \mathbb{C}[x]/(x) \times \mathbb{C}[x]/(1 - x) \\ &\cong \mathbb{C} \times \mathbb{C}.\end{aligned}$$

Now we instead take the scheme-theoretic intersection of the corresponding associated *reduced* affine schemes corresponding to Z_1 and Z_2. Then with $I_1 := I(Z_1) = (y - x^2)$ and $I_2 := I(Z_2) = (y - x)$, and $R := \mathbb{C}[x, y]$, we find that

$$\begin{aligned}\mathscr{O}(Z_1 \cap Z_2) &\cong R/I_1 \otimes_R R/I_2 \\ &\cong R/(I_1, I_2) \\ &= \mathbb{C}[x, y]/(y - x^2, y - x) \\ &\cong \mathbb{C} \times \mathbb{C}.\end{aligned}$$

We see that in this example, the classical and scheme-theoretic pictures agree. In this case, one notes that the reason is that $(y - x^2, y - x)$ is *radical.*

We provide another motivating example, where the classical and scheme-theoretic disagree. The next example is (scheme-theoretically) called a **fat point** or a **double point** over $\mathbb{C}$.

Example 1.2.2. Fix the base-field $\mathbb{C}$ and consider the ideals $I_1 := (y - x^2)$ and $I_2 := (y)$ in the polynomial ring $\mathbb{C}[x, y]$. Taking the zero locus of these ideals, we get affine varieties $V(I_1) = \{(x, y) \in \mathbb{A}^2_{\mathbb{C}} \ : \ y = x^2\}$ and $V(I_2) = \{(x, y) \in \mathbb{A}^2_{\mathbb{C}} \ : \ y = 0\}$, which we may denote Z_1 and Z_2, respectively. Taking the scheme-theoretic intersection, we find that, with $R := \mathbb{C}[x, y]$,

$$\begin{aligned}\mathscr{O}(Z_1 \cap Z_2) &\cong R/(y - x^2) \otimes_R R/(y) \\ &\cong \mathbb{C}[x, y]/(y - x^2, y) \\ &\cong \mathbb{C}[x]/(x^2)\end{aligned}$$

where we in the last isomorphism used the 1^{st} isomorphism theorem applied to the (surjective) homomorphism $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[x]/(x^2)$ defined by $f(x, y) \mapsto \overline{f(x, 0)}$ with kernel $(y - x^2, y)$. This is a $\mathbb{C}$-algebra with basis $\{1, \overline{x}\}$ of dimension 2 as a $\mathbb{C}$ vector space.

However, taking the classical intersection, we see that

$$\begin{aligned}Z_1 \cap_{\mathsf{cl}} Z_2 &= V(I_1) \cap V(I_2) \\ &= V(I_1 + I_2),\end{aligned}$$

so that

$$\begin{aligned}\mathscr{O}(Z_1 \cap_{\mathsf{cl}} Z_2) &= \mathbb{C}[x, y]/I(V(I_1 + I_2)) \\ &= \mathbb{C}[x, y]/\sqrt{I_1 + I_2} \\ &= \mathbb{C}[x, y]/\sqrt{(x^2, y)}, \qquad \text{since } I_1 + I_2 = (y - x^2, y) = (x^2, y) \\ &= \mathbb{C}[x, y]/(x, y), \qquad \text{since } \sqrt{(x^2, y)} = (x, y) \\ &\cong \mathbb{C}, \qquad \text{by evaluation at the origin } (0, 0).\end{aligned}$$

We see that the fact that the classical intersection $Z_1 \cap_{\mathsf{cl}} Z_2$ does in this case *not* agree with the scheme-theoretic intersection is that $(y - x^2, y)$ is *not a radical ideal.*

However, as the example below tries to illustrate, there are cases when even the scheme-theoretic picture fails to capture essential data about how varieties intersect one another.

Example 1.2.3. Let $a, b \in \mathbb{C}$ be any pair of points. Then we may view them as subsets of the affine complex line $\{a\}, \{b\} \subset \mathbb{A}^1_{\mathbb{C}}$ are algebraic subsets, in that $X_a := V(x-a) = \{a\}$ and $X_b := V(x-b) = \{b\}$. [2] By taking the vanishing ideal of these subsets we get the ideals $I_a := I(\{a\}) = (x-a)$ and $I_b := I(\{b\}) = (x-b)$ in $\mathbb{C}[x]$. The associated ring of global regular functions of X_a, X_b are $\mathscr{O}(X_a) = \mathbb{C}[x]/(x-a)$ and $\mathscr{O}(X_b) = \mathbb{C}[x]/(x-b)$. These are both isomorphic (as rings, i.e. fields since $(x-a)$ and $(x-b)$ are *maximal*) to $\mathbb{C}$ via evaluation at a (respectively, b). Then, taking the scheme-theoretic intersection, we find that

$$\begin{aligned}
\mathscr{O}(X_a \cap X_b) &\cong \mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-b) \\
&\cong \mathbb{C}[x]/(x-a, x-b) \\
&\cong \mathbb{C}/(a-b), \qquad \text{realized through e.g. } \mathbb{C}[x] \xrightarrow{\mathsf{ev}_a} \mathbb{C} \overset{\pi}{\twoheadrightarrow} \mathbb{C}/(a-b) \\
&= \begin{cases} \mathbb{C}, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}.
\end{aligned}$$

Now, the point of the above example is that morally, one might want the intersection of two points $\{a\} \cap \{b\}$ to always give back the intersection-number 0, since perturbing one of the points a or b by however a small non-zero ε-term will give us back an intersection number of zero. One might say that this intersection is *unstable* (it is *not invariant* under however small perturbations). To "correct" this failure to detect the instability of the intersection, we may compute the *derived tensor* product $M \otimes^{\mathbb{L}}_{\mathbb{C}[x]} M$ where $M = \mathbb{C}[x]/(x-a)$ (for some more details, see § 5.2.1 or [Maz23, §7.5]). Here we view M really as a complex $\underset{\sim}{M}$ concentrated in degree zero (see 2.3.4) living inside the derived ∞-category $\mathbf{D}_{\mathbb{C}[x]}$ of $\mathbb{C}[x]$-modules (c.f. § 6.3). We consider the complex

$$P := \left(\cdots \to 0 \to \mathbb{C}[x] \xrightarrow{\cdot(x-a)} \underset{\sim}{\mathbb{C}[x]} \to 0 \to \cdots \right).$$

We claim that this complex is *projective* in the sense of definition 5.2.2. This follows by a direct application of theorem 5.3.9 since $\mathbb{C}[x]$ is free as a $\mathbb{C}[x]$-module and the complex is bounded below (observe that it is also free as a $\mathbb{C}$-module with basis $\{1, x, x^2, \dots\}$).

Now, we construct a quasi-isomorphism (see definition 2.3.4) $P \xrightarrow{\approx} M$: We let $P \xrightarrow{q} M$ be the chain map defined by the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{C}[x] & \xrightarrow{\cdot(x-a)} & \mathbb{C}[x] & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M = \mathbb{C}[x]/(x-a) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}.$$

[2] Observe however that as *closed subschemes* (c.f. [Gat24, Construction 12.28]) we have that $X_a = \mathsf{Spec}\,(\mathbb{C}[x]/(x-a))$, and similarly for X_b, with $\mathbb{A}^1_{\mathbb{C}} = \mathsf{Spec}(\mathbb{C}[x])$.

It is immediate that this defines a chain map. Furthermore, we have (see §2.3 for the definition of $\mathsf{H}_n(\,\cdot\,)$),

$$\mathsf{H}_n(P) = \begin{cases} M, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathsf{H}_n(M) = \begin{cases} M, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases} \quad .$$

Then we find that by construction (see the proof of theorem 2.3.3) $\mathsf{H}_n(q) = 0$ is the zero map in all degrees except 0 and in degree 0 we have $\mathsf{H}_0(q) = \mathsf{id}_M$. Hence q is a quasi-isomorphism, so that indeed $P \xrightarrow[q]{\approx} M$ is a projective resolution.

Since P and M are quasi-isomorphic, we first replace $- \otimes_{\mathbb{C}[x]} M$ with the correction $- \otimes_{\mathbb{C}[x]} P$, so that we may compute the ordinary relative tensor product of complexes (the subject of § 2.5) $M \otimes_{\mathbb{C}[x]} P$ instead of $M \otimes_{\mathbb{C}[x]} M$ which lives in $\mathbf{K}_{\mathbb{C}}$ (see § 4). Let $Q \xrightarrow[q]{\approx} M$ be a quasi-isomorphism from another projective complex Q. Then we note that by theorem 5.2.17 we may form the zig-zag of quasi-isomorphisms

$$M \otimes_{\mathbb{C}[x]} P \xleftarrow[q \otimes \mathsf{id}]{\approx} Q \otimes_{\mathbb{C}[x]} P \xrightarrow[\mathsf{id} \otimes p]{\approx} Q \otimes_{\mathbb{C}[x]} M \xrightarrow[\sigma_{Q,M}]{\approx} M \otimes_{\mathbb{C}[x]} Q,$$

where $\sigma_{Q,M}$ is the natural isomorphism coming from the symmetric monoidal structure on $\mathsf{Ch}_{\mathbb{C}[x]}$ (by theorem 2.5.4). Hence upon passing to the derived ∞-category $\mathbf{D}_{\mathbb{C}[x]}$ of $\mathbb{C}[x]$-modules (see § 6.3), these complexes become equivalent (in an ∞-categorical sense which we do not specify here). So we might say that at the level of *homology* these are the same complexes, and so (roughly) can't be distinguished based on their **Euler characteristic**, which we define as

$$\chi(M) := \sum_{n \in \mathbb{Z}} (-1)^n \mathsf{dim}_{\mathbb{C}}(\mathsf{H}_n(M)), \qquad \text{see [Maz23, §7.5.1]},$$

for a derived $\mathbf{D}_{\mathbb{C}}$-module M for which this is well-defined (for example, this is clearly defined for bounded below and bounded above complexes with finite-dimensional homology).[3] Derived $\mathbb{C}$-modules (or derived $\Bbbk$-modules for say any field $\Bbbk$) for which its Euler characteristic is a finite sum of integers, we call **perfect**.

By definition 2.5.1 we find that

$$\left(M \otimes_{\mathbb{C}[x]} P\right)_\ell = \begin{cases} \mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x], & \text{if } \ell = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

with differential given by 2.5.1. Since d_j^M is zero for all j and $d_1^P = \cdot(x-a)$ with $d_k^P = 0$ for $k \neq 1$, we see that the only differential which *could* be non-zero is in degree one, however, we compute that

$$\begin{aligned} \overline{f} \otimes g &\xmapsto{d_1} \overline{f} \otimes (x-a) \cdot g \\ &= \overline{f} \cdot (x-a) \otimes g, \qquad \text{see definition 2.1.(3)} \\ &= 0. \end{aligned}$$

[3] We may here assume that $\mathbf{D}_{\mathbb{C}[x]}$ is *enriched* in $\mathbf{D}_{\mathbb{C}}$ (see [Maz23, § 6.3.2]).

Hence we get the complex

$$M \otimes_{\mathbb{C}[x]} P = \left(\cdots \to 0 \to \mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \xrightarrow{0} \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]} \to 0 \to \cdots \right).$$

We then see that by the diagram

$$\begin{array}{ccccccccc}
\cdots \to & 0 & \longrightarrow & \mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x] & \xrightarrow{0} & \underset{\sim\sim\sim}{\mathbb{C}[x]/(x-a) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]} & \longrightarrow & 0 & \to \cdots \\
& \downarrow & & \downarrow \approx & & \downarrow \approx & & \downarrow & \\
\cdots \to & 0 & \longrightarrow & \mathbb{C}[x]/(x-a) & \xrightarrow{0} & \underset{\sim\sim\sim}{\mathbb{C}[x]/(x-a)} & \longrightarrow & 0 & \to \cdots \\
& \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow & \\
\cdots \to & 0 & \longrightarrow & \mathbb{C} & \xrightarrow{0} & \underset{\sim}{\mathbb{C}} & \longrightarrow & 0 & \to \cdots
\end{array}$$

with $\approx$ the natural isomorphism $\overline{f} \otimes g \mapsto \overline{f} \cdot \overline{g}$ by "contraction", and $\cong$ realized by $\overline{f} \overset{\mathsf{ev}_a}{\longmapsto} f(a)$ by evaluation at a (for both maps: in degrees as indicated above), $M \otimes_{\mathbb{C}[x]} P$ is isomorphic to the complex

$$\left(\cdots \to 0 \to \mathbb{C} \xrightarrow{0} \underset{\sim}{\mathbb{C}} \to 0 \to \cdots \right).$$

We may then compute the Euler-characteristic of this complex as a representative of the derived tensor product, and so we find that

$$\begin{aligned}
\chi \left(\mathbb{C}[x]/(x-a) \otimes^{\mathbb{L}}_{\mathbb{C}[x]} \mathbb{C}[x]/(x-a) \right) &= \mathsf{dim}_{\mathbb{C}}(\mathbb{C}) - \mathsf{dim}_{\mathbb{C}}(\mathbb{C}) \\
&= 0.
\end{aligned}$$

Hence one might say that the **derived intersection**, using the derived tensor product, captures the instability of the intersection.

For more on intersection-theory, our understanding is that [Ful98] is a good source.

Remark 1.2.4. Although we have not said much about this, the actual computations above could equally well have taken place in the *derived* category of $\mathbb{C}$-modules, i.e. we did not really need to use any data captured by the ∞-category $\mathbf{D}_{\mathbb{C}}$ beyond the data encoded the "ordinary" derived category of $\mathbb{C}$-modules, which we may denote as $\mathsf{H}_0(\mathbf{D}_{\mathbb{C}})$. That, is, $M \otimes^{\mathbb{L}}_{\mathbb{C}[x]} M$ naturally inhabits $\mathsf{H}_0(\mathbf{D}_{\mathbb{C}})$ as well.

We may exchange each instance of $\mathbb{C}[x]$ with $\mathbb{C}$ in what we wrote in the previous paragraph since at the level of dg-complexes (see § 4) any complex in the dg-category of complexes of $\mathbb{C}[x]$-modules naturally also lives in the dg-category of complexes of $\mathbb{C}$-modules. We must be careful however to note that the actual computation of the Euler-characteristic took place with respect to $\mathbf{D}_{\mathbb{C}}$.

We will not say much more about derived algebraic geometry in this thesis. Instead we refer the reader to consult e.g. [Toë14] and [Kha23].

Chapter 2

Tensor products, complexes and homology

2.1 (Relative) tensor product

We will start by recalling the (relative) tensor product $A \otimes_R B$ of R-modules A and B. Observe that there are different ways to phrase this definition; one may for example either define it through the universal property it satisfies, or (roughly) construct it explicitly as a quotient of the *free* R-mdoule on the set

$$\bigoplus_{(a,b)\in A\times B} R$$

with basis $\{e_{a,b}\}_{(a,b)\in A\times B}$ and mod out" the relations defining the tensor product. Note however that defining it in terms of a universal property does not necessarily guarantees its existence. Here we give the definition in terms of the universal property it is supposed to satisfy. Whenever $R = \mathbb{Z}$, one may call $A \otimes_R B$ the **absolute** tensor product; it is then also customary to just write $A \otimes B$.

Definition 2.1.1 (Relative tensor product)**.** Given R-modules A and B, then we define the **relative tensor product** of A and B over R as the pair

$$(A \otimes_R B, \iota : A \times B \to A \otimes_R B)$$

where $A \otimes_R B$ is an abelian group (in the case that R is commutative, it also has a natural R-module structure), with ι satisfying

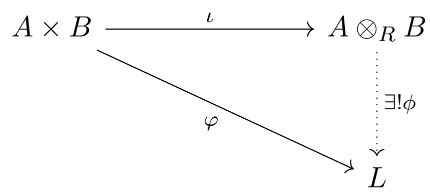
(1) $\iota(a + a', b) = \iota(a, b) + \iota(a', b)$ for all $a, a' \in A$ and all $b \in B$.

(2) $\iota(a, b + b') = \iota(a, b) + \iota(a, b')$ for all $a \in A$ and all $b, b' \in B$.

(3) $\iota(ar, b) = \iota(a, rb)$ for all $a \in A, b \in B$ and all $r \in R$.

Furthermore, $(A \otimes_R B, \iota)$ should satisfy the following *universal property*: For any R-bilinear map $\varphi : A \times B \to L$ with L another R-module, there is a *unique* R-module homomorphism

$\phi : A \otimes_R B \to L$ such that $\phi \circ \iota = \varphi$. We may represent this with the following diagram:



Example 2.1.2. We claim that

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\mathsf{gcd}(m,n)\mathbb{Z}$$

for integers $m, n \geqslant 1$ as $\mathbb{Z}$-modules, i.e. as abelian groups. To see this, start with the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z} \to 0$$

where $\cdot m$ is multiplication by m and the last non-zero map is the canonical projection down to the quotient. If we tensor with $(\,\cdot\,) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ we get the right-exact sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m \otimes \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to 0,$$

which, by the canonical isomorphism

$$R \otimes_R A \cong A, \qquad a \otimes r \mapsto ar,$$

is the same as the right-exact sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to 0.$$

By right exactness, it follows that

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} &\cong (\mathbb{Z}/n\mathbb{Z})\,/m\,(\mathbb{Z}/n\mathbb{Z}) \\ &\cong (\mathbb{Z}/n\mathbb{Z})\,/\,((\mathsf{gcd}(m,n)\mathbb{Z})/n\mathbb{Z}) \\ &\cong \mathbb{Z}/\mathsf{gcd}(m,n), \end{aligned}$$

where we used the 3^{rd} isomorphism theorem for groups in the last isomorphism.

2.2 Modules, Currying and Adjunctions

Let $\Bbbk$ be a commutative ring and let R be a $\Bbbk$-algebra and for simplicity assume that R is also commutative. At the level of non-derived homological algebra, we relate the following statements:

(a) If we let $\mathsf{Mod}_{\Bbbk}$ denote the category of $\Bbbk$-modules then $(\mathsf{Mod}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ is a **tensor category**, in the sense of [Mil25, Chap I, Def. 2.1].

(b) Given right R-modules M and N, there is a natural (right) $\Bbbk$-action defined on $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$ by $(f, k) \mapsto f \cdot k$ where

$$(f \cdot k)(m) = f(km) \tag{2.2.1}$$

where the action on of $\Bbbk$ on M from the right is induced by using *restriction of scalars* along the (central) ring homomorphism $\Bbbk \xrightarrow{i} R$. By *centrality* of $\Bbbk$, this naturally endows M with *left* $\Bbbk$-action compatible with the right-action, giving M a $(\Bbbk, \Bbbk)$-bimodule structure. It is this left-action we use in 2.2.1, i.e. $km = mi(k)$.

(c) It is sometimes customary to denote that a module M carries an $(\Bbbk, R)$-bimodule structure by ${}_{\Bbbk}M_R$. We usually suppress this and just write M as long as the bimodule-structure on M has been specified or is clear from context, to unclutter notation.

(d) For any right R-modules A, B, we have that $\mathsf{hom}_{\mathsf{Mod}_R}(A, B)$ has a $\mathbb{Z}$-module structure coming from the fact that M, N are abelian groups so have a natural $\mathbb{Z}$-module structure. This in turn means that $\mathsf{hom}_{\mathsf{Mod}_R}(A, B)$ is an abelian group with $(f, g) \mapsto f + g$ as group-operation.

(e) We have a natural right action of R on $\mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}(T, N)$ for right R-modules T, N as $(\psi \cdot r)(t) := \psi(t) \cdot r$.

The next theorem has an intuition that comes from "**currying**" with sets, in the sense that if $f : A \times B \to C$ is a function, we may define (suppressing Set in hom)

$$\Phi : \mathsf{hom}(A \times B, C) \to \mathsf{hom}(A, \mathsf{hom}(B, C))$$

so that $\Phi(f)(a) = f(a, -) : B \to C$, with inverse

$$\Psi : \mathsf{hom}(A, \mathsf{hom}(B, C)) \to \mathsf{hom}(A \times B, C)$$

defined by $\Psi(f)(a, b) = g(a)(b)$. Here $\Psi(f)$ is really the composition $\mathrm{ev} \circ (f \times \mathrm{id}_B)$ where $(f \times \mathrm{id}_B)(a, b) = (f(a), b)$ and $\mathrm{ev}(f(a), b) = f(a)(b)$. We check that they are inverses. For $f : A \times B \to C$ and $(a, b) \in A \times B$, we have

$$\begin{aligned}
(\Psi \circ \Phi(f))(a, b) &= \mathrm{ev} \circ (\Phi(f) \times \mathrm{id}_B)(a, b) \\
&= \mathrm{ev}(\Phi(f)(a), b) \\
&= \Phi(f)(a)(b) \\
&= f(a, b),
\end{aligned}$$

and for $h : A \to \mathsf{hom}(B, C)$ and $a \in A$,

$$\begin{aligned}
(\Phi \circ \Psi(h))(a) &= (\Phi(\mathrm{ev} \circ (h \times \mathrm{id}_B)))(a) \\
&= \mathrm{ev}(h(a) \times \mathrm{id}_B) \\
&= h(a).
\end{aligned}$$

Although the above paragraph may seem like a tangent, in the proof below we use similar reasoning in showing that the constructed map τ is an isomorphism at components T, M, N of right R-modules.

Below, we will assume that R is a $\Bbbk$-algebra, with both R and $\Bbbk$ commutative rings. Going forward, we may write $\mathsf{Mod}_R(M, N)$ instead of $\mathsf{hom}_{\mathsf{Mod}_R}(M, N)$.

Theorem 2.2.1 (c.f. e.g. [Rot09, Theorem 2.75, Theorem 2.76])**.** *Let $T \in \mathsf{Mod}_{\Bbbk}$ and let $M, N \in \mathsf{Mod}_R$. Then we have the following isomorphisms of abelian groups*

$$\begin{aligned}
\mathsf{Mod}_R(T \otimes_{\Bbbk} M, N) &\approx \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M, N)) \\
&\approx \mathsf{Mod}_R(M, \mathsf{Mod}_{\Bbbk}(T, N)),
\end{aligned}$$

natural in all three variables.

Remark 2.2.2. Observe that we view $T \otimes_{\Bbbk} M$ as a *right* R-module by the action $(t \otimes m) \cdot r = t \otimes (m \cdot r)$.

Remark 2.2.3. The $\Bbbk$-module structure on $\mathsf{Mod}_R(M,N)$ is the one given in 2.2.1.

Proof. $\underline{\mathsf{Mod}_R(T\otimes_{\Bbbk} M,N)\approx \mathsf{Mod}_{\Bbbk}(T,\mathsf{Mod}_R(M,N))}$:

Let $\tau_{T,M,N}:\mathsf{Mod}_R(T\otimes_{\Bbbk} M,N)\to \mathsf{Mod}_{\Bbbk}(T,\mathsf{Mod}_R(M,N))$ be defined by

$$t \xmapsto{\tau_{T,M,N}(f)} (m\mapsto f(t\otimes m)).$$

Then

$$\begin{aligned}\tau_{T,M,N}(f+g)(t) &= m\mapsto (f+g)(t\otimes m)\\ &= m\mapsto f(t\otimes m)+g(t\otimes m)\\ &= \tau_{T,M,N}(f)(t)+\tau_{T,M,N}(g)(t),\end{aligned}$$

for $t\in T$ and $f,g\in \mathsf{Mod}_R(T\otimes_{\Bbbk} M,N)$, so that $\tau_{T,M,N}$ is $\mathbb{Z}$-linear.

Assume that $\tau_{T,M,N}(f)=0$, i.e. that $\tau_{T,M,N}(f)(t)=0$ for all $t\in T$. This means that $f(t\otimes m)=0$ for all $t\in T$ and $m\in M$. Since the *pure* tensors $t\otimes m$ generates $T\otimes_{\Bbbk} M$ in the sense that $T\otimes_{\Bbbk} M=\langle t\otimes m \;:\; t\in T, m\in M\rangle_R$ where $\langle\;\;\rangle_R$ denotes right R-generation, and since f is right R-linear it follows that $f=0$. Hence $\tau_{T,M,N}$ is *injective.*

To see that $\tau_{T,M,N}$ is *surjective*: let $F:T\to \mathsf{Mod}_R(M,N)$ be a $\Bbbk$-linear map. We define a map $\varphi: T\times M\to N$ by $\varphi(t,m)=F(t)(m)$. Consider the following diagram,

$$\begin{array}{ccc} T\times M & \xrightarrow{\quad\iota\quad} & T\otimes_{\Bbbk} M \\ & \overset{\circlearrowleft}{\underset{\varphi}{\searrow}} & \Big\downarrow {\scriptstyle \exists!\Phi} \\ & & N \end{array},$$

with $\iota(t,m)=t\otimes m$. We note that

$$\begin{aligned}\varphi(tk,m) &= F(tk)(m)\\ &= (F(t)\cdot k)(m)\\ &= F(t)(km)\\ &= \varphi(t,km),\end{aligned}$$

so that φ is $\Bbbk$-balanced. Linearity in both arguments follows from the linearity of F and the linearity of $F(t)$. By the universal property of $T\otimes_{\Bbbk} M$, there is a unique $\Bbbk$-module homomorphism $\Phi: T\otimes_{\Bbbk} M\to N$ such that $\Phi(t\otimes m)=\varphi(t,m)$. We have that

$$\begin{aligned}\Phi((t\otimes mr)) &= \varphi(t,mr)\\ &= F(t)(mr)\\ &= F(t)(m)r\\ &= \varphi(t,m)r\\ &= \Phi(t\otimes m)r,\end{aligned}$$

so that $\Phi\in \mathsf{Mod}_R(T\otimes_{\Bbbk} M,N)$, hence $\tau_{T,M,N}$ is indeed surjective. Therefore, $\tau_{T,M,N}$ is an isomorphism.

It remains to show that τ is a *natural* transformation $\tau : F \Rightarrow G$ between the functors $F, G : \mathsf{Mod}_{\Bbbk} \times \mathsf{Mod}_R \times \mathsf{Mod}_R \rightrightarrows \mathsf{Ab}$, *contravariant* in the first two arguments. Here F takes a triplet of morphisms

$$\begin{cases} \alpha : T' \to T \text{ in } \mathsf{Mod}_{\Bbbk} \\ \beta : M' \to M \text{ in } \mathsf{Mod}_R \\ \gamma : N \to N' \text{ in } \mathsf{Mod}_R. \end{cases} \tag{2.2.2}$$

to $F(\alpha, \beta, \gamma)$ defined on morphisms $f \in \mathsf{Mod}_R(T \otimes_{\Bbbk} M, N)$ by

$$F(\alpha, \beta, \gamma)(f) := \gamma \circ f \circ (\alpha \otimes \beta) : T' \otimes_{\Bbbk} M' \to N'.$$

For G, we have that a triplet of morphisms as above instead gives that if $h : T \to \mathsf{Mod}_R(M, N)$ is a $\Bbbk$-linear map, then since β and γ are R-linear they are also $\Bbbk$-linear by *restriction of scalars*. Using this fact together with how the $\Bbbk$-module structure on $\mathsf{Mod}_R(M', N')$ is defined, a routine (but somewhat tedious) calculation then gives that $G(\alpha, \beta, \gamma)(h)$ is $\Bbbk$-linear, where

$$G(\alpha, \beta, \gamma) : \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M, N)) \to \mathsf{Mod}_{\Bbbk}(T', \mathsf{Mod}_R(M', N'))$$

is defined by

$$G(\alpha, \beta, \gamma)(h)(t') = \gamma \circ h(\alpha(t')) \circ \beta$$

for each $t' \in T'$. This is a composition of R-linear maps, hence is R-linear.

For naturality, we want to show that the following square commutes:

$$\begin{array}{ccc} \mathsf{Mod}_R(T \otimes_{\Bbbk} M, N) & \xrightarrow{\tau_{T,M,N}} & \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M, N)) \\ \downarrow{\scriptstyle F(\alpha,\beta,\gamma)} & & \downarrow{\scriptstyle G(\alpha,\beta,\gamma)} \\ \mathsf{Mod}_R(T' \otimes_{\Bbbk} M', N') & \xrightarrow[\tau_{T',M',N'}]{} & \mathsf{Mod}_{\Bbbk}(T', \mathsf{Mod}_R(M', N')) \end{array}.$$

For any $f \in \mathsf{Mod}_R(T \otimes_{\Bbbk} M, N), t' \in T'$ and any $m' \in M'$, we find that

$$\begin{aligned} \Big((G(\alpha, \beta, \gamma)(\tau_{T,M,N}(f)))(t')\Big)(m') &= (\gamma \circ \tau_{T,M,N}(f)(\alpha(t')) \circ \beta)(m') \\ &= \gamma(f(\alpha(t') \otimes \beta(m'))), \end{aligned}$$

while

$$\begin{aligned} \Big(((\tau_{T',M',N'} \circ F(\alpha, \beta, \gamma))(f))(t')\Big)(m') &= \Big((\tau_{T',M',N'}(\gamma \circ f \circ (\alpha \otimes \beta)))(t')\Big)(m') \\ &= \Big((x \mapsto \gamma \circ f(\alpha(t') \otimes \beta(x)))\Big)(m') \\ &= \gamma(f(\alpha(t') \otimes \beta(m'))). \end{aligned}$$

By comparison of the two computations, we find that they agree, and so we draw the conclusion that naturality holds.

$\underline{\mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N)) \approx \mathsf{Mod}_R(M, \mathsf{Mod}_{\Bbbk}(T,N))}$: For $h \in \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N))$, let

$$\Big(\sigma_{T,M,N}(h)(m)\Big)(t) := h(t)(m), \tag{2.2.3}$$

for any $t \in T$ and $m \in M$. If $h_1 = h_2$ in $\mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N))$, then

$$\begin{aligned} \sigma_{T,M,N}(h_1)(m)(t) &= h_1(t)(m) \\ &= h_2(t)(m) \\ &= \sigma_{T,M,N}(h_2)(m)(t) \end{aligned}$$

for all t and m. Furthermore, routine (but again, quite tedious) calculations shows that $\sigma_{T,M,N}(h)(m)$ is in $\mathsf{Mod}_{\Bbbk}(T,N)$ and that $m \mapsto \sigma_{T,M,N}(h)(m)$ is R-linear for fixed $h \in \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N))$, using that $h(t) \in \mathsf{Mod}_R(M,N)$ is R-linear.

Assume that $\sigma(h)$ is such that $\sigma(h)(m)(t) = 0$ for all m and t. This means that $h(t) \in \mathsf{Mod}_R(M,N)$ must be the zero morphism for each $t \in T$, which in turn means that $h = 0$. Hence σ is *injective*.

Let $f \in \mathsf{Mod}_R(M, \mathsf{Mod}_{\Bbbk}(T,N))$ be arbitrary. We want to construct h such that $\sigma(h) = f$. Let γ be defined as the composition (where we are abusing notation somewhat, in that our f in $f \times \mathrm{id}_T$ is really the image $U(f)$ of the forgetful-functor $U : \mathsf{Mod}_R \to \mathsf{Set}$).

$$T \times M \xrightarrow{\simeq} M \times T \xrightarrow{f \times \mathrm{id}_T} \mathsf{Set}(T,N) \times T \xrightarrow{\mathrm{ev}} N.$$

On elements $(t,m) \in T \times M$, we have that

$$\begin{aligned} \gamma(t,m) &= \mathrm{ev}(f(m), t) \\ &= f(m)(t). \end{aligned}$$

By *currying*, there is a unique map $h \in \mathsf{Set}(T, \mathsf{Set}(M,N))$ such that

$$\begin{aligned} \Psi(h)(t,m) &= \gamma(t,m) \\ &= f(m)(t). \end{aligned}$$

γ is by construction additive, $\Bbbk$-balanced and R-linear in M. By R-linearity in M it follows that $h(t) \in \mathsf{Mod}_R(M,N)$ for each $t \in T$, so that $h : T \to \mathsf{Mod}_R(M,N)$. Since γ is $\Bbbk$-balanced, it follows that h is $\Bbbk$-linear. Lastly, we find that for all $m \in M$ and $t \in T$,

$$\begin{aligned} (\sigma_{T,M,N}(h))(m)(t) &= h(t)(m) \\ &= \gamma(t,m) \\ &= f(m)(t), \end{aligned}$$

so that $\sigma_{T,M,N}$ is surjective, hence an isomorphism.

We want to show that $\sigma_{T,M,N}$ defines a *natural* isomorphism. We let $A(-,-,-)$ be defined on a $\Bbbk$-module T and (right) R-modules M, N as $A(T,M,N) = \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N))$ and triple of morphisms $\alpha : T' \to T, \beta : M' \to M$ and $\ell : N \to N'$ as in 2.2.2, where $A(\alpha,\beta,\gamma)(f)(t') = \gamma \circ f(\alpha(t')) \circ \beta$ for $f \in A(T,M,N)$ and $t' \in T'$. We let $B(T,M,N) = \mathsf{Mod}_R(M, \mathsf{Mod}_{\Bbbk}(T,N))$ on objects and we let $B(\alpha,\beta,\gamma)(f)(m') = \gamma \circ f(\beta(m')) \circ \alpha$ for $f \in B(T,M,N)$ and $m' \in M'$. It is straightforward to check that A and B are both functors $\mathsf{Mod}_R \times \mathsf{Mod}_R \times \mathsf{Mod}_R \to \mathsf{Ab}$ that are contravariant in the first two variables.

We check that the following diagram commutes,

$$\begin{array}{ccc} A(T,M,N) & \xrightarrow{\sigma_{T,M,N}} & B(T,M,N) \\ \Big\downarrow{\scriptstyle A(\alpha,\beta,\gamma)} & & \Big\downarrow{\scriptstyle B(\alpha,\beta,\gamma)} \\ A(T',M',N') & \xrightarrow[\sigma_{T',M',N'}]{} & B(T',M',N'), \end{array}$$

for $f \in A(T,M,N) = \mathsf{Mod}_{\Bbbk}(T, \mathsf{Mod}_R(M,N))$. We then find that

$$\begin{aligned} \big(B(\alpha,\beta,\gamma)(\sigma_{T,M,N})(f))(m')\big)(t') &= (\gamma \circ \sigma_{T,M,N}(f)(\beta(m'))(\alpha(t')) \\ &= \gamma(f(\alpha(t'))(\beta(m'))), \end{aligned}$$

while

$$\begin{aligned} (\sigma_{T',M',N'}(A(\alpha,\beta,\gamma)(f))(m')(t') &= A(\alpha,\beta,\gamma)(f)(t')(m') \\ &= \gamma \circ f(\alpha(t')) \circ \beta)(m') \\ &= \gamma(f(\alpha(t'))(\beta(m')). \end{aligned}$$

Comparison gives that the above computations agree. The conclusion follows. □

2.3 Homology

We recall a crucial definition of importance for algebraic topology, in the special case of a *chain complex* of R-modules.

Definition 2.3.1 (Chain complex of R-modules)**.** Let $(M_\bullet, d_\bullet)$, or even more succinctly, $M_\bullet$, denote the data

$$M_\bullet := ((M_n)_{n\in\mathbb{Z}}, (d_n : M_n \to M_{n-1})_{n\in\mathbb{Z}})$$

where M_n are R-modules and the d_n are R-module homomorphisms, such that

$$d_n \circ d_{n+1} = 0$$

for all n, or more compactly $d^2 = 0$. Then we call $M_\bullet$ a **chain complex** (of R-modules). We may represent this as

$$M_\bullet : \ldots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \ldots, \qquad (d^2 = 0).$$

We call the integer n in M_n or d_n the **degree** or **dimension**, and one may write $\deg(m) = n$ for $m \in M_n$. We call the R-module homomorphisms d_n the **differentials**.

Definition 2.3.2 (Morphism of chain complexes)**.** Let $M_\bullet = ((M_n)_{n\in\mathbb{Z}}, (d_n^M)_{n\in\mathbb{Z}})$ and $N_\bullet = ((N_n)_{n\in\mathbb{Z}}, (d_n^N)_{n\in\mathbb{Z}})$ be two chain complexes of R-modules. Then we say that the data $f_\bullet = (f_n : M_n \to N_n)_{n\in\mathbb{Z}}$ of R-module homomorphisms f_n such that the following square commutes for each n,

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n \\ \Big\downarrow{\scriptstyle f_{n+1}} & \circlearrowleft & \Big\downarrow{\scriptstyle f_n} \\ N_{n+1} & \xrightarrow[d_{n+1}^N]{} & N_n \end{array}$$

is a **chain map**, or without a diagram, as

$$d_{n+1}^N \circ f_{n+1} = f_n \circ d_{n+1}^M. \tag{2.3.1}$$

We may denote this as $f_\bullet : M_\bullet \to N_\bullet$.

We let Ch_R denote the *category* where objects are chain complexes (of R-modules) $M_\bullet$ and morphisms are chain maps $f_\bullet : M_\bullet \to N_\bullet$. The *identity* morphism is the map $\mathrm{id}_{M_\bullet} : M_\bullet \to M_\bullet$ defined by

$$(\mathrm{id}_{M_\bullet})_n := \mathrm{id}_{M_n} \tag{2.3.2}$$

in each degree, and we define the *composition* of chain maps $f_\bullet : M_\bullet \to N_\bullet$ and $g : N_\bullet \to L_\bullet$ as

$$(g_\bullet \circ f_\bullet)_n := g_n \circ f_n. \tag{2.3.3}$$

Checking that this is again a chain map is straightforward and amounts to checking that the big rectangle below commutes, which follows from the fact that the smaller squares commute.

$$\begin{CD}
M_{n+1} @>{d_{n+1}^M}>> M_n \\
@V{f_{n+1}}VV \circlearrowleft @VV{f_n}V \\
N_{n+1} @>>{d_{n+1}^N}> N_n \\
@V{g_{n+1}}VV \circlearrowleft @VV{g_n}V \\
L_{n+1} @>>{d_{n+1}^L}> L_n
\end{CD}$$

Associativity follows from the fact that it holds degree-wise on the R-module homomorphism level.

Given any R-module $M \in \mathsf{Mod}_R$, this naturally gives a chain complex of R-modules

$$\ldots \longrightarrow 0 \longrightarrow \underset{\sim}{M} \longrightarrow 0 \longrightarrow \ldots, \tag{2.3.4}$$

which we say is **concentrated in degree zero**. By abusing notation, we may label the complex in 2.3.4 as $\underset{\sim}{M}$, i.e. not distinguishing the complex from the degree zero module. One checks that any R-module homomorphism $f : M \to N$ determines a chain map $f_\bullet : M \to N$ by the data $(f_n)_{n\in\mathbb{Z}}$ such that

$$f_k = \begin{cases} f, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}. \tag{2.3.5}$$

This is a chain map (as below) due to the fact that f is a homomorphism,

$$\begin{CD}
\ldots @>>> 0 @>>> M @>>> 0 @>>> \ldots \\
@. @V{0}VV @V{f}VV @V{0}VV \\
\ldots @>>> 0 @>>> N @>>> 0 @>>> \ldots
\end{CD}.$$

We may therefore define a fully faithful embedding $\mathsf{Mod}_R \overset{\iota}{\hookrightarrow} \mathsf{Ch}_R$ taking objects M to the complex 2.3.4 and R-module homomorphisms $f : M \to N$ to the family of differentials defined in 2.3.5.

In the context of **homology**, with respect to some chain complex $(M_\bullet, d_\bullet)$, we will use the notation $\mathsf{Z}_n(M_\bullet)$ to denote the *kernel* $\mathsf{ker}(d_n) \subset M_n$ of the n^{th} differential, $\mathsf{B}_n(M_\bullet)$ to denote the *image* $\mathsf{im}(d_{n+1}) \subset M_n$ of the $n+1$-differential and the n^{th} homology $\mathsf{H}_n(M_\bullet)$ as

$$\begin{aligned} \mathsf{H}_n(M_\bullet) &:= \mathsf{Z}_n(M_\bullet)/\mathsf{B}_n(M_\bullet) \\ &= \mathsf{ker}(d_n)/\,\mathsf{im}(d_{n+1}). \end{aligned}$$

Theorem 2.3.3. *The constructions* $\mathsf{Z}_n, \mathsf{B}_n, \mathsf{H}_n$ *define functors*

$$\mathsf{Z}_n, \mathsf{B}_n, \mathsf{H}_n : \mathsf{Ch}_R \rightrightarrows \mathsf{Mod}_R.$$

Proof. It is clear from construction how $\mathsf{Z}_n, \mathsf{B}_n, \mathsf{H}_n$ are defined on objects. Recall that a morphism $f_\bullet : M_\bullet \to N_\bullet$ in Ch_R is defined as a family of maps $(f_n : M_n \to N_n)_{n\in\mathbb{Z}}$ such that 2.3.1 holds. We then define $\mathsf{Z}_n, \mathsf{B}_n, \mathsf{H}_n$ on $f_\bullet$ by in the first two cases restricting to $\mathsf{ker}\left(d_n^M\right), \mathsf{im}\left(d_{n+1}^M\right)$ (and co-restricting to $\mathsf{ker}\left(d_n^N\right), \mathsf{im}\left(d_{n+1}^N\right)$), while in the last case we let $\mathsf{H}_n(f_\bullet) : \mathsf{H}_n(M_\bullet) \to \mathsf{H}_n(N_\bullet)$ be defined by

$$\mathsf{H}_n(f_\bullet)([z]) := [f_n(z)] \tag{2.3.6}$$

We observe that if $z \in \mathsf{ker}\left(d_n^M\right)$ then $f_n(z) \in N_n$ is such that

$$\begin{aligned} d_n^N(f_n(z)) &= f_{n-1}d_n^M(z) \\ &= f_{n-1}(0) \\ &= 0, \end{aligned}$$

so that $f_n(z)$ lands in $\mathsf{ker}\left(d_n^N\right)$, i.e. so that corestriction to $\mathsf{ker}\left(d_n^N\right)$ is fine. We may abuse notation and still call the restricted and corestricted maps $f_\bullet$ and f_n.

If $z \in \mathsf{im}\left(d_{n+1}^M\right)$ then we may write $z = d_{n+1}^M(a)$ for some $a \in M_{n+1}$ so that

$$\begin{aligned} f_n(z) &= f_n(d_{n+1}^M(a)) \\ &= d_{n+1}^N(f_{n+1}(a)), \end{aligned}$$

so that $f_n(z) \in \mathsf{im}\left(d_{n+1}^N\right)$.

Lastly, since $f\left(\mathsf{ker}\left(d_n^M\right)\right) \subset \mathsf{ker}\left(d_n^N\right)$, by the universal property of the quotient we get a unique map $\mathsf{H}_n(f_\bullet) : \mathsf{H}_n(M_\bullet) \to \mathsf{H}_n(N_\bullet)$ as below, which we see is defined on homology classes $[z]$ as in 2.3.6,

$$\begin{array}{ccc} \mathsf{ker}\left(d_n^M\right) & \xrightarrow{\quad f_n \quad} & \mathsf{ker}\left(d_n^N\right) \\ \big\downarrow & \circlearrowleft & \big\downarrow \\ \mathsf{H}_n(M_\bullet) & \underset{\exists!\ \mathsf{H}_n(f_\bullet)}{\dashrightarrow} & \mathsf{H}_n(N_\bullet) \end{array} . \tag{2.3.7}$$

It remains to show that these constructions are *functorial*. Recall that the identity morphism in Ch_R was defined as 2.3.2 in each degree. Therefore, it is clear that

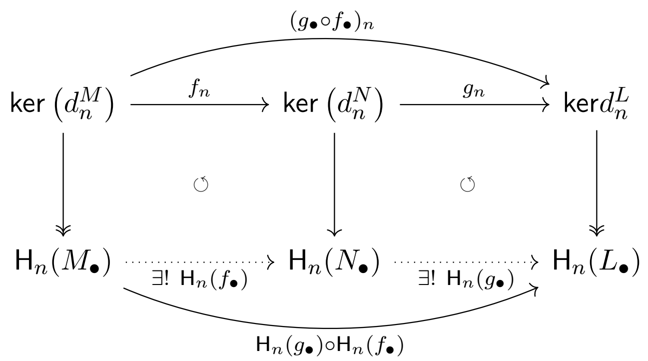
$$\begin{cases} \mathsf{Z}_n(\mathrm{id}_{M_\bullet}) & = \mathsf{id}_{\mathsf{ker}(d_n^M)} \\ \mathsf{B}_n(\mathrm{id}_{M_\bullet}) & = \mathsf{id}_{\mathsf{im}\, d_{n+1}^M} \\ \mathsf{H}_n(\mathrm{id}_{M_\bullet}) & = \mathsf{id}_{\mathsf{H}_n(M_\bullet} \end{cases},$$

where the last one follows by *uniqueness* of the map $H_n(f_\bullet)$ so that diagram 2.3.7 commutes, applied to $f_\bullet = \mathsf{id}_{M_\bullet}$. Since for chain maps $f_\bullet : M_\bullet \to N_\bullet$ and $g_\bullet : N_\bullet \to L_\bullet$ we have that $(g_\bullet \circ f_\bullet)_n = g_n \circ f_n$, it follows that

$$\begin{aligned} \mathsf{Z}_n(g_\bullet \circ f_\bullet) &= (g_n \circ f_n)|_{\mathsf{ker}(d_n^M)} \\ &= g_n|_{\mathsf{ker}(d_n^N)} \circ f_n|_{\mathsf{ker}(d_n^M)} \\ &= \mathsf{Z}(g_\bullet) \circ \mathsf{Z}(f_\bullet). \end{aligned}$$

Similarly, we may show that $\mathsf{B}_n(g_\bullet \circ f_\bullet) = \mathsf{B}_n(g_\bullet) \circ \mathsf{B}_n(f_\bullet)$.

Lastly, for H_n, we contemplate the following diagram:



Since the smaller squares commute, the larger rectangle commute. Since $g_n(f_n(\mathsf{ker}d_n^M)) \subset \mathsf{ker}\left(d_n^L\right)$, we also get a (unique) map $\mathsf{H}_n(g_\bullet \circ f_\bullet) : \mathsf{H}_n(M_\bullet) \to \mathsf{H}_n(L_\bullet)$. By *uniqueness* of R-module homomorphisms making this larger rectangle commute, it follows that

$$\mathsf{H}_n(g_\bullet \circ f_\bullet) = \mathsf{H}_n(g_\bullet) \circ \mathsf{H}_n(f_\bullet).$$

□

Definition 2.3.4 (Quasi-isomorphism)**.** We say that a morphism $f_\bullet : M_\bullet \to N_\bullet$ in Ch_R is a **quasi-isomorphism**, if the induced morphisms $\mathsf{H}_n(f_\bullet) : \mathsf{H}_n(M_\bullet) \to \mathsf{H}_n(N_\bullet)$ coming from diagram 2.3.7 in the proof of theorem 2.3.3 are *isomorphisms* for all $n \in \mathbb{Z}$. Following [Maz23] we decorate such a morphism as $\xrightarrow{\approx}$.

Definition 2.3.5 (Acyclic complex)**.** We say that a complex $M_\bullet \in \mathsf{Ch}_R$ is **acyclic** if $\mathsf{H}_n(M_\bullet) = 0$ for all n, i.e. if $\mathsf{ker}\left(d_n^M\right) = \mathsf{im}\left(d_{n+1}^M\right)$ for all n, or equivalently, if $0_\bullet \to M_\bullet$ or $M_\bullet \to 0_\bullet$ are quasi-isomorphisms, where $0_\bullet$ is the complex with all objects zero and differentials the zero maps.

Observe that a quasi-isomorphism $f_\bullet : M_\bullet \to N_\bullet$ in Ch_R is not necessarily *formally* an isomorphism in the (general) sense of having a two-sided inverse $g_\bullet : N_\bullet \to M_\bullet$, whenever R is not the zero ring.

Example 2.3.6. Assume that R is a commutative, *non-zero* unital ring. Consider the morphism $f_\bullet : R_{1,0} \to \underset{\sim}{0}$ in Ch_R defined as below:

$$
\begin{array}{lccccccccccc}
R_{01} : & \dots & \longrightarrow & 0 & \xrightarrow{d_2} & R & \xrightarrow{d_1} & R & \xrightarrow{d_0} & 0 & \longrightarrow & \dots \\
\downarrow f_\bullet & & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f_{-1} & & \\
\underset{\sim}{0} : & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array},
$$

with $d^1 = \mathrm{id}_R$. Since there is only one R-module homomorphism from R to 0, this chain map is the unique chain map with the given domain and codomain, i.e. $\mathsf{hom}_{\mathsf{Ch}_R}\left(R_{1,0}, \underset{\sim}{0}\right) = 0$. We observe that $\mathsf{H}_n(f_\bullet) : \mathsf{H}(R_{1,0}) \to \mathsf{H}_n\left(\underset{\sim}{0}\right)$ is the zero map for all $n \in \mathbb{Z}$. We observe that we have

$$
\begin{aligned}
\mathsf{H}_2(R_{1,0}) &= \mathsf{ker}\,(d_2) \,/\, \mathsf{im}\,(d_3) \\
&= (0)/(0) \\
&= (0),
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{H}_1(R_{1,0})_1 &= \mathsf{ker}\,(d_1) \,/\, \mathsf{im}\,(d_2) \\
&= (0)/(0) \\
&= (0),
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{H}_0(R_{1,0}) &= \mathsf{ker}\,(d_0) \,/\, \mathsf{im}\,(d_1) \\
&= R/R \\
&= (0),
\end{aligned}
$$

and all other $\mathsf{H}_n(R_{1,0}) = 0$. It follows that $\mathsf{H}_n(f_\bullet)$ is an isomorphism for all n, so that $f_\bullet$ is a quasi-isomorphism. We claim that it can not have an inverse $g_\bullet : \underset{\sim}{0} \to R_{1,0}$ in Ch_R. We see this by "completing" the diagram as below:

$$
\begin{array}{lccccccccccc}
R_{1,0} : & \dots & \longrightarrow & 0 & \xrightarrow{d_2} & R & \xrightarrow{d_1 = \mathrm{id}_R} & R & \xrightarrow{d_0} & 0 & \longrightarrow & \dots \\
\downarrow f_\bullet & & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f_{-1} & & \\
\underset{\sim}{0} : & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
\downarrow g_\bullet & & & g_2 \downarrow & & g_1 \downarrow & & g_0 \downarrow & & g_{-1} \downarrow & & \\
R_{1,0} : & \dots & \longrightarrow & 0 & \xrightarrow[d_2]{} & R & \xrightarrow[d_1 = \mathrm{id}_R]{} & R & \xrightarrow[d_0]{} & 0 & \longrightarrow & \dots
\end{array}
$$

The reason being that this would force

$$
\begin{aligned}
(g_\bullet \circ f_\bullet)_1 &= (\mathrm{id}_{R_{1,0}})_1 \\
&= \mathrm{id}_R,
\end{aligned}
$$

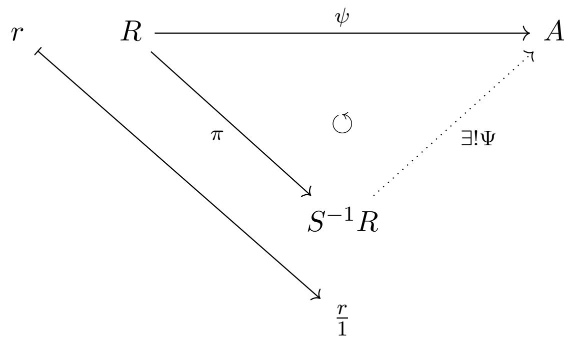
but $(g_\bullet \circ f_\bullet)_1 = g_1 \circ f_1$, which is necessarily the zero-morphism. Since $R \neq 0$, this is impossible.

The intuition behind constructing the **derived category of R-modules** (observe: *not* the derived ∞-category, but the "ordinary" derived category of R-modules) is to take the category Ch_R, and then adjoin formal inverses for every quasi-isomorphism $f_\bullet$ in Ch_R. In a sense, we would like to turn quasi-isomorphism into *actual* isomorphisms. We give a short intro in the next chapter. First, we will talk about how things look when they are "well-behaved" in that there is a so-called "calculus of fractions" available. Then we will mention the in a sense most general procedure (at least at the level of ordinary categories) that we are aware of, for localizing "freely", the *Gabriel-Zisman* localization.

2.4 Derived categories (of R-modules)

2.4.1 Localization of categories

Recall that for a commutative ring R, with S a **multiplicatively closed** set (that is, $1 \in S$ and whenever $r, s \in S$ then $rs \in S$) one may create a new ring $S^{-1}R$ by "localizing at S", such that all elements $s \in S$ have (multiplicative) inverses in $S^{-1}R$. Furthermore, this construction is *universal* in the sense that for any ring homomorphism $R \xrightarrow{\psi} A$ such that $\psi(s)$ has a multiplicative inverse, for all $s \in S$, then there exists a *unique* ring homomorphism $\Psi : S^{-1}R \to A$ such that ψ factors as $\psi = \Psi \circ \pi$, where π is the "canonical" localization-morphism $\pi : R \to S^{-1}R$ sending r to $\frac{r}{1}$ as in the diagram below,



, (2.4.1)

where one may show that $\psi\left(\frac{r}{s}\right) = \psi(r) \cdot \psi(s)^{-1}$ for all elements $\frac{r}{s} \in S^{-1}R$.

Now, we want to do something *roughly* similar with categories, where our "multiplicative system" corresponds to a class S of *morphisms* in a category $\mathcal{C}$. Our treatment follows [KS90, Section 1.6].

Definition 2.4.1 (Multiplicative system of morphisms)**.** Let $\mathcal{C}$ be a category and let S be a class of morphisms in $\mathcal{C}$. We then say that S is a **multiplicative system** if S satisfies the following properties:

(i) For any object $X \in \mathsf{Ob}(\mathcal{C})$ the identity morphism id_X belongs to S.

(ii) For any pair of morphisms f, g that belong to S, where the composition $g \circ f$ is defined, we have that $g \circ f$ also belongs to S.

(iii) Any diagram on the form to the left below where g belongs to S, may be completed to a commutative square as to the right below, where h belongs to S,

$$\begin{array}{ccc} & & A \\ & & \Big\downarrow g \\ B & \xrightarrow{f} & C \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} D & \dashrightarrow & A \\ {\scriptstyle h}\Big\downarrow & \circlearrowleft & \Big\downarrow g \\ B & \xrightarrow{f} & C \end{array} . \tag{2.4.2}$$

Similarly, the following diagram to the left below with g belonging to S must be able to be completed to a commutative square as indicated below, with h belonging to S.

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\Big\downarrow & & \\ C & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\Big\downarrow & \circlearrowleft & \Big\downarrow \\ C & \underset{h}{\dashrightarrow} & D \end{array} .$$

(iv) If f and g are morphisms belonging to $\mathsf{hom}_{\mathcal{C}}(X, Y)$, then the following conditions are equivalent:

(a) There exists a morphism $Y \xrightarrow{t} Y'$ in $\mathcal{C}$ that belongs to S, such that $t \circ f = t \circ g$.

(b) There exists a morphism $X' \xrightarrow{s} X$ in $\mathcal{C}$ that belongs to S, such that $f \circ s = g \circ s$.

We now introduce the definition of a "category localized at a multiplicative system S".

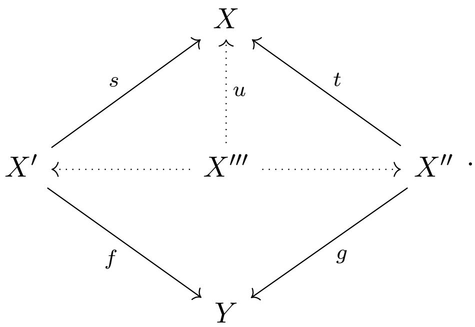
Definition 2.4.2 (The category $\mathcal{C}_S$)**.** Let $\mathcal{C}$ be a category and let S be a multiplicative system of morphisms in $\mathcal{C}$. Then the category $\mathcal{C}_S$ is the category with objects $\mathsf{Ob}(\mathcal{C}_S) = \mathsf{Ob}(\mathcal{C})$, and where, for any pair X, Y of objects of $\mathcal{C}_S$, we have

$$\mathsf{hom}_{\mathcal{C}_S}(X, Y) := \left\{ \left(X \xleftarrow{s} X' \xrightarrow{f} Y \right) \,\middle|\, s \text{ belongs to } S \right\} \Big/ \sim, \tag{2.4.3}$$

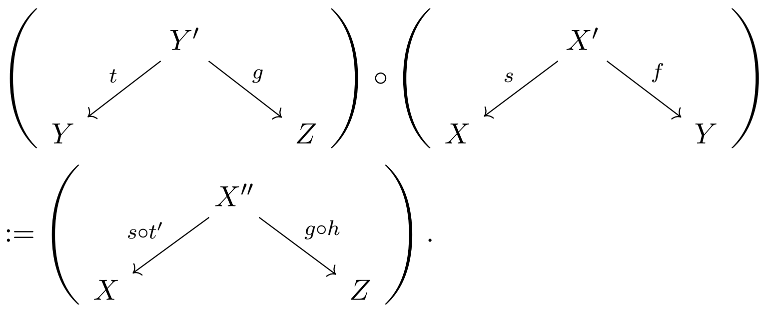
where $\sim$ identifies two "roofs"

$$\left(X \xleftarrow{s} X' \xrightarrow{f} Y \right) \quad \text{and} \quad \left(X \xleftarrow{t} X'' \xrightarrow{g} Y \right) \tag{2.4.4}$$

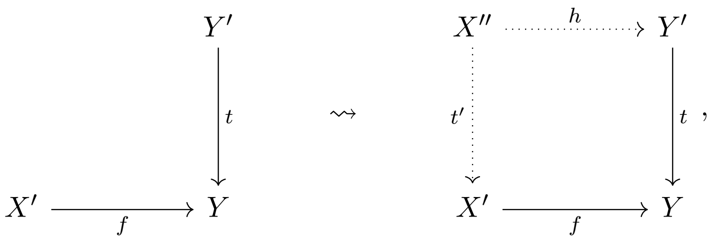
iff there exists a morphism $X''' \xrightarrow{u} X$ in S together with dashed morphisms in $\mathcal{C}$ as indicated, such that the diagram below commutes,



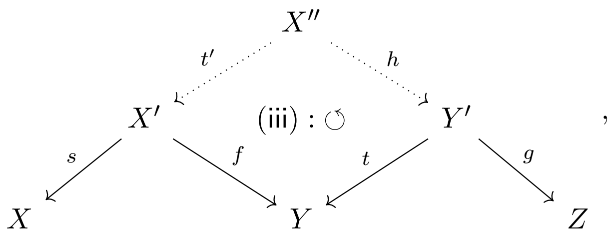
Composition of morphisms in $\mathcal{C}_S$ is defined as follows:



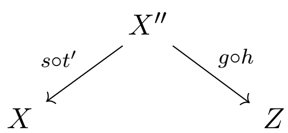
Explicating on how we get t' and h above: We use (iii) in definition 2.4.1 as indicated to complete the cospan $X' \xrightarrow{f} Y \xleftarrow{t} Y'$ as below



so this gives us the following datum



where we note that t' is in S by (iii) in definition 2.4.1. Then we see that $s \circ t'$ is in S by (ii) in definition 2.4.1, so that



is indeed an element of $\mathsf{hom}_{\mathcal{C}_S}(X, Z)$.

Remark 2.4.3. To see that the composition does not depend on which completion we choose in the diagrams in 2.4.2: Assume we have two different completions as below,

$$\begin{array}{ccc} A & \overset{b}{\dashrightarrow} & Y' \\ {\scriptstyle a}\,\vdots & & \downarrow{\scriptstyle t} \\ X' & \underset{f}{\longrightarrow} & Y \end{array} \qquad \begin{array}{ccc} \tilde{A} & \overset{\tilde{b}}{\dashrightarrow} & Y' \\ {\scriptstyle \tilde{a}}\,\vdots & & \downarrow{\scriptstyle t} \\ X' & \underset{f}{\longrightarrow} & Y \end{array}, \tag{2.4.5}$$

where a and $\tilde{a}$ belongs to S. We then have two composite roofs $X \xleftarrow{s \circ a} A \xrightarrow{g \circ b} Z$ and $X \xleftarrow{s \circ \tilde{a}} \tilde{A} \xrightarrow{g \circ \tilde{b}} Z$. We may then complete as indicated below (with note, $\tilde{a}$ belonging to S)

$$\begin{array}{ccc} & & \tilde{A} \\ & & \downarrow{\scriptstyle \tilde{a}} \\ A & \underset{a}{\longrightarrow} & X \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} W & \overset{q}{\dashrightarrow} & \tilde{A} \\ {\scriptstyle p}\,\vdots & & \downarrow{\scriptstyle \tilde{a}} \\ A & \underset{a}{\longrightarrow} & X \end{array}. \tag{2.4.6}$$

Then p belongs to S, and $a \circ p = \tilde{a} \circ q$. Furthermore, we have

$$\begin{aligned} t \circ b \circ p &= f \circ a \circ p, && \text{by the commutativity of the left square in 2.4.5} \\ &= f \circ \tilde{a} \circ q, && \text{by commutativity of the square to the right in 2.4.6} \\ &= t \circ \tilde{b} \circ q, && \text{by commutativity of the right square in 2.4.5.} \end{aligned}$$

Since t belongs to S, we by (iv) in 2.4.1 get a morphism $W' \xrightarrow{r} W$ belonging to S such that $b \circ p \circ r = \tilde{b} \circ q \circ r$, and by commutativity of the square in 2.4.6 it follows that also $a \circ p \circ r = \tilde{a} \circ q \circ r$. We may then consider the following diagram

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle s\circ a}\nearrow & \uparrow{\scriptstyle u} & \nwarrow{\scriptstyle s\circ\tilde{a}} & \\ A & \xleftarrow{p\circ r} & W' & \xrightarrow{q\circ r} & \tilde{A} \\ & {\scriptstyle g\circ b}\searrow & & \swarrow{\scriptstyle g\circ\tilde{b}} & \\ & & Z & & \end{array},$$

with $u = s \circ a \circ p \circ r = s \circ \tilde{a} \circ q \circ r$, which belongs to S, by repeated application of condition (ii) in definition 2.4.1, since all morphisms in the composition belongs to S. Furthermore, we have that $g \circ b \circ p \circ r = g \circ \tilde{b} \circ q \circ r$ since the two sides of the identity are equal before postcomposition with g. Hence it follows that

$$\left(\begin{array}{ccc} & A & \\ {\scriptstyle s\circ a}\swarrow & & \searrow{\scriptstyle g\circ b} \\ X & & Z \end{array} \right) = \left(\begin{array}{ccc} & \tilde{A} & \\ {\scriptstyle s\circ\tilde{a}}\swarrow & & \searrow{\scriptstyle g\circ\tilde{b}} \\ X & & Z \end{array} \right)$$

as elements of $\mathsf{hom}_{\mathcal{C}_S}(X, Z)$. One must further show that if two roofs are equivalent, then they yield the same equivalence class of roofs after composition with another roof. We omit this proof.

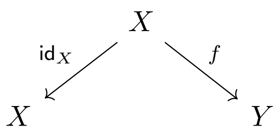
Proposition 2.4.4. *The definition given in 2.4.2 defines an actual category.*

Proof. The proof should be very similar to the one given for *left* multiplicative systems in [Lemma 4.27.2.(3), Sta26, Tag 04VB]. □

Almost completely analogous to the ring case expounded upon at the beginning of the chapter (in particular, c.f. with diagram 2.4.1) we have the following situation:

Proposition 2.4.5 ([KS90, Prop. 1.6.3])**.** *There is a functor $\mathcal{C} \xrightarrow{Q} \mathcal{C}_S$ whenever $\mathcal{C}$ is a category with a multiplicative system S of morphisms, that satisfies the following:*

(a) *$Q(X) = X$ for all objects X in $\mathcal{C}$, and*

$Q(f) :=$  ,

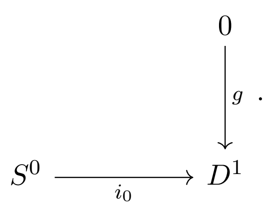
for all f that belongs to $\mathsf{hom}_{\mathcal{C}}(X, Y)$, for arbitrary objects X, Y in $\mathcal{C}$. [1]

(b) *If a morphism s belongs to S, then $Q(s)$ is an isomorphism in $\mathcal{C}_S$.*

(c) *Let $\mathcal{D}$ be another category and let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor such that $F(s)$ is an isomorphism for all morphisms s that belongs to S. Then F factors* uniquely *through Q, i.e. there is a functor $\mathcal{C}_S \xrightarrow{\tilde{F}} \mathcal{D}$ such that $F = \tilde{F} \circ Q$.*

Now, one might think that the class qiso of quasi-isomorphisms in Ch_R forms a multiplicative system, but this is not the case! Hence we may not (at least directly) apply the above construction to form $(\mathsf{Ch}_R)_{\mathsf{qiso}}$. Note however that *it is entirely possible* to localize Ch_R directly with respect to the class qiso of quasi-isomorphisms (see [GZ67, p. 6]), but our understanding is that this does not provide us into much insight into how morphisms in the resulting category behaves. We provide an example below showing that qiso in Ch_R does *not* form a multiplicative system of morphisms in the sense of definition 2.4.1

Example 2.4.6. Fix a commutative non-zero unital ring R and let S^0 be the complex with $(S^0)_n = R$ if $n = 0$ and $(S^0)_n = 0$ if $n \neq 0$ (with all differentials zero), and let D^1 be the complex with R in degree $n = 1, 0$ and zero otherwise, with $d_1 = \mathsf{id}_R$ and $d_n = 0$ if $n \neq 1$. Then 0 and D^1 are acyclic so g is a quasi-isomorphism and so belongs to qiso. Consider the following diagram



[1] Observe that this is well-defined since id_X belongs to S by (i) in definition 2.4.1.

with i_0 defined as in 5.5.1. By definition 2.4.1, condition (iii) we would need to be able to complete this to a commuting square

$$\begin{array}{ccc} W & \overset{a}{\dashrightarrow} & 0 \\ {\scriptstyle h}\,\big\downarrow & & \big\downarrow\,{\scriptstyle g} \\ S^0 & \underset{i_0}{\longrightarrow} & D^1 \end{array},$$

with h a quasi-isomorphism. The commutativity in the square in degree zero would mean that $(i_0)_0 \circ h_0 = \mathsf{id}_R \circ h_0 = h_0 = 0$. It would follow that $\mathsf{H}_0(h) : \mathsf{H}_0(W) \approx \mathsf{H}_0\left(S^0\right)$ is the zero map. But $\mathsf{H}_0\left(S^0\right) = \mathsf{ker}\left(d_0\right)/\mathsf{im}(d_1) = R/(0) = R$. Since $R \neq 0$, this is impossible (the map can not be surjective hence not an isomorphism!).

Although we will not provide all the details here, how one "corrects" the failure of qiso to be a multiplicative system of morphisms in Ch_R, is to first pass to the dg-category of complexes of R-modules (the subject of §4) $\mathbf{K}_R$. Then from $\mathbf{K}_R$ one may pass to the **homotopy category** $\mathsf{Ho}\left(\mathbf{K}_R\right)$ (c.f. [Yek19, Definition 3.4.6.(2)]). This homotopy category has the *same* objects as $\mathbf{K}_R$, but with morphisms

$$\begin{aligned} \mathsf{hom}_{\mathsf{Ho}(\mathbf{K}_R)}(M,N) &:= \mathsf{H}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) \\ &\cong \mathsf{hom}_{\mathsf{Ch}_R}(M,N)/\left(\text{null-homotopy}\right), \qquad \text{see 4.1.2.} \end{aligned}$$

Now, the claim is that the class of quasi-isomorphims in $\mathsf{Ho}(\mathbf{K}_R)$ *do form* a multiplicative system. We will not try to prove this here. For proofs, see e.g. [Sta26, Tag 05RT] or perhaps [Yek19, Def. 7.2.7, Prop. 7.1.1, Theorem 7.1.3].

A further claim is that one may then show that we have a functor $\mathsf{Ch}_R \to \mathsf{H}_0(\mathbf{D}_R)$, where formally $\mathsf{H}_0(\mathbf{D}_R) := \mathsf{Ho}(\mathbf{K}_R)_{\mathsf{qiso}}$, with the localization on the right-hand side the one described in 2.4.2. One may refer to this as the **derived category of R-modules**, and where the *objects* of $\mathsf{H}_0(\mathbf{D}_R)$ are called **derived R-modules**. Observe that since quasi-isomorphic complexes may have non-isomorphic modules degreewise (take for example D^1 and the zero-complex 0 from 2.4.6), one can not talk about the degree zero module of a derived R-module (it is *not* preserved under quasi-isomorphism, by the provided example). However, since quasi-isomorphic complexes have the same homology, one *should* be able to talk about the homology of a derived R-module. Therefore, morally, there should be a factorization as below

$$\begin{array}{ccc} \mathsf{Ch}_R & \xrightarrow{\mathsf{H}_n(\,\cdot\,)} & \mathsf{Mod}_R \\ \big\downarrow & \overset{\mathsf{H}_n(\,\cdot\,)}{\nearrow} & \\ \mathsf{H}_0(\mathbf{D}_R) & & \end{array}.$$

We will not say much more about $\mathsf{H}_0(\mathbf{D}_R)$ in this thesis. What is perhaps more interesting is $\mathbf{D}_R$ itself, which we will call the **derived ∞-category of R-modules**. The claim is that the objects of $\mathbf{D}_R$ are still derived R-modules with hom-complexes (cf. §4) derived

$\Bbbk$-modules, i.e. objects of $\mathbf{D}_\Bbbk$, the derived ∞-category of $\Bbbk$-modules (omitting further explicating on a possible *symmetric monoidal* structure on $\mathbf{D}_\Bbbk$). The notation $\mathsf{H}_0(\mathbf{D}_R)$ is meant to indicate that one may pass to the derived category of R-modules from the derived ∞-category of R-modules, by taking the 0^{th}-homology of the hom-complexes in $\mathbf{D}_R$.

In the case of this localization-scheme where one first passes to $\mathsf{Ho}(\mathbf{K}_R)$ before inverting qiso, we see that we by [Sta26, Tag 04VB]) get that each morphism $M \to N$ in $\mathsf{Ho}(\mathbf{K}_R)_{\mathsf{qiso}}$ will have a representative on the form

$$M \xleftarrow{\approx} A \longrightarrow N,$$

with the left-downward arrow $\approx$ indicating that it is a quasi-isomorphism.

On the other hand, we may, by starting directly at the class qiso of quasi-isomorphisms in Ch_R, "freely" invert the quasi-isomorphisms, and get the resulting category $\mathsf{Ch}_R\left[\mathsf{qiso}^{-1}\right]$, by the so called **Gabriel-Zisman** localization (see e.g. [GZ67], [Sim05]). Then the resulting category has the same objects as Ch_R, and any morphism in $\mathsf{Ch}_R[\mathsf{qiso}^{-1}]$ may be represented on the form

$$M \xleftarrow{\approx} A_1 \longrightarrow B_2 \xleftarrow{\approx} A_2 \longrightarrow B_3 \cdots B_m \xleftarrow{\approx} A_m \longrightarrow N. \tag{2.4.7}$$

Observation: By [Wei94, Example 10.3.2.(1)-(2)] we have that

$$\mathsf{Ho}(\mathbf{K}_R)_{\mathsf{qiso}} = \mathsf{Ch}_R[\mathsf{qiso}^{-1}].$$

This means that it does not *in theory* really matter whether we localize Ch_R at the quasi-isomorphisms or if we first pass to its associated homotopy category $\mathsf{Ho}(\mathbf{K}_R)$ before inverting the quasi-isomorphism. *Note* however that we have been abusing notation somewhat, in that the quasi-isomorphisms qiso in $\mathsf{Ho}(\mathbf{K}_R)$ are really equivalence classes modulo null-homotopy, which is not the case in Ch_R.

The above observation means that we may equally define (following [Maz23, §2.4]) the derived category of R-modules $\mathsf{H}_0(\mathbf{D}_R)$ as $\mathsf{Ch}_R\left[\mathsf{qiso}^{-1}\right]$.

2.5 Tensor product complex

Definition 2.5.1 (Tensor product of chain complexes)**.** Let $M, N \in \mathsf{Ch}_\Bbbk$ be chain complexes of $\Bbbk$-modules (suppressing the $\bullet$-notation). Then we define the **tensor product complex** $M \otimes N$ or $(M \otimes_\Bbbk N)_\bullet$ as having degree ℓ $\Bbbk$-module

$$(M \otimes_\Bbbk N)_\ell := \bigoplus_{i+j=\ell} M_i \otimes_\Bbbk N_j,$$

and degree ℓ differential

$$d_\ell : \bigoplus_{i+j=\ell} M_i \otimes_\Bbbk N_j \to \bigoplus_{i+j=\ell-1} M_i \otimes_\Bbbk N_j,$$

defined on *pure* tensors $m \otimes n \in M_i \otimes N_j$ as

$$m \otimes n \stackrel{d_\ell}{\longmapsto} d_i^M(m) \otimes n + (-1)^i \cdot m \otimes d_j^N(n). \tag{2.5.1}$$

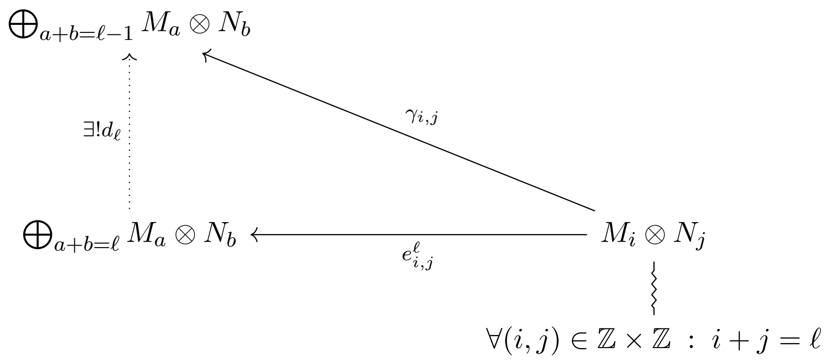
Remark 2.5.2. For later purposes (in particular, the proof of 2.5.4) we need to be a bit more precise about how d_ℓ is defined. We let

$$e_{i,j}^{\ell} : M_i \otimes N_j \to \bigoplus_{a+b=\ell} M_a \otimes N_b, \quad e_{i-1,j}^{\ell-1} : M_{i-1} \otimes N_j \to \bigoplus_{a+b=\ell-1} M_a \otimes N_b,$$
$$e_{i,j-1}^{\ell-1} : M_i \otimes N_{j-1} \to \bigoplus_{a+b=\ell-1} M_a \otimes N_b$$

be the canonical *coproduct* maps for each pair (i, j) such that $i + j = \ell$. Then with the *universal property* of the coproduct applied to the family of maps

$$\gamma_{i,j} := e_{i-1,j}^{\ell-1} \circ (d_i^M \otimes \mathrm{id}_{N_j}) + (-1)^i e_{i,j-1}^{\ell-1} \circ (\mathrm{id}_{M_i} \otimes d_j^N), \tag{2.5.2}$$

we get a uniquely specified map d_ℓ such that the diagram below commutes:



,

i.e. so that $d_\ell \circ e_{i,j}^{\ell} = \gamma_{i,j}$ for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $i + j = \ell$. As we will see in the proof of claim I below, one may suppress $e_{i,j}$ to unclutter notation, and just think of $d_\ell(e_{i,j}^{\ell}(m \otimes n)) = d_\ell(m \otimes n) = \gamma_{i,j}(m \otimes n)$ (then expanding the right-most side using the definition of $\gamma_{i,j}$ as in 2.5.2). Hence one may think of d_ℓ as being defined directly as in 2.5.1, as long as one keeps in mind that a pure tensor $m \otimes n$ in $(M \otimes N)_\ell$ is really $e_{i,j}(m \otimes n)$ for appropriate pair (i, j). Observe that this definition also then generalizes to *iterated* tensor products, e.g. $M \otimes (N \otimes P)$. We will later want to relate iterated tensor products on the form $M \otimes (N \otimes P)$ to $(M \otimes N) \otimes P$ by a (natural) isomorphism.

Claim I: $(M \otimes_{\Bbbk} N)_\bullet$ is a complex.

Proof. By definition of the differentials of M and N, by construction of $(M \otimes N)_\bullet$ and how we defined d_ℓ, it is clear that the image lands in $(M \otimes N)_{\ell-1}$. It remains to show that $d_\ell \circ d_{\ell+1} = 0$ for all $\ell \in \mathbb{Z}$: Let $m \otimes n$ be any pure tensor in $M_i \otimes N_j$ for $i + j = \ell + 1$.

Then we have

$$\begin{aligned}
(d_\ell \circ d_{\ell+1})(m \otimes n) &= d_\ell \left(d_i^M(m) \otimes n + (-1)^i \cdot m \otimes d_j^N(n) \right) \\
&= d_\ell(d_i^M(m) \otimes n) + (-1)^i d_\ell(m \otimes d_j^N(n)) \\
&= \underbrace{\underbrace{d_{i-1}^M d_i^M(m)}_{=0} \otimes n}_{=0} + (-1)^{i-1} d_i^M(m) \otimes d_j^N(n) \\
&\quad + (-1)^i \cdot d_i^M(m) \otimes d_j^N(n) + \underbrace{(-1)^{2i} \cdot m \otimes \underbrace{d_{j-1}^N d_j^N(n)}_{=0}}_{=0} \\
&= (-1)^{i-1} d_i^M(m) \otimes d_j^N(n) + (-1)^i d_i^M(m) \otimes d_j^N(n) \\
&= (-1)^{i-1} d_i^M(m) \otimes d_j^N(n) \underbrace{(1-1)}_{=0} \\
&= 0,
\end{aligned}$$

so that $M \otimes_{\Bbbk} N$ is indeed a complex. $\square$

We may want to promote the tensor product of complexes construction to a functor, by defining $\otimes$ on morphisms $f : M \to M', g : N \to N' \in \mathsf{Ch}_{\Bbbk}$ as follows: $f \otimes g : M \otimes N \to M' \otimes N'$ is the *unique* map we get from first defining, at the ℓ^{th}-level,

$$\Gamma_{ij} : M_i \times N_j \to M_i' \otimes_{\Bbbk} N_j', \qquad (m, n) \xmapsto{\Gamma_{i,j}} f_i(m) \otimes g_j(n). \tag{2.5.3}$$

The *universal property* of the tensor product $M_i \otimes_{\Bbbk} N_j$ then gives us a unique $\Bbbk$-linear map $\psi_{i,j} : M_i \otimes_{\Bbbk} N_j \to M_i' \otimes_{\Bbbk} N_j'$ such that $\Gamma_{i,j}$ factors through the "inclusion" $(m, n) \mapsto m \otimes n$ and $\psi_{i,j}$. We have canonical coproduct maps $e'^{\ell}_{i,j} : M_i' \otimes_{\Bbbk} N_j' \to \bigoplus_{i+j=\ell} M_i' \otimes_{\Bbbk} N_j'$. Let $\tilde{\psi}_{i,j} := e'^{\ell}_{i,j} \circ \psi_{i,j}$. By the *universal property* of the *coproduct* $\left(\bigoplus_{i+j=\ell} M_i \otimes_{\Bbbk} N_j, \{e^{\ell}_{i,j}\}_{i+j=\ell} \right)$ applied to the family of maps $\{\tilde{\psi}_{ij}\}_{i+j=\ell}$, we get a *unique* $\Bbbk$-module homomorphism $\varphi_\ell : \bigoplus_{i+j=\ell} M_i \otimes_{\Bbbk} N_j \to \bigoplus_{i+j=\ell} M_i' \otimes_{\Bbbk} N_j'$, as below, such that $\varphi_\ell \circ e^{\ell}_{i,j} = \tilde{\psi}_{i,j}$ for all $(i, j) \in \mathbb{Z}^2$ such that $i + j = \ell$.

$$\begin{array}{ccc}
\bigoplus_{i+j=\ell} M_i' \otimes_{\Bbbk} N_j' & & \\
\uparrow {\scriptstyle \exists! \varphi_\ell} & \nwarrow^{\tilde{\psi}_{i,j}} & \\
\bigoplus_{i+j=\ell} M_i \otimes_{\Bbbk} N_j & \xleftarrow[e^{\ell}_{i,j}]{} & M_i \otimes_{\Bbbk} N_j
\end{array} \tag{2.5.4}$$

We *define* $(f \otimes g)_\ell := \varphi_\ell$ for each $\ell \in \mathbb{Z}$. We need to show that $(f \otimes g) := (f \otimes g)_{\ell \in \mathbb{Z}}$ is a *chain*-map, i.e. that it is indeed a morphism in $\mathsf{Ch}_{\Bbbk}$. Diagrammatically, our situation is the following, in that we would like to show that each square of the following form commutes,

$$\begin{array}{ccc}
\bigoplus_{i+j=\ell+1} M_i \otimes_{\Bbbk} N_j & \xrightarrow{d_{\ell+1}} & \bigoplus_{i+j=\ell} M_i \otimes_{\Bbbk} N_j \\
\Big\downarrow{\scriptstyle (f\otimes g)_{\ell+1}} & & \Big\downarrow{\scriptstyle (f\otimes g)_{\ell}} \\
\bigoplus_{i+j=\ell+1} M_i' \otimes_{\Bbbk} N_j' & \xrightarrow{d'_{\ell+1}} & \bigoplus_{i+j=\ell} M_i' \otimes_{\Bbbk} N_j'
\end{array} .$$

We check this on *pure* tensors. Let $m \otimes n \in M_i \otimes_{\Bbbk} N_j$ for $i + j = \ell + 1$. Then we find that

$$\begin{aligned}
d'_{\ell+1}((f \otimes g)_{\ell+1}(m \otimes n)) &= d'_{\ell+1}(f_i(m) \otimes g_j(n)) \\
&= d'_i f_i(m) \otimes g_j(n) + (-1)^i \cdot f_i(m) \otimes d'_j g_j(n) \\
&= f_{i-1}(d_i(m)) \otimes g_j(n) + (-1)^i \cdot f_i(m) \otimes g_{j-1} d_j(n) \quad (f, g \text{ chain maps}) \\
&= (f \otimes g)_\ell(d_i(m) \otimes n) + (f \otimes g)_\ell((-1)^i \cdot m \otimes d_j(n)) \\
&= (f \otimes g)_\ell(d_i(m) \otimes n + (-1)^i \cdot m \otimes d_j(n)) \qquad (\text{by linearity of } (f \otimes g)_\ell) \\
&= (f \otimes g)_\ell(d_{\ell+1}(m \otimes n)).
\end{aligned}$$

By linearity of d, d' and $(f \otimes g)$ it follows that the diagram commutes for each ℓ, so that $(f \otimes g)$ is indeed a chain map. Having defined the tensor product on complexes and morphisms, we want to show that this in fact defines a *bifunctor*.

Claim II: $\otimes : \mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk} \to \mathsf{Ch}_{\Bbbk}$ is a bifunctor.

Remark 2.5.3. Observe that $\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk}$ is the product category of two copies of $\mathsf{Ch}_{\Bbbk}$.

Proof. Given $M, N \in \mathsf{Ch}_{\Bbbk}$, with identity morphisms $\mathrm{id}_M, \mathrm{id}_N \in \mathsf{Mor}(\mathsf{Ch}_{\Bbbk})$, we find that $\mathrm{id}_M \otimes \mathrm{id}_N$ is just defined as the identity on pure tensors in all degrees, so by extension it is the identity morphism of $M \otimes N$, i.e. $\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes N}$ as functions.

By the remark (i.e. by how composition is defined in $\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_{\Bbbk}$), we want to check that $\otimes$ respects composition, in the sense that for morphisms $f_1 : M' \to M, g_1 : N \to N'$ and $f_2 : M \to M'', g_2 : N \to N''$, we have

$$((f_2 \circ f_1) \otimes (g_2 \circ g_1)) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1).$$

This is immediate on pure tensors by construction, and the conclusion follows. □

Theorem 2.5.4. *The category* $\mathsf{Ch}_{\Bbbk}$ *with tensor product* $\otimes$ *is a tensor category, with unit object* $\underset{\sim}{\Bbbk}$ *and with symmetrizer* $\sigma(m \otimes n) = (-1)^{i \cdot j} n \otimes m$ *for* $m \otimes n \in M_i \otimes N_j$*.*

Remark 2.5.5. We will suppress $\Bbbk$ in $\otimes_{\Bbbk}$ in the following proof.

Remark 2.5.6. We will only try to give a *sketch* of how one may go about proving this. Perhaps the intuition to have in mind is that one is using the natural maps coming from the symmetric monoidal structure on the tensor category $(\mathsf{Mod}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ together with universal properties to extend these natural maps to $\mathsf{Ch}_{\Bbbk}$. We first tried to be as formal as possible in our proof, but the notation quickly became unwieldy. Perhaps this is a proof which should be formalized in e.g. `Lean` (unless it has already been done).

Remark 2.5.7. When we say that a map *preserves degrees* we mean that $f : M \to N$ preserves degrees if $f(M_\ell) \subseteq N_\ell$ for all $\ell \in \mathbb{Z}$ for complexes M, N in $\mathsf{Ch}_{\Bbbk}$.

Proof. Associator α: Define $\alpha_{M,N,P} : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$ on pure tensors by $\alpha_{M,N,P}(m \otimes (n \otimes p)) = ((m \otimes n) \otimes p)$ with $m \in M_i, n \in N_j, p \in P_k$, and observe that this map *preserves* total degree $i + j + k$.

We see that

$$\begin{aligned} d(m \otimes (n \otimes p)) &= d(m) \otimes (n \otimes p) + (-1)^i m \otimes d(n \otimes p) \\ &= d(m) \otimes (n \otimes p) + (-1)^i m \otimes (dn \otimes p + (-1)^j n \otimes d(p)) \\ &= d(m) \otimes (n \otimes p) + (-1)^i m \otimes (d(n) \otimes p) + (-1)^{i+j} m \otimes (n \otimes d(p)). \end{aligned}$$

Upon applying $\alpha_{M,N,P}$ to this we get

$$(d(m) \otimes n) \otimes p + (-1)^i (m \otimes d(n)) \otimes p + (-1)^{i+j} (m \otimes n) \otimes d(p).$$

Now instead first apply $\alpha_{M,N,P}$ to $m \otimes (n \otimes p)$ so that we get $(m \otimes n) \otimes p$. Then upon applying the differential d we get

$$\begin{aligned} d((m \otimes n) \otimes p) &= d(m \otimes n) \otimes p + (-1)^{i+j} (m \otimes n) \otimes d(p) \\ &= (d(m) \otimes n + (-1)^i m \otimes d(n)) \otimes p + (-1)^{i+j} (m \otimes n) \otimes d(p) \\ &= (d(m) \otimes n) \otimes p + (-1)^i (m \otimes d(n)) \otimes p + (-1)^{i+j} (m \otimes n) \otimes d(p). \end{aligned}$$

Hence $d \circ \alpha_{M,N,P} = \alpha_{M,N,P} \circ d$, modeling the chain-map condition. There is an "obvious" mutal inverse map given by $\alpha^{-1}_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ with $\alpha^{-1}_{M,N,P}((m \otimes n) \otimes p) = m \otimes (n \otimes p)$. With respect to naturality, one may check by an easy computation that on pure tensors $m \otimes (n \otimes p)$ the following square commutes

$$\begin{array}{ccc} M \otimes (N \otimes P) & \xrightarrow{\alpha_{M,N,P}} & (M \otimes N) \otimes P \\ \downarrow {\scriptstyle f \otimes (g \otimes h)} & & \downarrow {\scriptstyle (f \otimes g) \otimes h} \\ M' \otimes (N' \otimes P') & \xrightarrow[\alpha_{M',N',P'}]{} & M' \otimes (N' \otimes P') \end{array}$$

with chain maps $M \xrightarrow{f} M', N \xrightarrow{g} N', P \xrightarrow{h} P'$ in $\mathsf{Ch}_{\mathbb{k}}$.

Left unitor λ_M: Define $\lambda_M : \underset{\sim}{\mathbb{k}} \otimes M \to M$ on pure tenors $r \otimes m$ by the "contraction" $r \otimes m \mapsto rm$ with $r \in \mathbb{k}$ and $m \in M_i$. Since $rm \in M_i$, this map *preserves degree.* Since $d(r) = 0$ for any $r \in \mathbb{k}$, it follows that $d \circ \lambda_M = \lambda_m \circ d$, modeling the chain-map condition. Furthermore, in each degree i, λ_m is (up to canonical isomorphism) precisely the left-unitor coming from $\mathsf{Mod}_{\mathbb{k}}$, hence an isomorphism.

With respect to naturality, one checks that the following diagram commutes on pure tensors $r \otimes m$,

$$\begin{array}{ccc} \underset{\sim}{\mathbb{k}} \otimes M & \xrightarrow{\lambda_M} & M \\ \downarrow {\scriptstyle \mathsf{id} \otimes f} & & \downarrow {\scriptstyle f} \\ \underset{\sim}{\mathbb{k}} \otimes M' & \xrightarrow[\lambda_{M'}]{} & M' \end{array},$$

where $M \xrightarrow{f} M'$ is a chain map in $\mathsf{Ch}_{\Bbbk}$, using that f is $\Bbbk$-linear.

Right unitor ρ_M: Defining $\rho_M : M \otimes \underset{\sim}{\Bbbk} \to M$ by the contraction $m \otimes r \mapsto mr$ for $r \in \Bbbk$ and $m \in M_i$, up to a permutation of the factors the rest of the computations are essentially the same as for the left unitor λ_M.

Symmetrizer σ: We define $\sigma_{M,N} : M \otimes N \to N \otimes M$ on pure tensors $m \otimes n$ by

$$m \otimes n \mapsto (-1)^{ij} n \otimes m,$$

with $m \in M_i$ and $n \in N_j$. This map is degree preserving. We see that for $m \in M_i$ and $n \in N_j$,

$$\begin{aligned} d(\sigma_{M,N}(m \otimes n)) &= (-1)^{ij} d(n \otimes m) \\ &= (-1)^{ij} d(n) \otimes m + (-1)^{ij+j} n \otimes d(m), \end{aligned}$$

while

$$\begin{aligned} \sigma_{M,N}(d(m \otimes n)) &= \sigma_{M,N}(d(m) \otimes n + (-1)^i m \otimes d(n)) \\ &= (-1)^{(i-1)j} n \otimes d(m) + (-1)^i (-1)^{i(j-1)} d(n) \otimes m \\ &= (-1)^{ij} d(n) \otimes m + (-1)^{ij-j} n \otimes d(m) \\ &= (-1)^{ij} d(n) \otimes m + (-1)^{ij+j} n \otimes d(m), \end{aligned}$$

where we in the last equality used that

$$\begin{aligned} (-1)^{ij-j} &= (-1)^{ij+j} \\ \Leftrightarrow 1 &= (-1)^{2j}, \qquad \text{by multiplying both sides by } (-1)^{-ij+j}. \end{aligned}$$

This models the chain condition $d \circ \sigma_{M,N} = \sigma_{M,N} \circ d$. One checks that on pure tensors there is an inverse to $\sigma_{M,N}$ given by $\sigma_{N,M}$, since

$$\begin{aligned} \sigma_{N,M}(\sigma_{M,N}(m \otimes n)) &= \sigma_{N,M}\left((-1)^{ij} n \otimes m\right) \\ &= (-1)ji(-1)^{ij} m \otimes n \\ &= (-1)^{2ij} m \otimes n \\ &= m \otimes n, \end{aligned}$$

with $m \in M_i$ and $n \in N_j$.

With respect to naturality, one checks that on pure tensors $m \otimes n$ with chain maps $M \xrightarrow{f} M'$ and $N \xrightarrow{g} N'$, the following square commutes

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\sigma_{M,N}} & N \otimes M \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle g \otimes f} \\ M' \otimes N' & \xrightarrow[\sigma_{M',N'}]{} & N' \otimes M' \end{array},$$

using that $f \otimes g$ and $g \otimes f$ preserve the *total* degree of any pure tensor (following from the fact that f and g *individually* preserve degrees).

What is left to check is that:

(a) The functors $M \rightsquigarrow \underset{\sim}{\mathbb{k}} \otimes M$ and $M \rightsquigarrow M \otimes \underset{\sim}{\mathbb{k}}$ are *fully faithful.*

(b) The *pentagon* axiom is satisfied, i.e. that the following diagram commutes, for all objects $M, N, P, T \in \mathsf{Ch}_{\mathbb{k}}$,

$$\begin{array}{ccccc} & & M \otimes (N \otimes (P \otimes T)) & & \\ & \swarrow^{\mathsf{id} \otimes \alpha_{N,P,T}} & & \searrow^{\alpha_{M,N,P \otimes T}} & \\ M \otimes ((N \otimes P) \otimes T) & & & & (M \otimes N) \otimes (P \otimes T) \\ \downarrow{\scriptstyle \alpha_{M,N \otimes P,T}} & & & & \downarrow{\scriptstyle \alpha_{M \otimes N,P,T}} \\ (M \otimes (N \otimes P)) \otimes T & & \xrightarrow[\alpha_{M,N,P} \otimes \mathsf{id}]{} & & ((M \otimes N) \otimes P) \otimes T \end{array}.$$

(c) The *hexagon* axiom is satisfied, i.e. that the following diagram commutes, for all objects $M, N, P \in \mathsf{Ch}_{\mathbb{k}}$,

$$\begin{array}{ccccc} & M \otimes (N \otimes P) & \xrightarrow{\alpha_{M,N,P}} & (M \otimes N) \otimes P & \\ \swarrow^{\mathsf{id} \otimes \sigma_{N,P}} & & & & \searrow^{\sigma_{M \otimes N,P}} \\ M \otimes (P \otimes N) & & & & P \otimes (M \otimes N) \\ \searrow_{\alpha_{M,P,N}} & & & & \swarrow_{\alpha_{P,M,N}} \\ & (M \otimes P) \otimes N & \xrightarrow[\sigma_{M,P} \otimes \mathsf{id}]{} & (P \otimes M) \otimes N & \end{array}.$$

<u>(a)</u>: Since we gave a sketch for why the left and right unitor λ and ρ respectively are natural isomorphisms, in particular naturality says precisely that there are natural isomorphisms

$$\left(\underset{\sim}{\mathbb{k}} \otimes (\,\cdot\,)\right) \overset{\lambda}{\Rightarrow} \mathsf{id}_{\mathsf{Ch}_{\mathbb{k}}} \qquad \text{and} \qquad \left((\,\cdot\,) \otimes \underset{\sim}{\mathbb{k}}\right) \overset{\rho}{\Rightarrow} \mathsf{id}_{\mathsf{Ch}_{\mathbb{k}}}.$$

With respect to λ, we need to show that $\mathsf{hom}_{\mathsf{Ch}_{\mathbb{k}}}(M, N) \xrightarrow{F_{M,N}} \mathsf{hom}_{\mathsf{Ch}_{\mathbb{k}}}\left(\underset{\sim}{\mathbb{k}} \otimes M, \underset{\sim}{\mathbb{k}} \otimes N\right)$ defined by $f \mapsto \mathsf{id}_{\mathbb{k}} \otimes f$ is *bijective* for all objects $M, N \in \mathsf{Ch}_{\mathbb{k}}$. Naturality then says that $F_{M,N}(f) = \mathsf{id}_{\mathbb{k}} \otimes f = \lambda_N^{-1} \circ f \circ \lambda_M$. There is therefore an obvious two-sided inverse defined by

$$\mathsf{hom}_{\mathsf{Ch}_{\mathbb{k}}}\left(\underset{\sim}{\mathbb{k}} \otimes M, \underset{\sim}{\mathbb{k}} \otimes N\right) \xrightarrow{G_{M,N}} \mathsf{hom}_{\mathsf{Ch}_{\mathbb{k}}}(M, N), \qquad f \mapsto \lambda_N^{-1} \circ f \circ \lambda_M.$$

Hence $F_{M,N}$ is a bijection. We claim that (up to permuting the factors in the tensor product) the same argument works for ρ.

<u>(b)</u>: The computations on pure tensors follow through for essentially the same reason that $\mathsf{Mod}_{\mathbb{k}}$ with $\otimes_{\mathbb{k}}$ and unit object $\mathbb{k}$, is symmetric monoidal and hence satisfies the pentagon axiom, using that associator in $\mathsf{Ch}_{\mathbb{k}}$ is essentially just the natural extension of the corresponding associator in $\mathsf{Mod}_{\mathbb{k}}$.

<u>(c)</u>: We check this on pure tensors $m \otimes (n \otimes p)$ with $m \in M_i, n \in N_j$ and $p \in P_k$. We have

that

$$\begin{aligned}\alpha_{P,M,N}(\sigma_{M\otimes N,P}(\alpha_{M,N,P}(m\otimes(n\otimes p)))) &= \alpha_{P,M,N}(\sigma_{M\otimes N,P}((m\otimes n)\otimes p)\\ &= (-1)^{(i+j)k}\alpha_{P,M,N}(p\otimes(m\otimes n))\\ &= (-1)^{(i+j)k}(p\otimes m)\otimes n,\end{aligned}$$

and

$$\begin{aligned}\sigma_{M,P}\otimes \mathsf{id}(\alpha_{M,P,N}(\mathsf{id}\otimes\sigma_{N,P}(m\otimes(n\otimes p)))) &= (-1)^{j+k}\sigma_{M,P}\otimes\mathsf{id}(\alpha_{M,P,N}(m\otimes(p\otimes n))\\ &= (-1)^{jk}\sigma_{M,P}\otimes\mathsf{id}((m\otimes p)\otimes n)\\ &= (-1)^{jk}\cdot(-1)^{ik}(p\otimes m)\otimes n\\ &= (-1)^{(i+j)k}(p\otimes m)\otimes n.\end{aligned}$$

Comparing the end result of the two computations above, we see that they agree.

□

Theorem 2.5.8. *Fixing two complexes $M, N \in \mathsf{Ch}_{\Bbbk}$, the map $[m]\otimes[n]\mapsto[m\otimes n]$ defines a $\Bbbk$-module homomorphism $\mathsf{H}_i(M)\otimes\mathsf{H}_j(N)\to\mathsf{H}_{i+j}(M\otimes N)$.*

This $\Bbbk$-module homomorphism then induces a map $\bigoplus_{i+j=\ell}\mathsf{H}_i(M)\otimes\mathsf{H}_j(N)\to\mathsf{H}_\ell(M\otimes N)$.

Proof. We start by defining a map

$$\varphi:\mathsf{H}_i(M)\times\mathsf{H}_j(N)\to\mathsf{H}_{i+j}(M\otimes N),\qquad ([m],[n])\mapsto[m\otimes n].$$

We must then show that $[m\otimes n]$ is well-defined in the sense that $m\otimes n$ is a cycle if m,n are cycles. This follows directly from the computation

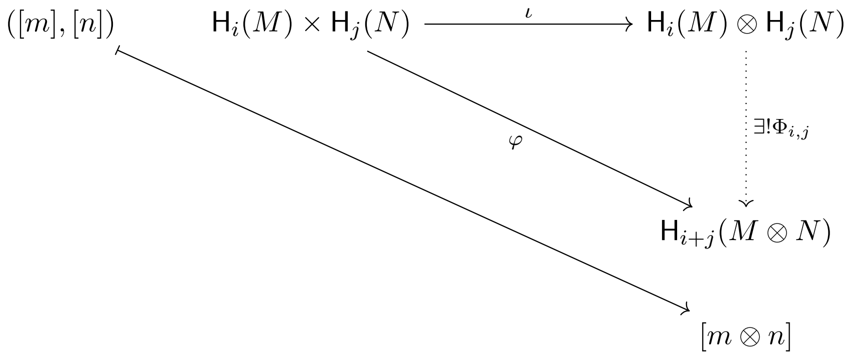
$$\begin{aligned}d(m\otimes n) &= \underbrace{\underbrace{d_i^M(m)}_{=0}\otimes n}_{=0} + \underbrace{(-1)^i\, m\otimes \underbrace{d_j^N(n)}_{=0}}_{=0}\\ &= 0.\end{aligned}$$

We show that the map is *well-defined*: Let $[m]=[m']$ and $[n]=[n']$, with representatives $m,m'\in\mathsf{ker} d_i^M$ and $n,n'\in\mathsf{ker}\left(d_j^N\right)$. Then $m-m'\in\mathsf{im}\left(d_{i+1}^M\right)$ and $n-n'\in\mathsf{im}\, d_{j+1}^N$, which means that we may write $m=m'+d_{i+1}^M(a)$ and $n=n'+d_{j+1}^N(b)$ for $a\in M_{i+1}$ and $b\in N_{j+1}$. We want to show that it follows that $[m\otimes n]=[m'\otimes n']$ i.e. that $m\otimes n-m'\otimes n'\in\mathsf{im}\left(d_{i+j+1}^{M\otimes N}\right)$. We find that

$$\begin{aligned}d_{i+j+1}^{M\otimes N}(a\otimes n'+(-1)^i m\otimes b) &= d_{i+j+1}^{M\otimes N}(a\otimes n')+(-1)^i d_{i+j+1}^{M\otimes N}(m\otimes b)\\ &= d_{i+1}^M(a)\otimes n'+\underbrace{(-1)^{i+1}a\otimes\underbrace{d_j^N(n')}_{=0}}_{=0}+\underbrace{(-1)^i\underbrace{d_i^M(m)}_{=0}\otimes b}_{=0}+(-1)^{2i}m\otimes d_{j+1}^N(b)\\ &= (m-m')\otimes n'+m\otimes(n-n')\\ &= \cancel{m\otimes n'}-m'\otimes n'+m\otimes n-\cancel{m\otimes n'}\\ &= m\otimes n-m'\otimes n',\end{aligned}$$

so that $m\otimes n$ and $m'\otimes n'$ differ by a boundary. We conclude that the map is well-defined.

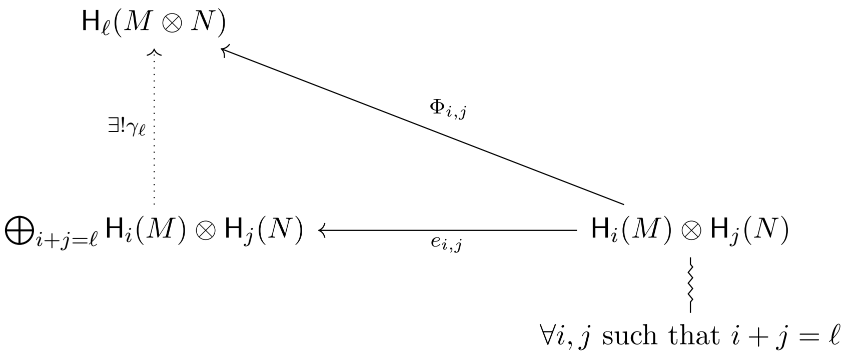
Routine calculations show that φ is a $\Bbbk$-bilinear map, so by the universal property of the tensor product, we get a unique $\Bbbk$-module homomorphism $\Phi_{i,j} : \mathsf{H}_i(M) \otimes \mathsf{H}_j(N) \to \mathsf{H}_{i+j}(M \otimes N)$ so that our original map φ factors through the "inclusion" $([m],[n]) \overset{\iota}{\mapsto} [m] \otimes [n]$ and the dashed arrow $\Phi_{i,j}$ as below,



(2.5.5)

Then, by definition, $\Phi_{i,j}([m] \otimes [n]) = [m \otimes n]$.

By the universal property of the coproduct $\bigoplus_{i+j=\ell} \mathsf{H}_i(M) \otimes \mathsf{H}_j(N)$, we get maps γ_ℓ as below,



, (2.5.6)

where $e_{i,j}$ are the canonical coproduct maps and where $\gamma_\ell \circ e_{i,j} = \Phi_{i,j}$ for all pairs (i,j) with $i + j = \ell$. □

The below statement, is sometimes classified as a "Künneth theorem".

Theorem 2.5.9. *If* $\Bbbk$ *is a field, then the maps* γ_ℓ *constructed in 2.5.6 are isomorphisms.*

Proof. Since the differentials are $\Bbbk$-module homomorphisms, $\mathsf{Z}_n(M)$ and $\mathsf{B}_n(M)$ are $\Bbbk$-modules. Since $\Bbbk$ is a field, they are vector spaces over $\Bbbk$. Since $\mathsf{B}_n(M) \subseteq \mathsf{Z}_n(M)$ we have that $\mathsf{B}_n(M)$ is a vector-subspace of $\mathsf{Z}_n(M)$ there is a complement $\mathsf{R}_n(M) \subseteq \mathsf{Z}_n(M)$ such that $\mathsf{Z}_n(M) = \mathsf{B}_n(M) \oplus \mathsf{R}_n(M)$. We observe that essentially by construction, $\mathsf{H}_n(M) \cong \mathsf{R}_n(M)$. By the same reasoning applied to M_n and the vector subspace $\mathsf{Z}_n(M) \subseteq M_n$ we can find a direct-sum complement $\mathsf{L}_n(M)$ to $\mathsf{Z}_n(M)$ such that

$$M_n = \mathsf{Z}_n(M) \oplus \mathsf{L}_n(M).$$

Taking the two direct-sum decompositions together we then see that

$$M_n = \mathsf{B}_n(M) \oplus \mathsf{R}_n(M) \oplus \mathsf{L}_n(M).$$

We then observe that $d^M_n(\mathsf{R}_n(M)) = d^M_n(\mathsf{B}_n(M)) = 0$ since $\mathsf{B}_n(M) \subseteq \mathsf{Z}_n(M)$ and $\mathsf{R}_n(M) \subseteq \mathsf{Z}_n(M)$.

The restriction (and corestriction) of d^M_n to $\mathsf{d}_n := (d^{\text{corest.}})^M_n|_{\mathsf{L}_n(M)} : \mathsf{L}_n(M) \to \mathsf{B}_{n-1}(M)$ is bijective: It is injective since $\mathsf{L}_n(M) \cap \mathsf{Z}_n(M) = 0$, and by the direct-sum decomposition $M_n = \mathsf{Z}_n(M) \oplus \mathsf{L}_n(M)$ we see that for each $x \in \mathsf{B}_{n-1}(M)$ there is some $l \in \mathsf{L}_n(M)$ such that $\mathsf{d}_n(l) = x$, so that d_n is surjective.

Then, up to canonical isomorphism by permuting factors, $M \cong \mathsf{R}(M) \oplus \mathsf{D}(M)$ with $\mathsf{R}(M)_n = \mathsf{R}_n(M)$ and $\mathsf{D}(M)_n = \mathsf{B}_n(M) \oplus \mathsf{L}_n(M)$. Here $\mathsf{D}(M)$ and $\mathsf{R}(M)$ are both subcomplexes of M with the differential inherited from M. Define $S_n : \mathsf{D}(M)_n \to \mathsf{D}(M)_{n+1}$ by $S_n(b+\ell) := \mathsf{d}^{-1}_{n+1}(b) \in \mathsf{L}_{n+1}(M) \subseteq \mathsf{D}(M)_{n+1}$. Then we see that

$$\begin{aligned}(d^M_{n+1}S_n + S_{n-1}d^M_n)(b+l) &= d^m_{n+1}S_n(b+l) + S_{n-1}d^M_n(b+l)\\ &= \underbrace{d^M_{n+1}\mathsf{d}^{-1}_{n+1}}_{\mathsf{id}}(b) + \mathsf{d}^{-1}_n d^M_n(l), \qquad \text{since } d^M_n(b) = 0\\ &= b + l,\end{aligned}$$

so that

$$d^{\mathsf{D}(M)}S + Sd^{\mathsf{D}(M)} = \mathsf{id}_{\mathsf{D}(M)},$$

i.e., $\mathsf{D}(M)$ is contractible. The same computations gives that $N = \mathsf{R}(N) \oplus \mathsf{D}(N)$ with $\mathsf{D}(N)$ contractible.

We will not prove this, but we may assume that the symmetric monoidal structure on $\mathsf{Ch}_{\Bbbk}$ interacts in the same fashion with the direct sum as $(\mathsf{Mod}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ does with $\oplus$, i.e., we have all the "nice" canonical isomorphisms in $\mathsf{Ch}_{\Bbbk}$ by extension from $\mathsf{Mod}_{\Bbbk}$ with respect to $\otimes_{\Bbbk}$ and $\oplus$, but the with the corresponding notions in $\mathsf{Ch}_{\Bbbk}$. We then see that

$$\begin{aligned}M \otimes N &= (\mathsf{R}(M) \oplus \mathsf{D}(M)) \otimes (\mathsf{R}(N) \oplus \mathsf{D}(N))\\ &\approx (\mathsf{R}(M) \otimes \mathsf{R}(N)) \oplus (\mathsf{R}(M) \otimes \mathsf{D}(N)) \oplus (\mathsf{D}(M) \otimes \mathsf{R}(N)) \oplus (\mathsf{D}(M) \otimes \mathsf{D}(N))\\ &= \mathsf{A} \oplus \mathsf{K},\end{aligned}$$

with $\mathsf{A} := \mathsf{R}(M) \otimes \mathsf{R}(N)$ and $\mathsf{K} = (\mathsf{R}(M) \otimes \mathsf{D}(N)) \oplus (\mathsf{D}(M) \otimes \mathsf{R}(N)) \oplus (\mathsf{D}(M) \otimes \mathsf{D}(N)$.

Lemma 2.5.10. *If $C \in \mathsf{Ch}_{\Bbbk}$ is contractible then $C \otimes M$ is contractible, for arbitrary $M \in \mathsf{Ch}_{\Bbbk}$.*

Proof. Sketch: Let $(s_n : C_n \to C_{n+1})_{n\in\mathbb{Z}}$ be a witness to C being contractible, so that $d^C s + sd^C = \mathsf{id}_C$. Define $S := (S_n)_{n\in\mathbb{Z}}$ on pure tensors $c \otimes m$ in degree ℓ with $c \in C_i$ and $m \in M_j$ as

$$S_\ell(c \otimes m) := s_i(c) \otimes m \in C_{i+1} \otimes M_j \subset (C \otimes M)_{\ell+1},$$

i.e. S_ℓ restricted to a direct summand $C_i \otimes M_j$ is $s_i \otimes \mathsf{id}_{M_j}$, which is $\Bbbk$-linear.

Then we check that, with d the differential for $C \otimes M$, we have

$$\begin{aligned}(d_{\ell+1}S_\ell + S_{\ell-1}d_\ell)(c \otimes m) &= d_{\ell+1}(s_i(c) \otimes m) + S_{\ell-1}d_\ell(c \otimes m)\\ &= d_{i+1}s_i(c) \otimes m + \cancel{(-1)^{i+1}s_i(c) \otimes d_j(m)} + s_{i-1}d_i(c) \otimes m + \cancel{(-1)^i s_i(c) \otimes d_j(m)}\\ &= (d_{i+1}s_i(c) + s_{i-1}d_i(c)) \otimes m, \qquad \text{by linearity in the first factor of } \otimes\\ &= c \otimes m, \qquad \text{since } ds + sd = \mathsf{id}_C,\\ \Rightarrow dS + Sd &= \mathsf{id}_{C\otimes M}.\end{aligned}$$

□

By two applications of the lemma, it follows that K is contractible.

Therefore, by functoriality, we see that

$$\begin{aligned}\mathsf{H}_\ell(M\otimes N) &\cong \mathsf{H}_\ell(\mathsf{A}\oplus\mathsf{K}) \qquad (2.5.7)\\ &\cong \mathsf{H}_\ell(A)\oplus\mathsf{H}_\ell(\mathsf{K}), \qquad \text{since } \mathsf{H}_\ell \text{ is \textit{additive}, i.e. commutes with direct sums}\\ &\cong \mathsf{H}_\ell(A), \qquad \text{using that } \mathsf{H}_\ell(\mathsf{K})=0 \text{ since } \mathsf{K} \text{ is contractible}\\ &= \mathsf{A}_\ell, \qquad \text{since the differentials of } A \text{ are zero}\\ &= \bigoplus_{i+j=\ell}\mathsf{R}i(M)\otimes\mathsf{R}_j(N) \qquad (2.5.8)\end{aligned}$$

Tensoring the isomorphisms

$$\begin{cases}\mathsf{R}_i(M)\xrightarrow[\cong]{f_i}\mathsf{H}_i(M), & r\mapsto[r]\\ \mathsf{R}_j(N)\xrightarrow[\cong]{g_j}\mathsf{H}_j(N), & r'\mapsto[r']\end{cases}$$

gives an isomorphism (since $(\,\cdot\,)\otimes(\,\cdot\,)$ is a bifunctor)

$$\mathsf{R}_i(M)\otimes\mathsf{R}_j(M)\xrightarrow{f\otimes g}\mathsf{H}_i(M)\otimes\mathsf{H}_j(N), \qquad r\otimes r'\mapsto[r]\otimes[r'].$$

By using the universal property of $\bigoplus$ (it is a coproduct in $\mathsf{Mod}_{\mathbb{k}}$) it follows that we have a unique isomorphism

$$A_\ell\xrightarrow[\cong]{\Theta_\ell}\bigoplus_{i+j=\ell}\mathsf{H}_i(M)\otimes\mathsf{H}_j(N),$$

such that $\Theta_{i,j}\circ\iota_{ij}=e_{i,j}\circ f_i\otimes g_j$ with $\mathsf{R}_i(M)\oplus\mathsf{R}_j(N)\xrightarrow{\iota_{i,j}}A_\ell$ the canonical coproduct inclusions, for all i,j such that $i+j=\ell$ and $e_{i,j}$ the canonical coproduct maps in 2.5.6.

Let $\zeta_\ell:\mathsf{H}_\ell(M\otimes N)\underset{\cong}{\rightarrow}A_\ell$ be the isomorphism coming from the composition of the isomorphisms in 2.5.7 to 2.5.8. Then one sees that an element

$$\mathsf{H}_\ell(M\otimes N)\ni[m\otimes n]=[(r_i+\mathfrak{d}_i)\otimes(r_j+\mathfrak{d}_j)]$$

is sent to $\iota_{i,j}(r_i\otimes r_j)\in\bigoplus_{i+j=\ell}\mathsf{R}_i(M)\otimes\mathsf{R}_j(N)$ with $r_i\in\mathsf{R}_i(M)$ and $\mathfrak{d}_i\in\mathsf{D}_i(M)$, $r_j\in\mathsf{R}_j(N)$ and $\mathfrak{d}_j\in\mathsf{D}_j(N)$ under ζ_ℓ. Let γ_ℓ be as in 2.5.6.

Then we find that

$$\begin{aligned}(\Theta_\ell\circ\iota_{i,j})(r\otimes s)&=e_{i,j}(f_i\otimes g_j)(r\otimes s)\\ &=e_{i,j}([r]\otimes[s])\\ \Rightarrow(\gamma_\ell\circ\Theta_\ell\circ\iota_{i,j})(r\otimes s)&=\gamma_\ell([r]\otimes[s])\\ &=[r\otimes s],\\ \Rightarrow(\zeta_\ell\circ\gamma_\ell\circ\Theta_\ell)(\iota_{i,j})(r\otimes s)&=\iota_{i,j}(r\otimes s),\end{aligned}$$

for all i,j with $i+j=\ell$ and all $r\otimes s$ in $\mathsf{R}_i(M)\oplus\mathsf{R}_j(N)$. By the universal property of A_ℓ as a coproduct it follows that

$$\begin{aligned}&\zeta_\ell\circ\gamma_\ell\circ\Theta_\ell=\mathrm{id}_{\mathsf{A}_\ell}\\ &\Leftrightarrow\gamma_\ell=\zeta_\ell^{-1}\circ\Theta_\ell^{-1},\end{aligned}$$

so that γ_ℓ is an isomorphism. □

Chapter 3

Homotopy, Homotopy kernels and Homotopy cokernels

3.1 Homotopies

If we let $f : M \to M_0$ be an R-module homomorphism, then this gives us a chain complex $(M_\bullet, (d_i)_{i\in\mathbb{Z}})$ with M_1 and M_0 in degree one and degree zero, respectively, and

$$d_i = \begin{cases} f, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases} .$$

Since Mod_R is an abelian category it has all cokernels, so there is a canonical cokernel map $M_0 \to \mathsf{coker}(f)$, which in turn induces a chain map $M_\bullet \xrightarrow{k} \underset{\sim}{\mathsf{coker}(f)}$,

$$\begin{array}{ccc} M_1 & \xrightarrow{\quad f \quad} & M_0 \\ {\scriptstyle k_1}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle k_0} \\ 0 & \xrightarrow{\qquad} & \mathsf{coker}(f) \end{array} .$$

For $n = 0$, we see that

$$\begin{aligned} \mathsf{H}_0(M_\bullet) &= \ker(d_0)/\operatorname{im} f \\ &= M_0/\operatorname{im}(f) \\ &= \mathsf{coker}(f). \end{aligned}$$

For $n = 1$ we get

$$\begin{aligned} \mathsf{H}_1(M_\bullet) &= \ker(d_1)/\operatorname{im}(d_2) \\ &= \ker(f)/(0) \\ &= \ker(f). \end{aligned}$$

If $n \notin \{0,1\}$ then $\mathsf{ker}(d_i) = 0$ so that $\mathsf{H}_n(M_\bullet) = 0$. Summarizing, we have

$$\mathsf{H}_n(M_\bullet) = \begin{cases} \mathsf{coker}(f), & \text{if } n = 0 \\ \mathsf{ker}(f), & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases} .$$

Claim: $\mathsf{H}_n(k) : \mathsf{H}_n(M_\bullet) \to \mathsf{H}_n\left(\underset{\sim}{\mathsf{coker}(f)}\right)$ is an isomorphism for all n, i.e. a quasi-isomorphism, if and only if f is injective.

Proof. $\Rightarrow$: We observe that $\mathsf{ker}\left(d_n^{\underset{\sim}{\mathsf{coker}(f)}}\right) = 0$ for $n \neq 0$ so that $\mathsf{H}_n\left(\underset{\sim}{\mathsf{coker}(f)}\right) = 0$ for $n \neq 0$. In particular, it follows that $\mathsf{ker}(f) \cong 0$.

$\Leftarrow$: If f is injective then $\mathsf{ker}(f) = 0$, so that $\mathsf{H}_n(M_\bullet) = 0$ for $n \neq 0$. Since $\mathsf{H}_n\left(\underset{\sim}{\mathsf{coker}(f)}\right) = 0$ for $n \neq 0$, $\mathsf{H}_n(k)$ is an isomorphism for $n \neq 0$. For $n = 0$, we see that

$$\begin{aligned} \mathsf{H}_0\left(\underset{\sim}{\mathsf{coker}(f)}\right) &= \mathsf{ker}\left(d_0^{\underset{\sim}{\mathsf{coker}(f)}}\right) \Big/ \mathsf{im}\left(d_1^{\underset{\sim}{\mathsf{coker}(f)}}\right) \\ &= \mathsf{coker}(f)/(0) \\ &= \mathsf{coker}(f). \end{aligned}$$

We observe that $\mathsf{H}_0(k)$ is induced from diagram 2.3.7. Since the cokernel map k_0 is surjective, it follows that the induced map on homology-groups in degree zero is an isomorphism that acts as the identity on elements. We conclude that k is a quasi-isomorphism. ☐

One way to view $M_\bullet$ is as a presentation of the R-module $\mathsf{H}_0(M_\bullet) \cong \mathsf{coker}(f)$, since the generators for this R-module is M_0, and M_1 is the source of the relations, in the sense that the image of f determines which elements gets killed off in the quotient $\mathsf{coker}(f)$, i.e. in the sense that each element m in M_1 gives that $f(m) \sim 0$ is a relation in $\mathsf{coker}(f)$. What $\mathsf{H}_1(M_\bullet)$ does is (in some sense) measures how overdetermined $\mathsf{coker}(f)$ is, in that if $\mathsf{ker}(f) \neq 0$ there is some $m \neq 0 \in M_1$ so that $f(m) = 0$, so that $\mathsf{ker}(f)$ in fact provides no further constraints for the construction of $\mathsf{coker}(f)$. We claim that this chain complex $M_\bullet \in \mathsf{Ch}_R$ is, in relation to $f : M_1 \to M_0$ of R-modules, an example of an object associated to something we call a **homotopy cokernel** of f.

Definition 3.1.1 (Chain-homotopy)**.** If $M_\bullet, N_\bullet$ are chain complexes in Ch_R and $f_\bullet, g_\bullet : M_\bullet \rightrightarrows N_\bullet$ are chain maps, such that there exists a family of R-module homomorphisms

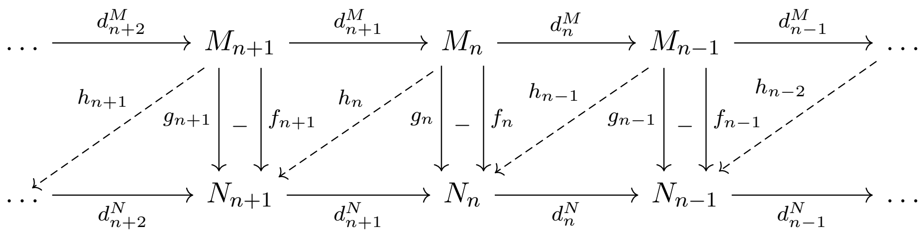
$$h_\bullet := (h_n : M_n \to N_{n+1})_{n \in \mathbb{Z}}$$

such that

$$g_n - f_n = d_{n+1}^N h_n + h_{n-1} d_n^M \tag{3.1.1}$$

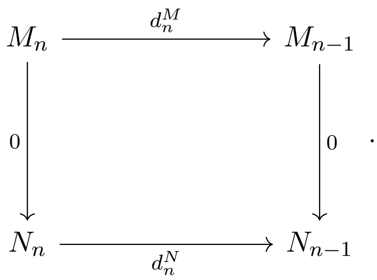
holds for all n, then we say that h is a **chain-homotopy** between $f_\bullet$ and $g_\bullet$, which we denote as $f_\bullet \overset{h_\bullet}{\Rightarrow} g_\bullet$.

The defining equation 3.1.1 may be depicted (mnemonically) as in the diagram below,



and can be understood as reading that the addition of the purple triangle (without f_n) in the second square from the left and the upper green triangle in the third square from the left (leaving out g_n) equals $g_n - f_n$.

Definition 3.1.2 (Null-homotopy)**.** Letting $0_\bullet : M_\bullet \to N_\bullet$ be the chain map defined in each degree n by the commutative square



Then if $g_\bullet : M_\bullet \to N_\bullet$ is a chain map such that there is a chain-homotopy $0_\bullet \overset{h_\bullet}{\Rrightarrow} g_\bullet$, then we say that $h_\bullet$ is a **null-homotopy** of $g_\bullet$.

Definition 3.1.3 (Contraction)**.** A **contraction** of a complex $M_\bullet \in \mathsf{Ch}_R$ is a nullhomotopy $0_\bullet \overset{h_\bullet}{\Rrightarrow} \mathrm{id}_{M_\bullet}$. Complexes that admit contractions are called **contractible**.

Theorem 3.1.4. *The property of being chain-homotopic defines an equivalence-relation on* $\mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$.

Proof. Reflexive: If we let $s_n : M_n \to N_{n+1}$ be the zero-homomorphism for all n, then

$$f_n - f_n = d^N_{n+1} s_n + s_{n-1} d^M_n = 0.$$

Hence $f_\bullet \sim f_\bullet$.

Symmetric: If $f_\bullet \sim g_\bullet$ realized by $f_\bullet \overset{h_\bullet}{\Rrightarrow} g_\bullet$ then define $\gamma_n := -h_n : M_n \to N_{n+1}$. We then see that

$$\begin{aligned} d^N_{n+1}\gamma_n + \gamma_{n-1} d^M_n &= -(d^N_{n+1} h_n + h_{n-1} d^M_n) \\ &= -(g_n - f_n) \\ &= f_n - g_n. \end{aligned}$$

Hence with $\gamma_\bullet := (\gamma_n)$ we have that $g_\bullet \overset{\gamma_\bullet}{\Rrightarrow} f_\bullet$ is a chain-homotopy.

Transitive: Assume $f_\bullet \overset{\theta_\bullet}{\Rrightarrow} g_\bullet$ and $g_\bullet \overset{\ell_\bullet}{\Rrightarrow} h_\bullet$. Then define $\gamma_\bullet := (\theta_n + \ell_n)_n$. We then see that

$$\begin{aligned} d^N_{n+1}\gamma_n + \gamma_{n-1}d^M_n &= d^N_{n+1} \circ (\theta_n + \ell_n) + (\theta_{n-1} + \ell_{n-1}) \circ d^M_n \\ &= \left(d^N_{n+1}\ell_n + \ell_{n-1}d^M_n\right) + \left(d^N_{n+1}\theta_n + \theta_{n-1}d^M_n\right) \\ &= (h_n - g_n) + (g_n - f_n) \\ &= h_n - f_n. \end{aligned}$$

Hence $f_\bullet \sim h_\bullet$ is witnessed by $f_\bullet \overset{\gamma_\bullet}{\Rrightarrow} h_\bullet$. □

The interval object $\mathbb{I} \in \mathsf{Ch}_{\Bbbk}$ introduced below should perhaps be thought of as an (algebraic or categorical) counterpart to the standard 1-simplex $\Delta^1 = \left\{(t_0, t_1) \ : \in \mathbb{R}^2 \ : \ t_0 + t_1 = 1, t_i \geqslant 0\right\}$ (for a more precise claim, see [Maz23, Footnote 33, p. 17]).

Definition 3.1.5 (Interval object $\mathbb{I}$)**.** Let $\mathbb{I} := (\mathbb{I}_n, d^{\mathbb{I}}_n)$ be defined on objects by

$$\mathbb{I}_n := \begin{cases} \Bbbk, \ \text{if } n = 1, \\ \Bbbk \oplus \Bbbk, \ \text{if } n = 0 \\ 0, \ \text{if } n \neq 0, 1 \end{cases},$$

with differentials

$$d^{\mathbb{I}}_n := \begin{cases} (-\operatorname{id}_{\Bbbk}, \operatorname{id}_{\Bbbk}) \text{ by } a \mapsto (-a, a), \ \text{if } n = 1, \\ 0, \ \text{if } n \neq 1 \end{cases}.$$

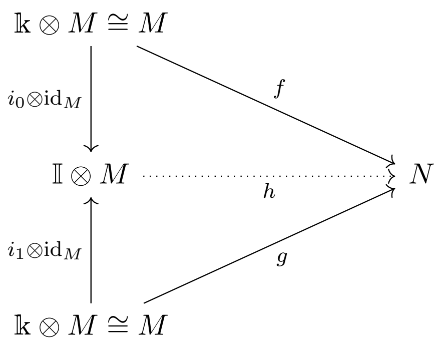
Second, we associate to this object two chain-maps $i_0, i_1 \in \hom_{\mathsf{Ch}_k}(\Bbbk, \mathbb{I})$ where i_0 and i_1 are the canonical coproduct maps associated with $\Bbbk \oplus \Bbbk$ into the first and second factor, respectively, in degree zero and otherwise are the zero maps.

We may represent this construction as

$$\begin{array}{ccccc} \Bbbk & & 0 & \longrightarrow & \underset{\sim}{\Bbbk} \\ \big\downarrow i_0 & & \big\downarrow & & \big\downarrow (\operatorname{id}_{\Bbbk},0) \\ \mathbb{I} & := & \Bbbk & \xrightarrow{(-\operatorname{id}_{\Bbbk},\operatorname{id}_{\Bbbk})} & \underset{\sim\sim}{\Bbbk \oplus \Bbbk} \\ \big\uparrow i_1 & & \big\uparrow & & \big\uparrow (0,\operatorname{id}_{\Bbbk}) \\ \Bbbk & & 0 & \longrightarrow & \underset{\sim}{\Bbbk} \end{array}$$

We call the data of $(\mathbb{I}, i_0, i_1)$ an **interval object** for the homotopy theory of chain-complexes over $\Bbbk$.

Theorem 3.1.6. *Given morphisms $f, g \in \hom_{\mathsf{Ch}_R}(\Bbbk \otimes M, N)$, a homotopy $f \Rightarrow g$ is* *equivalent to a chain map* $\mathbb{I} \otimes M \xrightarrow{h} N$ *such that the following diagram commutes,*



Remark 3.1.7. As in 2.2.2 we think of $\mathbb{I} \otimes M = \mathbb{I} \otimes_{\Bbbk} M$ as internal to Ch_R by *extension of scalars* in each degree.

Proof. $\underline{\Rightarrow}$: Assume first we are given a chain homotopy $f \overset{\gamma}{\Rightarrow} g$. By definition, we then have maps $(\gamma_n : \Bbbk \otimes M_n \to N_{n+1}) \in \mathsf{Mod}_R$ such that

$$g_n - f_n = d^N_{n+1}\gamma_n + \gamma_{n-1}d^{\Bbbk\otimes M}_n,$$

in each degree n. We note that

$$\begin{aligned}(\Bbbk \otimes M)_n &= \bigoplus_{i+j=n} (\underset{\sim}{\Bbbk})_i \otimes_{\Bbbk} M_j \\ &= \Bbbk \otimes_{\Bbbk} M_n \\ &\simeq M_n \qquad \text{by the canonical isomorphism } r \otimes m \mapsto rm.\end{aligned}$$

Then we see that the differential $d^{\Bbbk\otimes M}$ acts in degree n as

$$\begin{aligned}d^{\Bbbk\otimes M}_n(r \otimes m_n) &= d^{\Bbbk}_0(r) \otimes m + (-1)^0 r \otimes d^M_n(m_n) \\ &= r \otimes d^M(m_n).\end{aligned} \tag{3.1.2}$$

For $\mathbb{I} \otimes M$, we have that

$$\begin{aligned}(\mathbb{I} \otimes M)_n &= \bigoplus_{i+j=n} \mathbb{I}_i \otimes_{\Bbbk} M_j \\ &= (\mathbb{I}_1 \otimes_{\Bbbk} M_{n-1}) \oplus (\mathbb{I}_0 \otimes_{\Bbbk} M_n) \\ &= (\Bbbk \otimes_{\Bbbk} M_{n-1}) \oplus ((\Bbbk \oplus \Bbbk) \otimes_{\Bbbk} M_n).\end{aligned} \tag{3.1.3}$$

Then, in degree n, we find that

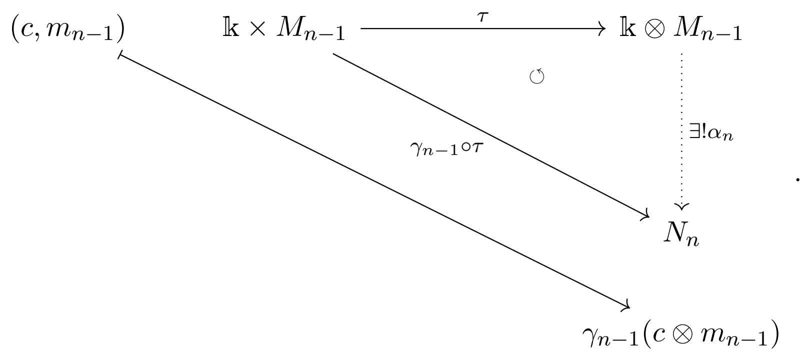
$$(i_0 \otimes \mathrm{id}_M)_n(c \otimes_{\Bbbk} m_n) = (0, (c, 0) \otimes m_n)$$

and

$$(i_1 \otimes \mathrm{id}_M)_n(c \otimes m_n) = (0, (0, c) \otimes m_n),$$

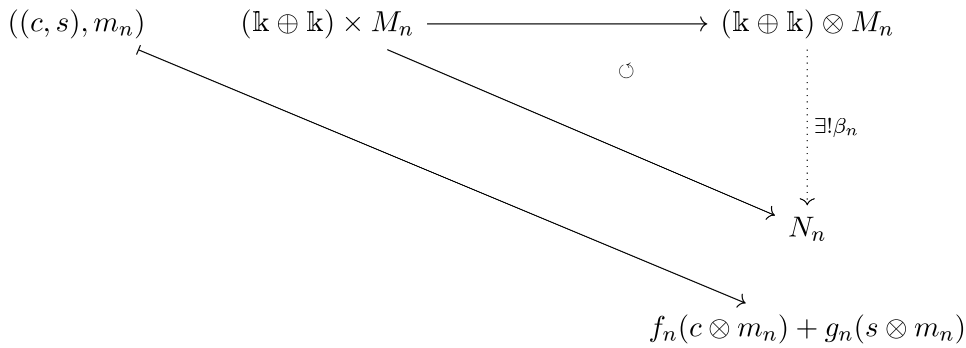
while the map from $\Bbbk \otimes M$ to N following f and g respectively, is defined on elements in degree n as $f_n(c \otimes m_n)$ and $g_n(c \otimes m_n)$.

We will construct the requisite map h by defining it degree-wise and then checking that it is a chain map. To do this, we first let α_n be induced from universal property of the tensor product of $\Bbbk$-modules,



An easy check gives that α_n is an R-module homomorphism since R is presumed to be a $\Bbbk$-algebra so that $\Bbbk$ acts centrally on R.

Let β_n be the map induced from the following diagram,



An easy check again gives that β_n defines an R-module homomorphism. We then define our target chain-map h in degree n as

$$h_n : (\Bbbk \otimes M_{n-1}) \oplus ((\Bbbk \oplus \Bbbk) \otimes M_n) \to N_n, \tag{3.1.4}$$

where h_n is the induced map we get from the universal property of the coproduct.

Comparison with 3.1.3 gives that h_n has domain $(\mathbb{I} \otimes M)_n$.

First we check that h_n satisfies

$$\begin{cases} h_n \circ (i_0 \otimes \mathrm{id}_M)_n = f_n \\ h_n \circ (i_1 \otimes \mathrm{id}_M)_n = g_n. \end{cases}$$

By R-linearity it is enough to check this on pure tensors. We find that

$$\begin{aligned} h_n \circ (i_0 \otimes \mathrm{id}_M)_n(c \otimes m_n) &= h_n(0, (c, 0) \otimes m_n) \\ &= \beta_n(0, (c, 0) \otimes m_n) \\ &= f(c \otimes m_n), \end{aligned}$$

and similarly $h_n \circ (i_1 \otimes \mathrm{id}_M)_n = g_n$.

Lastly, we need to check that $h := (h_n)_{n\in\mathbb{Z}}$ is a chain map, i.e. that the following diagram commutes for all n,

$$\begin{array}{ccc} (\mathbb{I}\otimes M)_n & \xrightarrow{d_n^{\mathbb{I}\otimes M}} & (\mathbb{I}\otimes M)_{n-1} \\ \Big\downarrow{\scriptstyle h_n} & & \Big\downarrow{\scriptstyle h_{n-1}} \\ N_n & \xrightarrow[d_n^N]{} & N_{n-1} \end{array} .$$

It is enough to check this on "simple" elements $(c\otimes m_{n-1}, (r,s)\otimes m_n)$. We find that

$$\begin{aligned} &h_{n-1}\left(d_n^{\mathbb{I}\otimes M}((c\otimes m_{n-1},(r,s)\otimes m_n))\right) \\ &= h_{n-1}((\underbrace{\underbrace{d_0^{\mathbb{I}}(r,s)}_{=0}\otimes m_n}_{=0} - c\otimes d_{n-1}^M(m_{n-1}), d_1^{\mathbb{I}}(c)\otimes m_{n-1} + (r,s)\otimes d_n^M(m_n))) \\ &= h_{n-1}(-c\otimes d_{n-1}^M(m_{n-1}), (-c,c)\otimes m_{n-1} + (r,s)\otimes d_n^M(m_n)) \\ &= -\gamma_{n-2}(c\otimes d^{M_{n-1}(m_{n-1})})) + (g_{n-1}(c\otimes m_{n-1}) - f_{n-1}(c\otimes m_{n-1})) \\ &+ f_{n-1}(r\otimes d_n^M m_n) + g_{n-1}(s\otimes d_n^M(m_n)), \end{aligned}$$

while

$$\begin{aligned} d_n^N(h_n((c\otimes m_{n-1}),((r,s)\otimes m_n))) &= d_n^N(\gamma_{n-1}(c\otimes m_{n-1})) + f_n(r\otimes m_n) + g_n(s\otimes m_n)) \\ &= d_n^N(\gamma_{n-1}(c\otimes m_{n-1})) + f_{n-1}(r\otimes d_n^M m_n) + g_{n-1}(s\otimes d^M m_n) \\ &= \left(-\gamma_{n-2}d_{n-1}^M(c\otimes m_{n-1}) + g_{n-1}(c\otimes m_{n-1}) - f_{n-1}(c\otimes m_{n-1})\right) \\ &+ f_n(r\otimes d_n^M m_n) + g_{n-1}(s\otimes d_n^M m_n), \qquad \text{since } g - f = d^N\gamma + \gamma d^M \end{aligned}$$

where we in the second equality used that f and g are chain maps and how the differential $d^{\Bbbk\otimes M}$ acts (cf. 3.1.2). Comparison gives that the two expressions above agree, so we conclude that h is a chain map and the main conclusion of this direction follows.

$\underline{\Leftarrow}$: Assume we have a map $\mathbb{I}\otimes M \xrightarrow{h} N$ such that the diagram in the statement of the theorem commutes. Define $\gamma_n : \Bbbk\otimes M_n \to N_{n+1}$ by

$$\gamma_n(c\otimes m) := h_{n+1}(c\otimes m, 0). \tag{3.1.5}$$

This is clearly well-defined and inherits R-linearity from h_{n+1}. Let $\gamma := (\gamma_n)_{n\in\mathbb{Z}}$ and observe that since h is a chain map we have that

$$d_{n+1}^N \circ h_{n+1} = h_n \circ d_{n+1}^{\mathbb{I}\otimes M}. \tag{3.1.6}$$

We find that

$$\begin{aligned} d_{n+1}^{\mathbb{I}\otimes M}(c\otimes m, 0) &= (-1\otimes d_n^M(m), d_1^{\mathbb{I}}(c)\otimes m) \\ &= (-1\otimes d_n^M(m), (-1,1)\otimes m) \in (\mathbb{I}\otimes M)_n, \end{aligned}$$

so that

$$\begin{aligned}
d^N_{n+1}\gamma_n(c\otimes m,0) &= d^N_{n+1}h_{n+1}(c\otimes m,0)\\
&= h_n(-1\otimes d^M_n(m),(-1,1)\otimes m)\\
&= h_n((-1\otimes d^M_n(m),0)+h_n(0,(-1,1)\otimes m))\\
&= h_n(0,(0,1)\otimes m-(1,0)\otimes m)-h_n(1\otimes d^M_n(m),0)\\
&= h_n(0,(0,1)\otimes m)-h_n(0,(1,0)\otimes m)-h_n(1\otimes d^M_n(m),0)\\
&= g_n(1\otimes m)-f_n(1\otimes m)-\gamma_{n-1}(1\otimes d^M_n(m))\\
\Leftrightarrow d^N_{n+1}\circ\gamma_n+\gamma_{n-1}\circ d^{\Bbbk\otimes M}_n &= g_n-f_n.
\end{aligned}$$

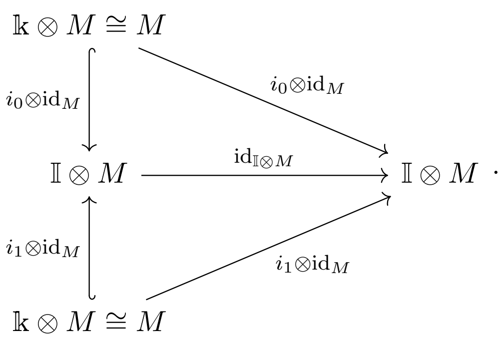
Lastly, we check that the constructions produced above are inverses to each other, hence giving a bijection (and hence uniqueness of the maps) in both directions. In one direction, starting with a chain homotopy $\gamma : f \Rightarrow g$, we then construct h as in the $\Rightarrow$-direction. Then upon applying the $\Leftarrow$-construction to h we get a maps $\tilde{\gamma}_n : \Bbbk\otimes M_n \to N_{n+1}$ such that $\tilde{\gamma}_n(c\otimes m) = h_{n+1}(c\otimes m,0) = \gamma_n(c\otimes m)$ for all n and all elements $c\otimes m\in\Bbbk\otimes M_n$. Hence it follows we get back γ.

On the other hand, if we start with a chain map $h : \mathbb{I}\otimes M \to N$ we first define $\gamma_n : \Bbbk\otimes M_n \to N_{n+1}$ by $\gamma_n(c\otimes m) = h_{n+1}(c\otimes m,0)$ on elements $c\otimes m\in\Bbbk\otimes M_n$. By then following the construction of the $\Rightarrow$-direction, we find that this gives us back a chain map $\tilde{h}$ which in degree n is $\tilde{h}_n(c\otimes m,0) = \gamma_{n-1}(c\otimes m) = h_n(c\otimes m,0)$. In each degree n, $\tilde{h}_n$ and h_n agrees on the second summand since both maps equals $f_n(c\otimes m)+g_n(s\otimes m)$ on elements $(0,(c,s)\otimes m)$. It follows by decomposition and linearity that they agree in each degree n. Hence the constructions realize a bijective correspondence as stated in the theorem. □

Remark 3.1.8. Observe that we are *silently* treating the canonical isomorphism removing zeroes from $(\mathbb{I}\otimes M)_n$ as an identity in the proof above.

To recapitulate, the theorem tells us that a chain map $\mathbb{I}\otimes M \xrightarrow{h} N$ gives a chain homotopy between $h\circ(i_0\otimes\mathrm{id}_M)\Rightarrow h\circ(i_1\otimes\mathrm{id}_M)$ on the one hand, and on the other hand a given triple (f,g,γ) with $f,g:\Bbbk\otimes M\rightrightarrows N$ chain maps and a chain homotopy $f\overset{\gamma}{\Rightarrow} g$ gives us a chain map $h:\mathbb{I}\otimes M\to N$.

Now consider the diagram



By the theorem we claim to have just proven (3.1.6), this gives us a chain homotopy $(i_0\otimes\mathrm{id}_M)\overset{\gamma}{\Rightarrow}(i_1\otimes\mathrm{id}_M)$. By inspecting the proof of said theorem, in particular by equation 3.1.5 we find that the chain homotopy γ is defined on degrees as

$$\begin{aligned}\gamma_n(m) &= (\mathrm{id}_{\mathbb{I}\otimes M})_{n+1}(1 \otimes m, 0)\\ &= (1 \otimes m, 0).\end{aligned}$$

Definition 3.1.9 (Finite category)**.** We say that a category $\mathcal{I}$ is **finite** if $\mathsf{Ob}(\mathcal{I})$ and $\mathsf{Mor}(\mathcal{I})$ are *finite* sets.

We briefly recall the definition of limits and colimits (as formulated in [Sta26, Tag 002D]). To do this, we specify some notational conventions. Whenever we have a diagram $F : \mathcal{I} \to \mathcal{C}$ (with $\mathcal{I}$ usually called the *index*-category) and i an object in $\mathcal{I}$, we let $F_i := F(i)$ denote the corresponding object in $\mathcal{C}$. When $\phi : i \to i'$ is a morphism in $\mathcal{I}$ we let $F(\phi) : F_i \to F_{i'}$ denote the corresponding morphism in $\mathcal{C}$. We will use capital letters, e.g. A, B, etc., for *objects* in the target category of the functor defining the diagram.

Definition 3.1.10 (Limit)**.** A **limit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ in the category $\mathcal{C}$, is an object $\mathsf{lim}_{\mathcal{I}}F$ in $\mathcal{C}$ together with specified morphisms $p_i : \mathsf{lim}_{\mathcal{I}}F \to F_i$ for each object i in $\mathcal{I}$ such that

(a) For any morphism $\phi : i \to j$ in the index-category $\mathcal{I}$, we have

$$p_j = F(\phi) \circ p_i, \tag{3.1.7}$$

or perhaps in clearer form, as

$$\left(\mathsf{lim}_{\mathcal{I}}F \xrightarrow{p_j} F_j\right) = \left(\mathsf{lim}_{\mathcal{I}}F \xrightarrow{p_i} F_i \xrightarrow{F(\phi)} F_j\right). \tag{3.1.8}$$

(b) For any object A in $\mathcal{C}$ and a collection $q_i : A \to F_i$ of morphisms indexed by the objects i in $\mathcal{I}$, that satisfy that for all morphisms $\phi : i \to j$ in $\mathcal{I}$ we have

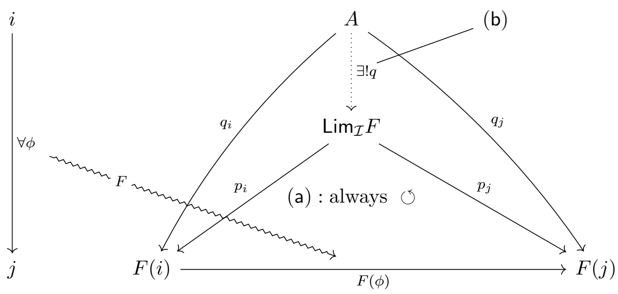
$$q_j = F(\phi) \circ q_i, \tag{3.1.9}$$

there exists a *unique* morphism $q : A \to \mathsf{lim}_{\mathcal{I}}F$ such that

$$q_i = p_i \circ q, \tag{3.1.10}$$

for all objects i in $\mathcal{I}$.

Diagrammatically, we may try to capture both conditions (a) and (b) by the following,



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Dually (in the sense that a limit in $\mathcal{C}$ is a colimit in $\mathcal{C}^{\mathrm{op}}$), we have the notion of colimit.

Definition 3.1.11 (Colimit)**.** A **colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ in the category $\mathcal{C}$, is an object $\mathsf{colim}_{\mathcal{I}} F$ in $\mathcal{C}$ together with specified morphisms $s_i : F_i \to \mathsf{colim}_{\mathcal{I}} F$ for each object i in $\mathcal{I}$, such that

(a) For any morphism $\phi : i \to j$ in the index-category $\mathcal{I}$,

$$s_i = s_j \circ F(\phi), \tag{3.1.11}$$

or in different format, as

$$\left(F_i \xrightarrow{s_i} \mathsf{colim}_{\mathcal{I}} F\right) = \left(F_i \xrightarrow{F(\phi)} F_j \xrightarrow{s_j} \mathsf{colim}_{\mathcal{I}} F\right). \tag{3.1.12}$$

(b) For any object A i $\mathcal{C}$ and a collection $f_i : F_i \to A$ indexed by objects i in $\mathcal{I}$, that satisfy that for all morphisms $\phi : i \to j$ in $\mathcal{I}$,

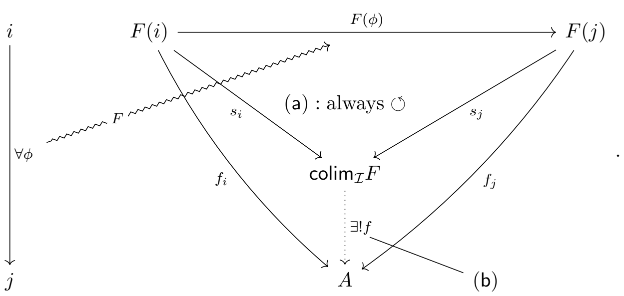
$$f_i = f_j \circ F(\phi), \tag{3.1.13}$$

there is a *unique* morphism $f : \mathsf{colim}_{\mathcal{I}} F \to A$ such that

$$f_i = f \circ s_i, \tag{3.1.14}$$

for all objects i in $\mathcal{I}$.

Schematically, one might try to capture both conditions (a) and (b) with the following diagram



Theorem 3.1.12. *The category* Ch_R *of chain complexes of a commutative ring* R *has all finite limits and colimits.*

Remark 3.1.13. The statement means that whenever $F : \mathcal{I} \to \mathsf{Ch}_R$ is a diagram (i.e. a functor) with $\mathcal{I}$ finite, then $\mathsf{lim}_{\mathcal{I}} F, \mathsf{colim}_{\mathcal{I}} F$ exists in Ch_R.

Proof.

Lemma 3.1.14. *An additive category with all kernels and cokernels has all finite limits and colimits.*

Proof. Realize that the proof given in [Sta26, Tag 010D] only depends on the category $\mathcal{A}$ being additive with all kernels and cokernels. □

By the lemma, it is enough to show that Ch_R is additive and has all kernels and cokernels.

$\underline{\mathsf{Ch}_R \text{ additive}}$: Let $f, g : M \rightrightarrows N$ be chain maps, so that $d^N f = f d^M$ and $d^N g = g d^M$ holds in all degrees. Define $f+g$ degree-wise as $(f+g)_n = f_n + g_n$. Since the differentials d^N, d^M are homomorphisms, $f+g$ is again a chain map. Then it is easy to see that $-f = (-f_n)_{n\in\mathbb{Z}}$ is the (additive) inverse of f and that $0_\bullet = (0)_{n\in\mathbb{Z}}$ the additive identity, so that $\mathsf{hom}_{\mathsf{Ch}_R}(M, N)$ is an abelian group.

Considering chain maps

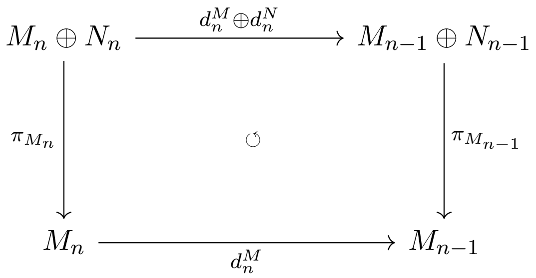
$$L \xrightarrow{f} M \overset{h,g}{\rightrightarrows} N \xrightarrow{k} P$$

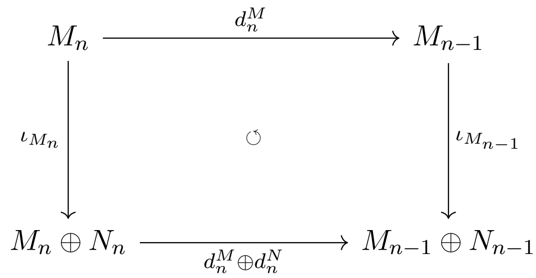
it is straightforward to check in degrees that the composition is bilinear with (recall) composition defined as $(g \circ f)_n = g_n \circ f_n$.

We have that $0_\bullet$ with the zero R-module in each degree as 0-morphisms as differentials, is both initial and terminal in Ch_R since, in degrees, $0 \in \mathsf{Mod}_R$ is initial and terminal.

Lastly, for chain complexes $M, N \in \mathsf{Ch}_R$, define $M \oplus N := \big((M_n \oplus N_n)_{n\in\mathbb{Z}}, (d_n^M \oplus d_n^N)_{n\in\mathbb{Z}}\big)$.

Then we may define $\pi_M : M \oplus N, \pi_N : M \oplus N \to N$ and $\iota_M : M \to M \oplus N, \iota_N : N \to M \oplus N$ in degrees as the projection $(\pi_M)_n = \pi_{M_n} : M_n \oplus N_n \to M_n, \quad (m, n) \mapsto m$ coming from Mod_R and $(\iota_M)_n = \iota_{M_n} : M_n \to M_n \oplus N_n, \quad m \mapsto (m, 0)$ making the following diagrams commute in each degree n,



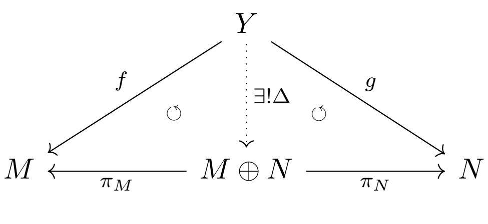


so that $\pi_M, \pi_N, \iota_M, \iota_N$ are chain maps.

It is clear that

1. $\pi_M \circ \iota_M = \mathrm{id}_M$ and $\pi_N \circ \iota_N = \mathrm{id}_N$,
2. $\pi_M \circ \iota_N = \pi_N \circ \iota_M = 0$,
3. $\iota_M \circ \pi_M + \iota_N \circ \pi_N = \mathrm{id}_{M\oplus N}$.

Furthermore, the commutative diagram below indicates $(M \oplus N, \pi_M, \pi_N)$ is a product, since $\Delta_n : Y_n \to M_n \oplus N_n, \quad y \mapsto (f_n(y), g_n(y))$ must be the unique chain map (that it is a chain map follows from f, g being chain maps).

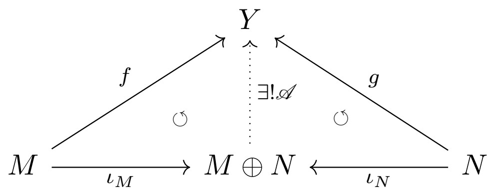


.

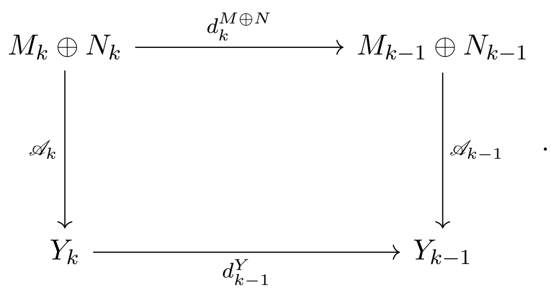
It is clear that $\pi_M \circ \Delta = f$ and $\pi_N \circ \Delta = g$. If also $h : Y \to M \oplus N$ so that $\pi_M \circ h = f$ and $\pi_N \circ h = g$ then this implies that in each degree k, we have that $u_k(y) = (m, n) \in M_k \oplus N_k$

is such that $(\pi_M)_k \circ u_k(y) = m = f_k(y)$ and $(\pi_N)_k \circ u_k(y) = n = g_k(y)$, for arbitrary $y \in Y_k$ and arbitrary k. Hence $\Delta_k = h_k$ so $\Delta = h$, so that Δ is unique.

Now instead consider the following diagram,



Here in each degree k we define $\mathscr{A}_k(m, n) = f_k(m) + g_k(n)$. We check that the following diagram commutes in each degree k,



We find that

$$\begin{aligned} d_{k-1}^Y(\mathscr{A}_k(m,n)) &= d_{k-1}^Y(f_k(m) + g_k(n)) \\ &= d_{k-1}^Y f_k(m) + d_{k-1}^Y g_k(n) \\ &= f_{k-1}\left(d_k^M(m)\right) + g_{k-1}\left(d_k^N(n)\right) \\ &= \mathscr{A}_{k-1}\left(d_k^M(m), d_k^N(n)\right) \\ &= \left(\mathscr{A}_{k-1} \circ d_k^{M\oplus N}\right)(m,n), \end{aligned}$$

so that $\mathscr{A}$ is indeed a chain map.

We then see that $\mathscr{A}_k \circ (\iota_M)_k(m) = \mathscr{A}_k(m, 0) = f_k(m)$ and $\mathscr{A}_k \circ (\iota_N)_k(n) = \mathscr{A}_k(0, n) = g_k(n)$ since f_k, g_k are homomorphisms. Assume we had a chain map $\mathscr{B} : M \oplus N \to Y$ such that $\mathscr{B}_k \circ (\iota_M)_k = f_k$ and $\mathscr{B}_k \circ (\iota_N)_k = g_k$. Let $(m, n) \in M_k \oplus N_k$ be arbitrary. Then

$$\begin{aligned} \mathscr{B}_k(m,n) &= \mathscr{B}_k((m,0) + (0,n)) \\ &= \mathscr{B}_k(m,0) + \mathscr{B}_k(0,n) \qquad \text{since } \mathscr{B}_k \text{ is a homomorphism} \\ &= \left(\mathscr{B}_k \circ (\iota_M)_k\right)(m) + \left(\mathscr{B}_k \circ (\iota_N)_k\right)(n) \\ &= f_k(m) + g_k(n) \\ &= \mathscr{A}_k(m,n), \end{aligned}$$

so that $\mathscr{A}_k = \mathscr{B}_k$ for all $k \in \mathbb{Z}$, hence $\mathscr{A} = \mathscr{B}$.

It follows that $(M \oplus N, \pi_M, \pi_N, \iota_M, \iota_N)$ defines the data of a biproduct in Ch_R.

<u>Ch_R has all kernels</u>: Let $M \xrightarrow{f} N$ be a chain map. Define $\mathsf{ker}(f)$ in degree k as $(\mathsf{ker}(f))_k := \mathsf{ker}(f_k) \subseteq M_n$. Then we see since for $x \in \mathsf{ker}(f_k)$ we have that

$$\begin{aligned} f_{k-1}(d_k^M(x)) &= d^N(f_k(x)) \\ &= 0, \end{aligned}$$

we have that $d_k^M(x) \in \mathsf{ker}(f_{k-1})$, and clearly $d^2 = 0$ holds for $x \in \mathsf{ker}(f_k) \subset M_k$. Thus we may take the differentials of $\mathsf{ker}(f)$ as the differentials of M restricted (and corestricted) to the kernels $\mathsf{ker}(f_k)$. We check that this satisfies the universal property of a kernel, with universal map the inclusion $i : \mathsf{ker}(f) \hookrightarrow M$. That the inclusion i is a chain map follows from f being a chain map. We check that this satisfies the properties characterizing a kernel.

(a) It is clear that $f_k \circ i_k(x) = 0$ for each $k \in \mathbb{Z}$ and $x \in \mathsf{ker}(f_k)$, so that $f \circ i = 0$.

(b) Assume that $g : L \to M$ is a chain map such that $f \circ g = 0$, i.e. so that $f_k \circ g_k = 0$ for each k. This means that $\mathsf{im}(g_k) \subset \mathsf{ker}(f_k)$. We then see that g_k factorizes as $g_k = i_k \circ \overline{g}_k$ with $\overline{g}_k : L_k \to \mathsf{ker}(f_k)$ the *corestriction* of g_k to $\mathsf{ker}(f_k)$. $\overline{g} := (\overline{g}_k)_{k\in\mathbb{Z}}$. If $h : L \to \mathsf{ker}(f)$ is another chain map such that $i_k \circ h_k = g_k$ in each degree, then since i_k is injective, it has the left-cancellation property, so it follows that $h_k = \overline{g}_k$, i.e. so that $\overline{g}_k$ is unique for each k, hence $\overline{g}$ is unique. Observe that

$$\begin{aligned} i_{k-1} \circ d_k^{\mathsf{ker}(f)} \circ \overline{g}_k &= d_k^M \circ i_k \circ \overline{g}_k \\ &= d_k^M \circ g_k \\ &= g_{k-1} \circ d_k^L \\ &= i_{k-1} \circ \overline{g}_{k-1} \circ d_k^L \\ \Rightarrow d_k^{\mathsf{ker}(f)} \circ \overline{g}_k &= \overline{g}_{k-1} \circ d_k^L, \end{aligned}$$

i.e., the following diagram commutes

$$\begin{array}{ccc} L_k & \xrightarrow{d_k^L} & L_{k-1} \\ \downarrow{\scriptstyle \overline{g}_k} & \circlearrowleft & \downarrow{\scriptstyle \overline{g}_{k-1}} \\ \mathsf{ker}(f)_k & \xrightarrow[d_k^{\mathsf{ker}(f)}]{} & \mathsf{ker}(f)_{k-1} \end{array}$$

so that $\overline{g}$ is a chain map.

We conclude that $\left(\mathsf{ker}(f), \mathsf{ker}(f) \xrightarrow{i} M\right)$ is a kernel of f.

<u>Ch_R has all cokernels</u>: Let $M \xrightarrow{f} N$ be a chain map. Define $\mathsf{coker}(f)$ in each degree as $(\mathsf{coker}(f))_k := \mathsf{coker}(f_k) = N_k / \mathsf{im}(f_k)$, with differentials $d_k^{\mathsf{coker}(f)}$ the unique maps induced from the universal property of the quotient, where we use that for $x \in \mathsf{im}(f_k)$, so that $x = f_k(y)$,

$$\begin{aligned} q_{k-1} d_k^N(x) &= q_{k-1} d_k^N(f_k(y)) \\ &= q_{k-1}\left(f_{k-1} d_k^M(y)\right), \qquad \text{since } f \text{ is a chain map} \\ &= 0, \qquad \text{since } f_{k-1} d_k^M(y) \in \mathsf{im}(f_{k-1}) \end{aligned}$$

so that $\mathsf{im}(f) \subset \mathsf{ker}\left(q_{k-1} \circ d_k^N\right)$. Hence we get the following diagram,

$$\begin{array}{ccc} N_k & \xrightarrow{\quad d_k^N \quad} & N_{k-1} \\ {\scriptstyle q_k}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle q_{k-1}} \\ N_k/\,\mathsf{im}(f_k) & \xrightarrow[\exists! d_k^{\mathsf{coker}(f)}]{} & N_{k-1}/\,\mathsf{im}(f_{k-1}) \end{array}$$

That $q := (q_k)_{k\in\mathbb{Z}}$ with $q_k : N_k \twoheadrightarrow N_k/\,\mathsf{im}(f_k)$ the quotient map, is a chain map then follows directly by construction from the diagram above.

We check that $\left(\mathsf{coker}(f), N \xrightarrow{q} \mathsf{coker}(f)\right)$ satisfies the properties characterizing a cokernel object in Ch_R.

(a) It is clear that $q \circ f = 0$ since by construction $q_k \circ f_k = 0$.

(b) Assume $g : N \to L$ is a chain map such that $g \circ f = 0$, i.e. so that $g_k \circ f_k = 0$, hence $\mathsf{im}(f_k) \subset \mathsf{ker}(g_k)$. Hence for each k there is a unique map $c_k : \mathsf{coker}(f)_k \to L_k$ such that the following diagram commutes,

$$\begin{array}{ccc} N_k & \xrightarrow{\qquad g_k \qquad} & L_k \\ & {\scriptstyle q_k}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle \exists! c_k} & \\ & \mathsf{coker}(f)_k & \end{array} .$$

We let $c := (c_k)_{k\in\mathbb{Z}}$. Then we find that

$$\begin{aligned} d_k^L \circ c_k \circ q_k &= d_k^L \circ g_k, && \text{since } c_k \circ q_k = g_k \\ &= g_{k-1} \circ d_k^N \\ &= c_{k-1} \circ q_{k-1} \circ d_k^N \\ &= c_{k-1} \circ d_k^{\mathsf{coker}(f)} \circ q_k \\ \Rightarrow d_k^L \circ c_k &= c_{k-1} \circ d_k^{\mathsf{coker}(f)}, && \text{since } q_k \text{ is surjective,} \end{aligned}$$

i.e., the following diagram commutes,

$$\begin{array}{ccc} \mathsf{coker}(f)_k & \xrightarrow{d_k^{\mathsf{coker}(f)}} & \mathsf{coker}(f)_{k-1} \\ {\scriptstyle c_k}\big\downarrow & & \big\downarrow{\scriptstyle c_{k-1}} \\ L_k & \xrightarrow[d_k^L]{} & L_{k-1} \end{array} ,$$

so that c is a chain map. By uniqueness in each degree of $\overline{g}$ we find that $\left(\mathsf{coker}(f), N \xrightarrow{q} \mathsf{coker}(f)\right)$ is a cokernel of f.

The conclusion now follows by lemma 3.1.14.

□

Consider the chain complex

$$C := \mathsf{colim}\left(\begin{array}{ccc} \Bbbk \otimes M & \xhookrightarrow{i_0 \otimes \mathrm{id}_M} & \mathbb{I} \otimes M \\ {\scriptstyle i_1 \otimes \mathrm{id}_M}\big\downarrow & & \\ \mathbb{I} \otimes M & & \end{array}\right).$$

where we note that the diagram inside $\mathsf{colim}\big(- \big)$ is the image of a functor

$$F : \mathcal{I} \to \mathsf{Ch}_R$$

with $\mathcal{I}$ the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects and two arrows, hence this is a finite category and so by theorem 3.1.12 this colimit exists in Ch_R. Observe that by definition 3.1.11, and contemplating the diagram below, the colimit of F is a **pushout**,

$$\begin{array}{ccccc} F_0 & \xrightarrow{F(\phi_1)} & F_1 & & \\ {\scriptstyle F(\phi_2)}\big\downarrow & \overset{s_0}{\searrow} & \big\downarrow{\scriptstyle s_1} & \searrow{\scriptstyle f_1} & \\ F_2 & \xrightarrow{s_2} & \mathsf{colim}_{\mathcal{I}} F & & \\ & \searrow{\scriptstyle f_2} & {\scriptstyle f_0} & \searrow{\scriptstyle \exists! f} & \\ & & & & A \end{array} . \tag{3.1.15}$$

This means that we may represent C as

$$\begin{array}{ccc} \Bbbk \otimes M & \xhookrightarrow{i_0 \otimes \mathrm{id}_M} & \mathbb{I} \otimes M \\ {\scriptstyle i_1 \otimes \mathrm{id}_M}\big\downarrow & & \big\downarrow{\scriptstyle h_0} \\ \mathbb{I} \otimes M & \xrightarrow[h_1]{} & C \end{array},$$

where suppress what corresponds to s_0 in diagram 3.1.15 since it essentially encodes that the diagram should commute, and when one gets an induced map $f : C \to A$. Hence we get chain maps h_0, h_1 such that

$$h_1 \circ (i_1 \otimes \mathrm{id}_M) = h_0 \circ (i_0 \otimes \mathrm{id}_M). \tag{3.1.16}$$

By theorem 3.1.6, h_0 gives us a chain homotopy $h_0 \circ (i_0 \otimes \mathrm{id}_M) \overset{\ell_0}{\Rightarrow} h_0 \circ (i_1 \otimes \mathrm{id}_M)$ and by the same theorem we get a chain homotopy $h_1 \circ (i_0 \otimes \mathrm{id}_M) \overset{\ell_1}{\Rightarrow} h_1 \circ (i_1 \otimes \mathrm{id}_M)$. By equation 3.1.16 we can rewrite this as

$$h_1 \circ (i_0 \otimes \mathrm{id}_M) \overset{\ell_1}{\Rightarrow} h_1 \circ (i_1 \otimes \mathrm{id}_M) \overset{3.1.16}{=} h_0 \circ (i_0 \otimes \mathrm{id}_M) \overset{\ell_0}{\Rightarrow} h_0 \circ (i_1 \otimes \mathrm{id}_M).$$

By theorem 3.1.6 it then follows that $h_1 \circ (i_0 \otimes \mathrm{id}_M)$ is chain homotopic to $h_0 \circ (i_1 \otimes \mathrm{id}_M)$, and one checks that the chain homotopy we want is $\ell := \ell_0 + \ell_1$.

Starting with the data of a triple $(h_1 \circ (i_0 \otimes \mathrm{id}_M), h_0 \circ (i_1 \otimes \mathrm{id}_M), \ell)$ where the first two are chain maps and ℓ is a chain homotopy from the first map to the second map, theorem 3.1.6 tells us that this should give us chain map $\mathbb{I} \otimes M \to C$.

Let $j_0 := h_1 \circ (i_0 \otimes \mathrm{id}_M)$ and $j_2 := h_0 \circ (i_1 \otimes \mathrm{id}_M)$. Inspecting the proof of 3.1.6 we find that in degree n, $h_n : (\mathbb{I} \otimes M)_n \to C_n$ will be computed on elements $(c \otimes m_{n-1}, (r,s) \otimes m_n) \in (\mathbb{I} \otimes M)_n$ as

$$\begin{aligned} h_n(c \otimes m_{n-1}, (r,s) \otimes m_n) &= \ell_{n-1}(c \otimes m_{n-1}) + (j_0)_n(r \otimes m_n) + (j_2)_n(s \otimes m_n) \\ &= (\ell_0)_{n-1}(c \otimes m_{n-1}) + (\ell_1)_{n-1}(c \otimes m_{n-1}) + (j_0)_n(r \otimes m_n) + (j_2)_n(s \otimes m_n). \end{aligned} \tag{3.1.17}$$

But by 3.1.6 we have an explicit description (see equation 3.1.5) of ℓ_0 and ℓ_1 in degrees, as

$$(\ell_0)_{n-1}(c \otimes m_{n-1}) = (h_0)_n(c \otimes m_{n-1}, 0)$$

and

$$(\ell_1)_{n-1}(c \otimes m_{n-1}) = (h_1)_n(c \otimes m_{n-1}, 0).$$

Unraveling equation 3.1.17 we find that

$$\begin{aligned} h_n(c \otimes m_{n-1}, (r,s) \otimes m_n) &= (h_0)_n(c \otimes m_{n-1}, 0) + (h_1)_n(c \otimes m_{n-1}, 0) \\ &\quad + (h_1)_n(0, (r,0) \otimes m_n) + (h_0)_n(0, (0,s) \otimes m_n) \\ &= (h_0)_n(c \otimes m_{n-1}, (0,s) \otimes m_n) + (h_1)_n(c \otimes m_{n-1}, (r,0) \otimes m_n). \end{aligned}$$

Definition 3.1.15. We say that a morphism $f : M \to N$ in Ch_R is a **chain-homotopy equivalence** if there exists a morphism $g : N \to M$ and chain-homotopies $\mathrm{id}_M \Rightarrow g \circ f$ and $\mathrm{id}_N \Rightarrow f \circ g$.

Theorem 3.1.16. *If $f_\bullet, g_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(M_\bullet, N_\bullet)$ are chain-homotopic maps, i.e. if there is a chain-homotopy $f_\bullet \overset{h}{\Rightarrow} g_\bullet$, then they induce the same maps on homology, i.e. $\mathsf{H}_n(f) = \mathsf{H}_n(g)$ holds for all n.*

Proof. By definition, $f_\bullet$ and $g_\bullet$ being chain-homotopic means that there exists homomorphisms $h_n : M_n \to N_{n+1}$ such that for each $m \in \mathsf{ker}\left(d_n^M\right)$, we have

$$\begin{aligned} g_n(m) - f_n(m) &= d_{n+1}^N h_n(m) + \underbrace{h_{n-1} \underbrace{d_n^M(m)}_{=0}}_{=0} \\ &= d_{n+1}^N h_n(m). \end{aligned}$$

Hence g_n and f_n differ by a boundary on $\mathsf{ker}\left(d_n^M\right)$, hence $[f_n(m)] = [g_n(m)]$ in $\mathsf{H}_n(N_\bullet)$. Since $\mathsf{H}_n(f)([m]) = [f_n(m)]$ on generators $[m]$ of $H_n(M_\bullet)$, and the same holds for $\mathsf{H}_n(g)$, it follows that they agree on all of $\mathsf{H}_n(M_\bullet)$. □

Below, we provide two examples that shows that a complex can be acyclic without being contractible. The second complex illuminates the fact that whether or not a chain complex is contractible depends on the module-structure imposed on it, i.e. in which category Ch_R one views the complex as inhabiting.

Example 3.1.17. We first check that the sequence

$$M_\bullet = \quad \ldots \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \ldots$$

of $\mathbb{Z}/4$-modules, where 2 is short-hand for $\overline{2} \in \mathbb{Z}/4$, is a chain complex. Observe that it is immediate that $\cdot 2$ defines a $\mathbb{Z}/4$-module homomorphism from endowing $\mathbb{Z}/4$ with the canonical $\mathbb{Z}/4$-module structure $\mu : \mathbb{Z}/4 \times \mathbb{Z}/4 \to \mathbb{Z}/4, (\overline{a}, \overline{b}) \mapsto \overline{a} \cdot \overline{b} = \overline{ab}$, since the map $\cdot 2$ is exactly what we get by fixing the second argument of μ at $\overline{2}$, and using the $\mathbb{Z}/4$-module structure together with the fact that $\mathbb{Z}/4$ is commutative.

Furthermore, associativity of multiplication μ gives that

$$\begin{aligned}(a \cdot 2) \cdot 2 &= a \cdot (\overline{2 \cdot 2}) \\ &= a \cdot 0 \\ &= 0,\end{aligned}$$

so that $d^2 = 0$ holds.

Acyclic: We note that $d_n = \cdot 2$ for all n, so that $\mathsf{ker}(d_n) = 2\mathbb{Z}/4\mathbb{Z}$ and $\mathsf{im}(d_{n+1}) = 2\mathbb{Z}/4\mathbb{Z}$, hence $\mathsf{H}_n(M_\bullet) = 0$. Therefore, $M_\bullet$ is acyclic.

Not contractible: If $M_\bullet$ was contractible, there would exist a sequence of $\mathbb{Z}/4$-module homomorphism $\mathbb{Z}/4 \xrightarrow{h_n} \mathbb{Z}/4$ such that

$$2h_n + 2h_{n-1} = \mathsf{id}_{\mathbb{Z}/4}.$$

But any $\mathbb{Z}/4$-module homomorphism h_n is multiplication by an element a_n in $\mathbb{Z}/4$, so we may rewrite this as

$$\begin{aligned}2a_n + 2a_{n-1} &= 2(a_n + a_{n-1}) \\ &= \mathsf{id}_{\mathbb{Z}/4}.\end{aligned}$$

This means that $2(a_n + a_{n-1}) = 1$ in $\mathbb{Z}/4$, which is impossible! We conclude that $M_\bullet$ is not contractible.

Example 3.1.18. Let $\Bbbk$ be a commutative, unital, non-zero ring. Consider the sequence

$$M_\bullet = \quad \ldots \xrightarrow{\cdot \overline{x}} \Bbbk[x]/(x^2) \xrightarrow{\cdot \overline{x}} \Bbbk[x]/(x^2) \xrightarrow{\cdot \overline{x}} \Bbbk[x]/(x^2) \xrightarrow{\cdot \overline{x}} \ldots.$$

For the same reason as in the previous example, this is a $\Bbbk[x]/(x^2)$-module homomorphism, and it is clear that with differentials $d_i = \cdot\overline{x}$ we have $d^2 = 0$ since $\overline{x} \cdot \overline{x} = \overline{x^2} = 0 \in \Bbbk[x]/(x^2)$, hence $M_\bullet \in \mathsf{Ch}_{\Bbbk[x]/(x^2)}$.

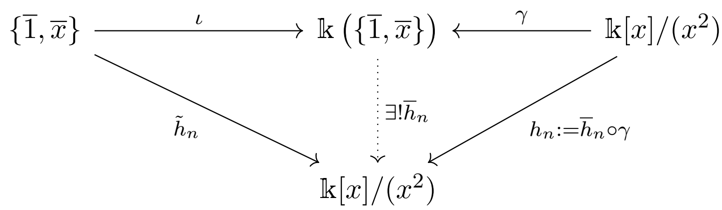
Acyclic as a complex of $\Bbbk[x]/(x^2)$-modules: This sequence is in fact *exact* in each degree, since if $\overline{m(x)} \cdot x = \overline{m(x)x} = \overline{0}$ then $m(x)x = p(x)x^2$ for some $p(x) \in \Bbbk[x]$ which means that $m(x) \cdot x = p(x)x \cdot x$. But multiplication by x is injective since if $xf(x) = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1} = 0$ with $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ then by definition of two polynomials being equal if they are equal in all degrees, $a_0 = a_1 = \ldots = a_n = 0$ so that $f(x) = 0$. Therefore, we have that $m(x) = p(x)x$ so that $m(x) \in (x)$, hence $\overline{m(x)} \in (\overline{x})$. We conclude that $\mathsf{ker}(d_n) \subset (\overline{x})$. Therefore, we have that $\mathsf{im}\, d_{n+1} = (\overline{x}) = \mathsf{ker}(d_n)$ for all n. Hence $\mathsf{H}_n(M_\bullet) = 0$ in all degrees n, so that $M_\bullet$ is acyclic.

Contractible as a complex of $\Bbbk$-modules: We let $\tilde{h}_n : \{\overline{1}, \overline{x}\} \to \Bbbk[x]/(x^2)$ be the function defined by $\overline{1} \xmapsto{h} \overline{x}$ and $\overline{x} \mapsto \overline{1}$.

We note that

$$\gamma : \Bbbk[x]/(x^2) \to \Bbbk\overline{1} \oplus \Bbbk\overline{x}$$

are isomorphic as $\Bbbk$-modules. As in the diagram below, we may extend $\tilde{h}_n$ to $\overline{h}_n$ first, and then define h_n as the composition $\overline{h}_n \circ \gamma$,



Then h_n is a $\Bbbk$-module homomorphism defined on the generators $\overline{1}, \overline{x}$ (as a $\Bbbk$-module) of $\Bbbk[x]/(x^2)$ as $h_n(\overline{1}) = \overline{x}$ and $h_n(\overline{x}) = \overline{1}$. Then we see that for $a + b\overline{x}$ in $\Bbbk[x]/(x^2)$ we have that

$$\begin{aligned} \overline{x} \cdot (h_n(a + b\overline{x})) + h_n(\overline{x} \cdot (a + b\overline{x})) &= \overline{x} \cdot (a\overline{x} + b) + h_n(a\overline{x}) \\ &= a + b\overline{x} \\ &= \mathrm{id}_{\Bbbk[x]/(x^2)}(a + b\overline{x}), \end{aligned}$$

so that $h := (h_n)_{n \in \mathbb{Z}}$ is a chain-homotopy $0_\bullet \overset{h}{\Rightarrow} \mathrm{id}_{F(M_\bullet)}$ in $\mathsf{Ch}_\Bbbk$ with $\mathsf{Ch}_{\Bbbk[x]/(x^2)} \xrightarrow{F} \mathsf{Ch}_\Bbbk$ the forgetful functor.

Not contractible as a complex of $\Bbbk[x]/(x^2)$-modules: On the other hand, we observe that it is a general fact that $\mathrm{End}_R(R) \cong R$ by $f \mapsto f(1)$, hence the $\Bbbk[x]/(x^2)$-module homomorphism $h_n : \Bbbk[x]/(x^2) \to \Bbbk[x]/(x^2)$ would have to be on the form of multiplication by an element $r_n \in \Bbbk[x]/(x^2)$. But this is impossible since then for any element $m \in \Bbbk[x]/(x^2)$,

$$d_{n+1}h_n(a + b\overline{x}) + h_{n-1}d_n(a + b\overline{x}) = (a + b\overline{x})r_n\overline{x} + (a + b\overline{x})r_{n-1}\overline{x} \in (\overline{x}),$$

so it can not be the identity on elements $a + b\overline{x} \in \Bbbk[x]/(x^2)$ with $b = 0$ and $a \neq 0$ (here is where we require it to be a non-zero ring).

In going from ordinary to *higher* category theory, we would like some way to keep track of homotopies between maps, homotopies between homotopies, and so on. The objects that keeps track of this data, hom-complexes, will be touched upon later. In a sense, we are upgrading or promoting "ordinary" $\mathsf{hom}(X, Y)$'s between objects X and Y to something *richer* that still retains the information of $\mathsf{hom}(X, Y)$ but keeps track of relations between maps, relations between maps between maps etc.

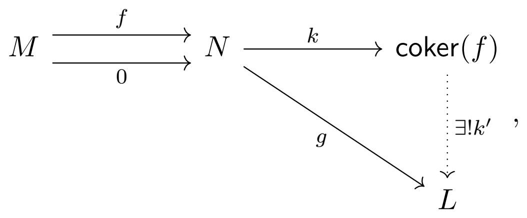
3.2 Homotopy cokernels

First we should note (as we understand it) that there are several "presentations" of the concept "homotopy cokernel", that we now want to present, at increasing levels of abstraction. For now we will use a definition that is perhaps easier to get a handle on, but is not dressed in the (modern) language of ∞-categories or model categories.

Recall that categorically, the definition of the **cokernel** of a morphism $f \in \mathsf{hom}_\mathcal{C}(M, N)$ of say an *abelian* category $\mathcal{C}$, is defined as the pair

$$\left(\mathsf{coker}(f), N \xrightarrow{k} \mathsf{coker}(f)\right)$$

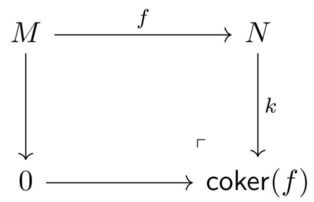
satisfying the universal property making the following diagram commute:



i.e. as the *coequalizer*

$$\mathsf{coeq}\left(M \overset{f}{\underset{0}{\rightrightarrows}} N \right)$$

of the parallel pair of arrows $f, 0 : M \rightrightarrows N$. Another way to phrase this is that the cokernel $\mathsf{coker}(f)$ is the **pushout**



of the span $0 \leftarrow M \overset{f}{\to} N$.

where we suppress writing out the maps going into or from 0, since 0 is the zero object in $\mathcal{C}$, i.e. it is both *initial* and *terminal* (hence the morphism is uniquely determined).

In Mod_R, we call $A \overset{u}{\to} B \overset{v}{\to} C$ a **cokernel sequence** if

(i) $v \circ u = 0$.

(ii) $(C, B \overset{v}{\to} C)$ is a cokernel of u.

We now define want to define the corresponding "homotopical" version of the cokernel, internal to the category of chain complexes Ch_R over a commutative ring R. To define it formally, we start at the other end, giving a concrete instance of a chain complex and a chain map which satisfies the (derived) universal property of the *homotopy cokernel* in Ch_R.

Definition 3.2.1 ((Mapping-) cone)**.** Let $f_\bullet : M_\bullet \to N_\bullet$ be a chain map. Then we define the (mapping-)**cone** of $f_\bullet$, $\mathsf{Cone}(f_\bullet)$, as the chain complex $(\mathsf{Cone}(f_\bullet), d_\bullet) \in \mathsf{Ch}_R$ with

$$\mathsf{Cone}(f_\bullet)_n := M_{n-1} \oplus N_n$$

and differentials

$$d_n(a, b) := \left(-d^M_{n-1}(a), d^N_n(b) + f_{n-1}(a) \right).$$

There is then a canonical chain map $k_\bullet : N_\bullet \to \mathsf{Cone}(f_\bullet)$ defined on degrees as $k_n(a) = (0, a)$.

We check that $d_\bullet$ in fact defines a system of differentials on the cone of f. For any pair $(a,b) \in M_{n-1} \oplus N_n$, we find that

$$\begin{aligned} d_{n-1}(d_n(a,b)) &= d_{n-1}\left(\left(-d^M_{n-1}(a), d^N_n(b) + f_{n-1}(a)\right)\right) \\ &= \left(\underbrace{-d^M_{n-2}\left(-d^M_{n-1}(b)\right)}_{=0}, d^N_{n-1}\left(d^N_n(b) + f_{n-1}(a)\right) + f_{n-2}(-d^M_{n-1}(a))\right) \\ &= \left(0, \underbrace{d^N_{n-1}d^N_n(b)}_{=0} + \underbrace{d^N_{n-1}f_{n-1}(a) - f_{n-2}d^M_{n-1}(a)}_{=0}\right) \\ &= (0,0), \end{aligned}$$

where we used that $f_\bullet$ is a chain map in the last equality.

We also want to check that $k_\bullet$ indeed defines a chain map. We see that if we follow the diagram

$$\begin{array}{ccc} N_n & \xrightarrow{\quad d^N_n \quad} & N_{n-1} \\ \downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_{n-1}} \\ N_n \oplus M_{n-1} & \xrightarrow[\quad d_n \quad]{} & N_{n-1} \oplus M_{n-2} \end{array}$$

right then down, an element $b \in N_n$ lands at the element $(0, d^N_n(b))$. If we instead follow the diagram down-right, a lands in

$$\left(-d^M_{n-1}(0), d^N_n(b) + f_{n-1}(0)\right) = \left(0, d^N_n(b)\right)$$

since f and d^M are homomorphisms in each degree, so that $k_\bullet$ indeed defines a chain map.

Definition 3.2.2 (Homotopy-coherent Cocone)**.** Let $F : \mathcal{I} \to \mathsf{Ch}_R$ be a diagram valued in Ch_R of shape $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$ and let $I_\bullet \in \mathsf{Ch}_R$. A **homotopy-coherent Cocone** from diagram F to $I_\bullet$ is a pair $(g_\bullet, h_\bullet)$ with $g_\bullet : \mathsf{codom}(f_\bullet) = N_\bullet \to I_\bullet$ a chain-map such that $0_\bullet \overset{h_\bullet}{\Rrightarrow} g_\bullet \circ f_\bullet$. Writing this out, this means that for every $n \in \mathbb{Z}$, we want the following identity to be satisfied,

$$g_n \circ f_n = d^{I_\bullet}_{n+1} \circ h_n + h_{n-1} d^{M_\bullet}_n.$$

Set-theoretically, we may denote the set of such homotopy-coherent Cocones from $f_\bullet$ to $I_\bullet$ as

$$\mathsf{Cocone}^h(f_\bullet, I_\bullet) := \left\{ (g_\bullet, h_\bullet) \;\middle|\; \begin{array}{l} g_\bullet : \mathsf{codom}(f_\bullet) = N_\bullet \to I_\bullet \text{ chain map} \\ 0_\bullet \overset{h_\bullet}{\Rrightarrow} g_\bullet \circ f_\bullet \text{ a null homotopy} \end{array} \right\}.$$

Definition 3.2.3 (Homotopy cokernel)**.** A **homotopy cokernel** of a chain-map $f_\bullet : M_\bullet \to N_\bullet \in \mathsf{Ch}_R$ is a triple

$$(\mathsf{hcoker}(f_\bullet), u_\bullet, \gamma_\bullet)$$

where $\mathsf{hcoker}(f_\bullet)$ is a chain complex in Ch_R, $u_\bullet : N_\bullet \to \mathsf{hcoker}(f_\bullet)$ is a chain map and $0_\bullet \overset{\gamma_\bullet}{\Rrightarrow} u_\bullet \circ f_\bullet$ is a witnessing null-homotopy for $u_\bullet \circ f_\bullet$, hence $(u_\bullet, \gamma_\bullet) \in \mathsf{Cocone}^h(f_\bullet, \mathsf{hcoker}(f_\bullet))$.

Furthermore, we require that the triple $(\mathsf{hcoker}(f_\bullet), u_\bullet, \gamma_\bullet)$ satisfies that for every chain complex $I_\bullet \in \mathsf{Ch}_R$ "precomposition" with $(u_\bullet, \gamma_\bullet)$ induces a *natural* bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f_\bullet), I_\bullet) \xrightarrow{\approx} \mathsf{Cocone}^h(f_\bullet, I_\bullet), \qquad \psi_\bullet \mapsto (\psi_\bullet \circ u_\bullet, \psi_\bullet \gamma_\bullet).$$

Another way to phrase this definition, is that $\mathsf{hcoker}(f_\bullet)$ is the *homotopy pushout*

$$\begin{array}{ccc} M_\bullet & \xrightarrow{\quad f_\bullet \quad} & N_\bullet \\ \big\downarrow & \overset{\gamma_\bullet}{\nearrow\!\!\!\nearrow} & \big\downarrow u_\bullet \\ 0 & \xrightarrow{\qquad} & \mathsf{hcoker}(f_\bullet) \end{array} \tag{3.2.1}$$

of the diagram $0 \leftarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet$.

A third perspective on the homotopy cokernel of $f_\bullet$, is as a *representing* object, when it exists, for the functor $\mathsf{Cocone}^h(f_\bullet, -) : \mathsf{Ch}_R \to \mathsf{Set}$, sending objects $I_\bullet$ to $\mathsf{Cocone}^h(f_\bullet, I_\bullet)$ and morphisms $\alpha_\bullet : I_\bullet \to J_\bullet$ to the map $(g_\bullet, h_\bullet) \to (\alpha_\bullet \circ g_\bullet, \alpha_\bullet h_\bullet)$ with $(\alpha_\bullet h_\bullet)_n := \alpha_{n+1} \circ h_n$. The definition is the appropriate definition since if $g_n f_n = d^{I_\bullet}_{n+1} h_n + h_{n-1} d^{M_\bullet}_n$ then

$$\begin{aligned} \alpha_n g_n f_n &= \alpha_n d^{I_\bullet}_{n+1} h_n + \alpha_n h_{n-1} d^{M_\bullet}_n \\ &= d^{J_\bullet}_{n+1} \alpha_{n+1} h_n + \alpha_n h_{n-1} d^{M_\bullet}_n, \end{aligned}$$

using that $\alpha_\bullet$ is a chain-map. But this precisely means that $0_\bullet \overset{\alpha_\bullet h_\bullet}{\Rightarrow} \alpha_\bullet \circ g_\bullet \circ f_\bullet : M_\bullet \to J_\bullet$ is a null-homotopy. Since

$$(\beta_\bullet(\alpha_\bullet h_\bullet))_n = \beta_{n+1} \circ (\alpha_\bullet h_\bullet)_n = \beta_{n+1} \circ \alpha_{n+1} \circ h_n = (\beta_{n+1} \circ \alpha_{n+1}) \circ h_n = ((\beta_\bullet \circ \alpha_\bullet) h_\bullet)_n,$$

using how composition in Ch_R is defined, as in definition 2.3.3.

Remark 3.2.4. The homotopy cokernel can perhaps be viewed as the *algebraic* counterpart of the *topological* **mapping cone** C_f, defined from the data of a continuous map $f : X \to Y$ between spaces X and Y, with $C_f := Y \sqcup_f CX$ where

$$CX := (X \times I)\big/\big((x, 0) \sim (x', 0),\ \forall x, x' \in X\big)$$

with $\iota : X \hookrightarrow CX, \quad x \overset{\iota}{\mapsto} [(x, 1)]$, and where f in $\sqcup_f$ denotes the fact that we identify $[(x, 1)]$ with $f(x)$.

Remark 3.2.5. Observe that the Yoneda embedding being fully faithful provides us with *canonical* isomorphisms between different *models* (i.e. instances) of the homotopy cokernel (see [Rie16, Prop. 2.3.1]), with the perspective of defining the homotopy cokernel as a certain *representing object.*

Theorem 3.2.6. *Let $f_\bullet : M_\bullet \to N_\bullet$ be a chain map. Then, with $\bullet$ suppressed in $M_\bullet$ and $N_\bullet$ of the differentials, we have*

$$\cdots \xrightarrow{\begin{pmatrix} -d^M_1 & 0 \\ f_1 & d^N_2 \end{pmatrix}} \begin{matrix} M_0 \\ \oplus \\ N_1 \end{matrix} \xrightarrow{\begin{pmatrix} -d^M_0 & 0 \\ f_0 & d^N_1 \end{pmatrix}} \begin{matrix} M_{-1} \\ \oplus \\ \underset{\sim\sim}{N_0} \end{matrix} \xrightarrow{\begin{pmatrix} -d^M_{-1} & 0 \\ f_{-1} & d^N_0 \end{pmatrix}} \begin{matrix} M_{-2} \\ \oplus \\ N_{-1} \end{matrix} \xrightarrow{\begin{pmatrix} -d^M_{-2} & 0 \\ f_{-2} & d^N_{-1} \end{pmatrix}} \cdots,$$

defines a chain complex of R-modules and that it is a homotopy cokernel of $f_\bullet$.

Remark 3.2.7. Note that the matrices should be read as

$$(a,b) \mapsto \left(-d_i^M(a), f_i(a) + d_{i+1}^N(b)\right)$$

for $(a,b) \in M_i \oplus N_{i+1}$, and that $\sim\sim$ in $\begin{matrix} M_{-1} \\ \oplus \\ \underset{\sim\sim}{N_0} \end{matrix}$ denotes that this is the *degree zero-module* of the complex, which also tells us the degree of the differentials of the complex. That is, this is precisely the complex $\mathsf{Cone}(f_\bullet)$ defined in definition 3.2.1.

Proof. The first statement is identical to the proof given in defining $\mathsf{Cone}(f_\bullet)$ (3.2.1).

To show that it is a homotopy cokernel of $f_\bullet$. Let $C_\bullet$ denote the above diagram. Then we need to show that there is a natural bijection

$$\mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, I_\bullet) \overset{\Phi_{I_\bullet}}{\simeq} \mathsf{Cocone}^h(f_\bullet, I_\bullet)$$

for all chain complexes $I_\bullet \in \mathsf{Ch}_R$. Let $u_\bullet : N_\bullet \to C_\bullet$ be defined in degree n as

$$u_n : N_n \to C_n = M_{n-1} \oplus N_n, \qquad b \mapsto (0, -b).$$

That $u_\bullet$ is a chain-map follows from the fact that k as defined in connection with the (mapping-)cone of a morphism f, was shown to be chain map, due to the fact that $u = -k$. Let $\gamma_\bullet := (\gamma_n)_{n\in\mathbb{Z}}$ be defined in degree n as

$$\gamma_n : M_n \to C_{n+1} = M_n \oplus N_{n+1}, \qquad a \mapsto (-a, 0).$$

It is (almost) immediate that γ_n is an R-module homomorphism. We then find that, for $a \in M_n$,

$$\begin{aligned} d_{n+1}^C\gamma_n(a) + \gamma_{n-1}d_n^M(a) &= d_{n+1}^C(-a,0) + (-d_n^M(a), 0) \\ &= (d_n^M(a), -f_n(a)) + (-d_n^M(a), 0) \\ &= (0, -f_n(a)) \\ &= u_n f_n(a). \end{aligned}$$

Hence $\gamma_\bullet$ is a null homotopy $0_\bullet \overset{\gamma_\bullet}{\Rightarrow} u_\bullet \circ f_\bullet$, so that $(u_\bullet, \gamma_\bullet) \in \mathsf{Cocone}^h(f_\bullet, C_\bullet)$.

Let $I_\bullet \in \mathsf{Ch}_R$ be arbitrary, let $\psi_\bullet \in \mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, I_\bullet)$, let

$$\Phi_I : \mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, I_\bullet) \to \mathsf{Cocone}^h(f_\bullet, I_\bullet), \qquad (\psi_\bullet) \mapsto (\psi_\bullet \circ u_\bullet, \psi_\bullet\gamma_\bullet),$$

and let

$$\Theta_I : \mathsf{Cocone}^h(f_\bullet, I_\bullet) \to \mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, I_\bullet), \qquad \Theta_I(g_\bullet, h_\bullet)_n(a,b) := -h_{n-1}(a) - g_n(b).$$

$\underline{\Phi_I \text{ and } \Theta_I \text{ are well-defined}}$: For Φ_I note that $\psi_\bullet \circ u_\bullet$ is a composition of chain maps, hence a chain map. We need to check that $\psi\gamma$ is a chain-homotopy $0_\bullet \Rightarrow (\psi_\bullet \circ u_\bullet) \circ f_\bullet$, with recall $\psi_\bullet : C_\bullet \to I_\bullet$ a chain map. Then we see that

$$\begin{aligned} d_{n+1}^I(\psi_{n+1}\gamma_n) + (\psi_n\gamma_{n-1})d_n^M &= \psi_n d_{n+1}^C\gamma_n + \psi_n\gamma_n d_n^M, && \text{since } \psi \text{ is a chain map} \\ &= \psi_n(d_{n+1}^C\gamma_n + \gamma_n d_n^M) \\ &= \psi_n(u_n f_n), && \text{since } 0_\bullet \overset{\gamma_\bullet}{\Rightarrow} u_\bullet \circ f_\bullet \text{ is a chain homotopy} \\ &= (\psi u)_n f_n. \end{aligned}$$

For Θ_I we check that $\Theta_I(g_\bullet, h_\bullet)$ is a chain map for all pairs $(g_\bullet, h_\bullet) \in \mathsf{Cocone}^h(f_\bullet, I_\bullet)$. It is enough to check in degrees. We then find that for $(a,b) \in C_n = M_{n-1} \oplus N_n$,

$$d_n^I \Theta_I(g_\bullet, h_\bullet)_n(a,b) = -d_n^I h_{n-1}(a) - g_{n-1} d_n^N(b), \qquad \text{since } g_\bullet \text{ is a chain map.} \tag{3.2.2}$$

Next,

$$\begin{aligned}\Theta_I(g_\bullet, h_\bullet)_{n-1} d_n^C(a,b) &= \Theta_I(g_\bullet, h_\bullet)_{n-1}(-d_{n-1}^M(a), f_{n-1}(a) + d_n^N(b)) \\ &= h_{n-2}\left(d_{n-1}^M(a)\right) - g_{n-1}(f_{n-1}(a)) - g_{n-1}\left(d_n^N(b)\right).\end{aligned}$$

Showing that this is equal to 3.2.2 after adding $g_{n-1}\left(d_n^N(b)\right)$ to both sides, amounts to checking that

$$d^I h_{n-1}(a) + h_{n-2} d_{n-1}^M(a) = g_{n-1} f_{n-1}(a),$$

which follows from the fact that $h_\bullet$ is a nullhomotopy $0_\bullet \Rightarrow g_\bullet \circ f_\bullet$.

<u>Φ_I and Θ_I are inverses</u>: First we check that $\Phi_I \circ \Theta_I = \mathrm{id}$. Let $(g_\bullet, h_\bullet) \in \mathsf{Cocone}^h(f_\bullet, I_\bullet)$ be arbitrary. Then

$$(\Phi_I \circ \Theta_I)(g_\bullet, h_\bullet) = (\Theta_I(g_\bullet, h_\bullet) \circ u_\bullet, \Theta_I(g_\bullet, h_\bullet)\gamma).$$

Checking the first argument in degree n, we have that

$$\begin{aligned}(\Theta_I(g_\bullet, h_\bullet) \circ u_\bullet)_n(b) &= (\Theta_I(g_\bullet, h_\bullet))_n(0, -b) \\ &= g_n(b).\end{aligned}$$

Checking the second argument, we find that

$$\begin{aligned}(\Theta_I(g_\bullet, h_\bullet)\gamma_\bullet)_n(a) &= (\Theta_I(g_\bullet, h_\bullet))_{n+1}(-a, 0) \\ &= h_n(a).\end{aligned}$$

The conclusion follows.

Next, we check that $\Theta_I \circ \Phi_I = \mathrm{id}$. Let $\psi_\bullet : C_\bullet \to I_\bullet$ be an arbitrary chain map. Then for $(a,b) \in C_n = M_{n-1} \oplus N_n$ we have

$$\begin{aligned}((\Theta_I \circ \Phi_I)(\psi))_n(a,b) &= (\Theta_I(\psi_\bullet \circ u_\bullet, \psi_\bullet\gamma))_n(a,b) \\ &= -(\psi_n \gamma_{n-1})(a) - \psi_n u_n(b) \\ &= \psi_n(a, 0) + \psi_n(0, b) \\ &= \psi_n(a,b).\end{aligned}$$

The conclusion follows.

<u>Naturality of Φ</u>: We claim that by definition this amounts to checking that for every chain map $\alpha_\bullet : I_\bullet \to J_\bullet$ with $I_\bullet, J_\bullet \in \mathsf{Ch}_R$, it holds that

$$G(\alpha_\bullet) \circ \Phi_I = \Phi_J \circ F(\alpha_\bullet), \tag{3.2.3}$$

where $F(\alpha_\bullet)(\psi_\bullet) = \alpha_\bullet \circ \psi_\bullet$ for $\psi_\bullet : C_\bullet \to I_\bullet$ a chain map, and

$$G(\alpha_\bullet)(g_\bullet, h_\bullet) = (\alpha_\bullet \circ g_\bullet, \alpha_\bullet h_\bullet)$$

for $(g_\bullet, h_\bullet) \in \mathsf{Cocone}^h(f_\bullet, I_\bullet)$, i.e. that the following diagram (to the right) commutes,

$$\begin{array}{ccccc}
I_\bullet & & \mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, I_\bullet) & \xrightarrow{\Phi_I} & \mathsf{Cocone}^h(f_\bullet, I_\bullet) \\
{\scriptstyle \alpha_\bullet}\big\downarrow & \rightsquigarrow & \big\downarrow{\scriptstyle F(\alpha_\bullet)} & & \big\downarrow{\scriptstyle G(\alpha_\bullet)} \\
J_\bullet & & \mathsf{hom}_{\mathsf{Ch}_R}(C_\bullet, J_\bullet) & \xrightarrow[\Phi_J]{} & \mathsf{Cocone}^h(f_\bullet, J_\bullet)
\end{array} .$$

It is not hard to show that $G(\,\cdot\,)$ is a (Set-valued) functor. If we let $\psi_\bullet \in \hom_{\mathsf{Ch}_R}(C_\bullet, I_\bullet)$ be arbitrary, then

$$\begin{aligned}
(\Phi_J \circ F(\alpha_\bullet))(\psi_\bullet) &= \Phi_J(\alpha_\bullet \circ \psi_\bullet) \\
&= ((\alpha_\bullet \circ \psi_\bullet) \circ u_\bullet, (\alpha_\bullet \circ \psi_\bullet)\gamma_\bullet).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
(G(\alpha_\bullet) \circ \Phi_I)(\psi_\bullet) &= G(\alpha_\bullet)(\psi_\bullet \circ u_\bullet, \psi_\bullet\gamma_\bullet) \\
&= (\alpha_\bullet \circ \psi_\bullet \circ u_\bullet, \alpha_\bullet(\psi_\bullet\gamma_\bullet))
\end{aligned}$$

We have that

$$\begin{aligned}
((\alpha_\bullet(\psi_\bullet\gamma_\bullet))_n &= \alpha_{n+1} \circ (\psi_\bullet\gamma_\bullet)_n \\
&= \alpha_{n+1} \circ \psi_{n+1} \circ \gamma_n \\
&= (\alpha_\bullet \circ \psi_\bullet)_{n+1} \circ \gamma_n \\
&= ((\alpha_\bullet \circ \psi_\bullet)\gamma_\bullet)_n,
\end{aligned}$$

and the conclusion follows. □

Theorem 3.2.8.

$$\mathsf{hcoker}(\mathrm{id}_{\Bbbk} \otimes f) \simeq \mathsf{colim} \left(\begin{array}{ccc}
 & & \Bbbk \otimes M \longrightarrow 0 \\
 & & \big\downarrow{\scriptstyle i_0 \otimes \mathrm{id}_M} \\
\Bbbk \otimes M & \xhookrightarrow{i_1 \otimes \mathrm{id}_M} & \mathbb{I} \otimes M \\
{\scriptstyle \mathrm{id}_{\Bbbk} \otimes f}\big\downarrow & & \\
\Bbbk \otimes N & &
\end{array} \right) . \tag{3.2.4}$$

Proof. We will show that for every chain complex $A \in \mathsf{Ch}_R$, we have a natural isomorphism

$$\Phi_A : \mathsf{hom}_{\mathsf{Ch}_R}(H, A) \simeq \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A), \tag{3.2.5}$$

where H is the colimit in the right-hand side of 3.2.4. Consider any chain map $\psi_\bullet : H \to A$. Recalling definition 3.1.11 the data of H comes with structure maps

$$\begin{cases} s_0 : \Bbbk \otimes M \to H \\ s_1 : \Bbbk \otimes M \to H \\ u : \Bbbk \otimes N \to H \\ v : \mathbb{I} \otimes M \to H \end{cases}$$

and of course $0_{0,H} : 0 \to H$ (but there is always such a morphism since 0 is initial in Ch_R). Observe that s_0 is associated with $i_0 \otimes \mathrm{id}_M$ and s_1 with $i_1 \otimes \mathrm{id}_M$, as expressed by the indices. Furthermore, straight from the properties of the structure maps (3.1.11[(a)]), it follows that

$$\begin{cases} s_0 = v \circ (i_0 \otimes \mathrm{id}_M) \\ s_1 = v \circ (i_1 \otimes \mathrm{id}_M) \\ s_1 = u \circ (\mathrm{id}_\Bbbk \otimes f) \\ s_0 = 0_{0,H} \circ 0_{\Bbbk \otimes M,0} = 0_{\Bbbk \otimes M,H}. \end{cases}$$

The last equality gives that $v \circ (i_0 \otimes \mathrm{id}_M) = 0$ and the second and third equality gives that $v \circ (i_1 \otimes \mathrm{id}_M) = u \circ (\mathrm{id}_\Bbbk \otimes f)$.

By theorem 3.1.6 the chain map v determines a chain homotopy $0 \overset{\gamma_\bullet}{\Rrightarrow} v \circ (i_1 \otimes \mathrm{id}_M) = u \circ (\mathrm{id}_\Bbbk \otimes f)$. Therefore, $(u, \gamma_\bullet) \in \mathsf{Cocone}^h(\mathrm{id}_\Bbbk \otimes f, H)$. Let A be an arbitrary chain complex in Ch_R. Let $\Phi_A : \hom_{\mathsf{Ch}_R}(H, A) \to \mathsf{Cocone}^h(\mathrm{id}_\Bbbk \otimes f, A)$ be defined by sending chain maps $\psi_\bullet : H \to A$ to $(\psi_\bullet \circ u, \psi_\bullet \gamma)$. Clearly $\psi_\bullet \circ u : \Bbbk \otimes N \to A$ is a chain map, and a standard check gives that $0 \overset{\psi_\bullet \gamma_\bullet}{\Rrightarrow} (\psi_\bullet \circ u) \circ \mathrm{id}_\Bbbk \otimes f$ is a null-homotopy. It follows that $(\psi_\bullet \circ u, \psi_\bullet \gamma) \in \mathsf{Cocone}^h(\mathrm{id}_\Bbbk \otimes f, A)$.

On the other hand, let $(g, \eta) \in \mathsf{Cocone}^h(\mathrm{id}_\Bbbk \otimes f, A)$ so that $g : \Bbbk \otimes N \to A$ is a chain map and $0 \overset{\eta}{\Rrightarrow} g \circ (\mathrm{id}_\Bbbk \otimes f)$ is a chain homotopy. By theorem 3.1.6 this determines a *unique* chain map $w : \mathbb{I} \otimes M \to A$ such that $w \circ (i_0 \otimes \mathrm{id}_M) = 0$ and $w \circ (i_1 \otimes \mathrm{id}_M) = g \circ (\mathrm{id}_\Bbbk \otimes f)$.

Now associate to diagram 3.2.4 the zero map $\Bbbk \otimes M \to A$ to the top $\Bbbk \otimes M$, $g \circ (\mathrm{id}_\Bbbk \otimes f)$ to the left copy of $\Bbbk \otimes M$, w to $\mathbb{I} \otimes M$, for $\Bbbk \otimes N$ choose g, and for 0 there is the unique map $0 \to A$. We claim that this system satisfies 3.1.13 in definition 3.1.11 so that there is a unique chain map $\Theta_A(g, \eta) : H \to A$ satisfying the property described by equation 3.1.14 in the definition of the colimit. The chain map $\Theta_A(g, \eta)$ satisfies $\Theta_A(g, \eta) \circ u = g$ and $\Theta_A(g, \eta) \circ v = w$.

We claim that Φ_A and Θ_A are inverses. If we let $\psi : H \to A$ be a chain map, then $\Phi_A(\psi) = (\psi \circ u, \psi\gamma)$. We let $w_\psi : \mathbb{I} \otimes M \to A$ be the chain map corresponding to $\psi\gamma$ by theorem 3.1.6. Then $w_\psi = \psi \circ v$: We have that $(\psi \circ v) \circ (i_0 \otimes \mathrm{id}_M)) = \psi \circ 0 = 0$ and that $(\psi \circ v) \circ (i_1 \otimes \mathrm{id}_M) = \psi \circ u \circ (\mathrm{id}_\Bbbk \otimes f)$. Since γ is the chain homotopy associated to v, $\psi\gamma$ is the chain homotopy associated with $\psi \circ v$. By the one-to-one correspondence in 3.1.6 it follows that $w_\psi = \psi \circ v$. Hence we see that for $\psi : H \to A$ a chain map, we have that $\Theta_A(\Phi_A(\psi)) = \Theta_A(\psi \circ u, \psi\gamma) =: \theta$ satisfies $\theta \circ u = \psi \circ u$ and $\theta \circ v = w_\psi = \psi \circ v$.

Furthermore we find that

$$\begin{aligned} \theta \circ s_0 &= \theta \circ v \circ (i_0 \otimes \mathrm{id}_M) \\ &= \psi \circ v \circ (i_0 \otimes \mathrm{id}_M) \\ &= \psi \circ s_0, \end{aligned}$$

and

$$\begin{aligned}\theta \circ s_1 &= \theta \circ v \circ (i_1 \otimes \mathrm{id}_M)\\ &= \psi \circ v \circ (i_1 \otimes \mathrm{id}_M)\\ &= \psi \circ s_1.\end{aligned}$$

We also have that $\theta \circ 0_{0,H} = \psi \circ 0_{0,H}$. By *uniqueness* of θ it follows that $\theta = \psi$, i.e.

$$\Theta_A(\Psi_A(\psi)) = \psi.$$

Now instead consider arbitrary $(g, \eta) \in \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A)$ and let $\Theta_A(g, \eta) := \lambda$, so that $\Phi_A(\Theta_A(g, \eta)) = (\lambda \circ u, \lambda\gamma)$. By construction we have that $\lambda \circ u = g$. If $w : \mathbb{I} \otimes M \to A$ is the unique chain map corresponding to the chain homotopy η then also $\lambda \circ v = w$. Since γ is the chain homotopy associated with v under theorem 3.1.6, by uniqueness it follows that $\lambda\gamma = \eta$. Hence

$$\begin{aligned}\Phi_A(\Theta_A(g, \eta)) &= (\lambda \circ u, \lambda\gamma)\\ &= (g, \eta).\end{aligned}$$

Naturality check on $\Phi_{(\,\cdot\,)}$: For any chain map $\alpha : A \to B$ it is straightforward to verify (by unraveling of definitions) that the following diagram commutes,

$$\begin{array}{ccccc} A & & \mathsf{hom}_{\mathsf{Ch}_R}(H, A) & \xrightarrow{\Phi_A} & \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A)\\ \downarrow{\scriptstyle\alpha} & \rightsquigarrow & \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle G(\alpha)}\\ B & & \mathsf{hom}_{\mathsf{Ch}_R}(H, B) & \xrightarrow[\Phi_B]{} & \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, B)\end{array} \tag{3.2.6}$$

with $F(\alpha) = \alpha_*$ the pushforward and $G(\alpha)$ defined on pairs $(g, \eta) \in \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk}\, a \otimes f, A)$ as $G(\alpha)(g, \eta) = (\alpha \circ g, \alpha\eta)$. □

Corollary 3.2.9. $\mathsf{hcoker}(f) \simeq H$ *where* H *is the colimit in 3.2.4.*

Remark 3.2.10. By $\simeq$, we mean "canonically isomorphic" in the (categorical) sense explained at the end of the proof.

Proof. We first show that $\mathsf{hcoker}(f)$ is canonically isomorphic to $\mathsf{hcoker}(\mathrm{id}_{\Bbbk} \otimes f)$. By appealing to theorem 2.5.4, there is a *left unitor* λ with isomorphism $\lambda_M : \Bbbk \otimes M \simeq M$ for every $M \in \mathsf{Ch}_R$. Given $f : M \to N$ a chain map, we then see that by naturality of λ, there is a commutative diagram

$$\begin{array}{ccc} \Bbbk \otimes M & \xrightarrow[\simeq]{\lambda_M} & M\\ \downarrow{\scriptstyle \mathrm{id}_{\Bbbk} \otimes f} & \circlearrowleft & \downarrow{\scriptstyle f}\\ \Bbbk \otimes N & \xrightarrow[\lambda_N]{\simeq} & N \end{array},$$

so that $\mathrm{id}_{\Bbbk} \otimes f = \lambda_N^{-1} \circ f \circ \lambda_M$.

We show that there is a natural bijection $\mathscr{X}_A : \mathsf{Cocone}^h(f, A) \simeq \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A)$ for each $A \in \mathsf{Ch}_R$, defined by $(g, \eta) \overset{\mathscr{X}_A}{\longmapsto} (g \circ \lambda_N, \eta \circ \lambda_M)$. Observe first that $g \circ \lambda_N : \Bbbk \otimes N \to A$ is a chain map with appropriate domain and codomain. By naturality of λ and λ_M being a chain map, we get from the null homotopy $0 \overset{\eta}{\Rightarrow} g \circ f$ a null homotopy $0 \overset{\eta\lambda_M}{\Rightarrow} (g \circ \lambda_N) \circ (\mathrm{id}_{\Bbbk} \otimes f)$.

Now define $\mathscr{Y}_A : \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A) \to \mathsf{Cocone}^h(f, A)$ by $(\tilde{g}, \tilde{\eta}) \overset{\mathscr{Y}_A}{\longmapsto} (\tilde{g} \circ \lambda_N^{-1}, \tilde{\eta} \circ \lambda_M^{-1})$. The first argument $\tilde{g} \circ \lambda_N^{-1}$ is easy to verify has the correct properties. For the second argument $\tilde{\eta}\lambda_M^{-1}$, essentially the same check (using naturality and that λ_N^{-1} is a chain map) as for $\mathscr{X}_A$ gives that this defines an honest map into $\mathsf{Cocone}^h(f, A)$. It is easy to verify that for each $A \in \mathsf{Ch}_R$, $\mathscr{X}_A$ and $\mathscr{Y}_A$ are inverses. Hence $\mathscr{X}_A$ defines a bijection.

Naturality check on $\mathscr{X}_{(\,\cdot\,)}$: We verify that the following diagram commutes,

$$\begin{array}{ccccc} A & & \mathsf{Cocone}^h(f, A) & \xrightarrow{\ \mathscr{X}_A\ } & \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, A) \\ {\scriptstyle\alpha}\Big\downarrow & \rightsquigarrow & \Big\downarrow{\scriptstyle F(\alpha)} & & \Big\downarrow{\scriptstyle G(\alpha)} \\ B & & \mathsf{Cocone}^h(f, B) & \xrightarrow[\ \mathscr{X}_B\ ]{} & \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, B) \end{array},$$

for each chain map $\alpha : A \to B$ and with $F(\alpha)$ and $G(\alpha)$ defined by "postcomposition" $F(\alpha)(g, \eta) = (\alpha \circ g, \alpha\eta)$. We find that for $(g, \eta) \in \mathsf{Cocone}^h(f, A)$ we have that

$$\begin{aligned} \mathscr{X}_B(F(\alpha)(g, \eta)) &= \mathscr{X}_B(\alpha \circ g, \alpha\eta) \\ &= (\alpha \circ g \circ \lambda_N, (\alpha\eta) \circ \lambda_M), \end{aligned}$$

while

$$\begin{aligned} G(\alpha)(\mathscr{X}_A(g, \eta)) &= G(\alpha)(g \circ \lambda_N, \eta \circ \lambda_M) \\ &= (\alpha \circ g \circ \lambda_N, \alpha(\eta \circ \lambda_M)). \end{aligned}$$

Observe that in degree n, we have that

$$\begin{aligned} (\alpha(\eta \circ \lambda_M))_n &= \alpha_{n+1} \circ \eta_n \circ (\lambda_M)_n \\ &= (\alpha\eta)_n \circ (\lambda_M)_n \\ &= ((\alpha\eta) \circ \lambda_M)_n, \end{aligned}$$

and naturality follows.

We have shown that $\mathscr{X} : F \Rightarrow G$, i.e. $\mathscr{X} : \mathsf{Cocone}^h(f, -) \simeq \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, -)$ is an isomorphism of functors $\mathsf{Ch}_R \to \mathsf{Set}$. Since we already have (in the functor category $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Set})$) the isomorphism

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), -) \simeq \mathsf{Cocone}^h(f, -)$$

and

$$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(\mathrm{id}_{\Bbbk} \otimes f), -) \simeq \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, -)$$

it follows that we have the isomorphism

$$\begin{aligned}\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), -) &\simeq \mathsf{Cocone}^h(f, -)\\ &\overset{\mathscr{X}}{\simeq} \mathsf{Cocone}^h(\mathrm{id}_{\Bbbk} \otimes f, -)\\ &\simeq \mathsf{hom}_{\mathsf{Ch}_R}(H, -) \qquad \text{by 3.2.5.}\end{aligned} \tag{3.2.7}$$

in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Set})$. By [Rie16, Prop. 2.3.1.(iii) ⇒ (i)] it follows that $\mathsf{hcoker}(f) \simeq H$. Indeed, Yoneda's lemma [Rie16, Theorem 2.2.4] provides a *unique* isomorphism

$$\gamma : H \simeq \mathsf{hcoker}(f),$$

such that $\gamma^* : \mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), -) \Rightarrow \mathsf{hom}_{\mathsf{Ch}_R}(H, -)$ is precisely the composite in 3.2.7, where γ^* denotes the **pullback** $g \mapsto g \circ \gamma$ (γ must be an isomorphism since full and faithful functors [the appropriate yoneda embedding, in this case] **reflects** isomorphisms [Rie16, Exercise 1.5.vi.(i)]).

□

Theorem 3.2.11. *Let $M \xrightarrow{f} N$ be a morphism in Ch_R that is injective in each degree. Then the canonical morphism $\mathsf{hcoker}(f) \to \mathsf{coker}(f)$ to the level-wise cokernel is a quasi-isomorphism.*

Proof. We use the explicit cone model $\mathsf{Cone}(f)$ (defined as in definition 3.2.1) for $\mathsf{hcoker}(f)$ coming from theorem 3.2.6. Let $q : N_\bullet \to \mathsf{coker}(f)$ be the canonical map defined in degree n as $q_n : N_n \twoheadrightarrow N_n/\mathsf{im}(f_n)$. Then $q \circ f = 0$. Hence we have the trivial zero homotopy $0 \Rightarrow q \circ f$ and since q has appropriate domain and codomain it follows that $(q, 0) \in \mathsf{Cocone}^h(f, \mathsf{coker}(f))$ which under the bijection $\mathsf{Cocone}^h(f, \mathsf{coker}(f)) \simeq \mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), \mathsf{coker}(f))$ gives us a *unique* chain map $\pi : \mathsf{hcoker}(f) \to \mathsf{coker}(f)$ such that

$$\pi \circ u_\bullet = q \tag{3.2.8}$$

and

$$\pi\gamma_\bullet = 0, \tag{3.2.9}$$

with $u_\bullet, \gamma_\bullet$ defined as in the proof of theorem 3.2.6. We then see that by in degree, we have that

$$\begin{aligned}(\pi \circ u_\bullet)_n &= \pi_n \circ u_n\\ &= q_n.\end{aligned}$$

Since we may write any element $(a, b) \in M_{n-1} \oplus N_n$ as

$$(a, b) = -\gamma_{n-1}(a) - u_n(b)$$

it follows that

$$\begin{aligned}\pi_n(a, b) &= \pi_n(-\gamma_{n-1}(a) - u_n(b))\\ &= -\pi_n(\gamma_{n-1}(a)) - \pi_n(u_n(b))\\ &= -\pi_n(u_n(b)), \qquad \text{since } \pi\gamma_\bullet = 0\\ &= \pi_n((0, b))\\ &= -q_n(b).\end{aligned} \tag{3.2.10}$$

We now show that the induced map $\mathsf{H}(\pi) : \mathsf{H}(\mathsf{hcoker}(f)) \to \mathsf{H}(\mathsf{coker}(f))$ on homology, is an isomorphism in each degree n, i.e. a quasi-isomorphism.

$\mathsf{H}(\pi)_n$ is surjective: Let $[\bar{b}] \in \mathsf{H}(\mathsf{coker}(f))_n$ with $\bar{b} = q_n(b) \in \mathsf{coker}(f)_n = N_n/\mathsf{im}(f_n)$. Since $\bar{b} \in \mathsf{ker}\left(d_n^{\mathsf{coker}(f)}\right)$ we find that

$$\begin{aligned} 0 &= d_n^{\mathsf{coker}(f)}(\bar{b}) \\ &= d_n^{\mathsf{coker}(f)}(q_n(b)) \\ &= q_{n-1}d_n^N(b), \qquad \text{since } q \text{ is a chain map,} \end{aligned}$$

so that $d_n^N(b) \in \mathsf{im}(f_{n-1})$, i.e., there is an $a \in M_{n-1}$ so that $f_{n-1}(a) = d_n^N(b)$. Upon applying d_{n-1}^N to this we find that

$$\begin{aligned} 0 &= d_{n-1}^N d_n^N(b) \\ &= d_{n-1}^N f_{n-1}(a) \\ &= f_{n-2}d_{n-1}^M(a), \qquad \text{since } f \text{ is a chain map.} \end{aligned}$$

We have that f_{n-2} is by assumption *injective*, hence $d_{n-1}^M(a) = 0$. It follows that

$$\begin{aligned} d_n^{\mathsf{Cone}(f)}(a, -b) &= (-d_{n-1}^M(a), -d_n^N(b) + f_{n-1}(a)) \\ &= (0, 0), \end{aligned}$$

so that $(a, -b) \in \mathsf{ker}\left(d_n^{\mathsf{Cone}(f)}\right) \rightsquigarrow [(a, -b)] \in \mathsf{H}(\mathsf{hcoker}(f))_n$. We then see that

$$\begin{aligned} \mathsf{H}(\pi)_n\left([(a, -b)]\right) &= [\pi_n(a, -b)] \\ &= [-q_n(-b)], \qquad \text{by 3.2.10} \\ &= [\bar{b}], \end{aligned}$$

so that $\mathsf{H}(\pi)_n$ is indeed surjective.

$\mathsf{H}(\pi)_n$ injective: Let $(a, b) \in \mathsf{ker}\left(d_n^{\mathsf{Cone}(f)}\right) \subset M_{n-1} \oplus N_n$, so that $\mathsf{H}(\pi)_n([(a, b)]) = 0$. This means that $[-q_n(b)] = 0 \in \mathsf{H}(\mathsf{coker}(f))_n$. Hence there is a $\bar{c} \in \mathsf{coker}(f)_{n+1} = N_{n+1}/\mathsf{im}(f_{n+1})$ such that $d_{n+1}^{\mathsf{coker}(f)}(\bar{c}) = -q_n(b)$. But $q_{n+1} : N_{n+1} \twoheadrightarrow \mathsf{coker}(f)_{n+1}$ is surjective, so there is a $c \in N_{n+1}$ such that

$$\begin{aligned} -q_n(b) &= d_{n+1}^{\mathsf{coker}(f)}(\bar{c}) \\ &= d_{n+1}^{\mathsf{coker}(f)}(q_{n+1}(c)) \\ &= q_n\left(d_{n+1}^N(c)\right), \qquad \text{since } q \text{ is a chain map} \\ \Leftrightarrow q_n(b + d_{n+1}^N(c)) &= 0. \end{aligned}$$

It follows that $b + d_{n+1}^N(c) \in \mathsf{im}(f_n)$ so there is some $x \in M_n$ such that

$$f_n(x) = b + d_{n+1}^N(c) \Leftrightarrow f_n(x) - d_{n+1}^N(c) = b \tag{3.2.11}$$

Since (a, b) was a cycle, we find that $-d_{n-1}^M(a) = 0$ and

$$d_n^N(b) + f_{n-1}(a) = 0 \Leftrightarrow d_n^N(b) = -f_{n-1}(a) = f_{n-1}(-a). \tag{3.2.12}$$

Upon applying d_n^N to equation 3.2.11 we find that

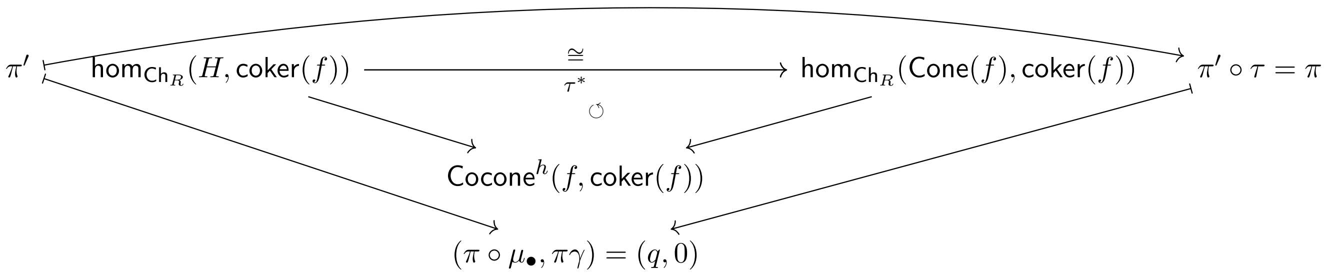
$$\begin{aligned} & d_n^N(f_n(x)) = d_n^N(b) \\ \Leftrightarrow\ & f_{n-1}\left(d_n^M(x)\right) = d_n^N(b), \qquad \text{since } f \text{ is a chain map} \\ \Leftrightarrow\ & f_{n-1}\left(d_n^M(x)\right) = f_{n-1}(-a), \qquad \text{by 3.2.12.} \end{aligned}$$

By injectivity of f_{n-1} it follows that $d_n^M(x) = -a$. Hence, we see that

$$\begin{aligned} d_{n+1}^{\mathsf{Cone}(f)}(x, -c) &= (-d_n^M(x), f_n(x) - d_{n+1}^N(c)) \\ &= (a, b), \end{aligned}$$

so that $(a, b) \in \mathsf{im}\left(d_{n+1}^{\mathsf{Cone}(f)}\right) \rightsquigarrow [(a, b)] = 0 \in \mathsf{H}(\mathsf{hcoker}(f))_n$. Therefore, $\mathsf{H}(\pi)_n$ is injective.

We conclude that $\mathsf{H}(\pi)$ is an isomorphism in each degree n, i.e. π is a quasi-isomorphism. To conclude that this is independent of the specific *model* for the homotopy cokernel we choose, we note that any two homotopy cokernels say H and $\mathsf{Cone}(f)$ of f, are related as in the diagram below,



Hence $\pi' = \pi \circ \tau^{-1}$ and since we showed that π is a quasi isomorphism and τ^{-1} is an isomorphism it follows that π' is a quasi-isomorphism. ☐

The example provided below shows that the canonical map $\mathsf{hocoker}(f) \to \mathsf{coker}(f)$ in 3.2.8 need not be a *homotopy equivalence* (recall definition 3.1.15).

Example 3.2.12. Let $R = \mathbb{Z}$ and consider the complex $\underset{\sim}{\mathbb{Z}}$ in Ch_R and let $f : \underset{\sim}{\mathbb{Z}} \to \underset{\sim}{\mathbb{Z}}$ be the chain map with degree zero homomorphism $f_0 : \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ multiplication by 2. Then f is injective in each degree, and with $\mathsf{coker}(f) = \underset{\sim}{\mathbb{Z}/2\mathbb{Z}}$.

If we take as model for $\mathsf{hcoker}(f)$ the cone $\mathsf{Cone}(f)$ of f, then we see that

$$\mathsf{Cone}(f)_n = \begin{cases} 0 \oplus \mathbb{Z} \simeq \mathbb{Z}, n = 0, \\ \mathbb{Z} \oplus 0 \simeq \mathbb{Z}, n = 1, \\ 0, \ \text{otherwise}. \end{cases},$$

with differentials

$$d_n^{\mathsf{Cone}(f)} = \begin{cases} (0, f_0), \ \text{if } n = 1 \\ 0, \ \text{otherwise}. \end{cases}$$

Then we see that the canonical map $\pi : \mathsf{Cone}(f) \to \mathsf{coker}(f)$ is given by in degree n on elements $(a,b) \in \mathsf{Cone}(f)_n$ by $\pi_n(a,b) = -q_n(b) = q_n(b) \mod 2 = \bar{b}$. We find then that $\pi_0 : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the projection homomorphism and that $\pi_n = 0$ otherwise. We claim that this is not a homotopy equivalence. If it were, there would exist a chain map $g : \mathsf{coker}(f) \to \mathsf{Cone}(f)$ and a chain homotopy $\mathrm{id}_{\mathsf{coker}(f)} \Rightarrow \pi \circ g$. Now consider that $g : \mathsf{coker}(f)_0 \to \mathsf{Cone}(f)_0$ is a map $\mathbb{Z}/2\mathbb{Z} \to 0 \oplus \mathbb{Z}$. The only possible such map is the zero homomorphism, since $\mathbb{Z}$ has no torsion elements. But then $\pi \circ g$ must be the zero homomorphism in each degree. Since all differentials of $\mathsf{coker}(f)$ are zero, this would force in particular

$$\begin{aligned} \underbrace{(\pi \circ g)_0}_{=0} - \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} &= 0, \\ \Leftrightarrow \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} &= 0, \end{aligned}$$

which is impossible. Hence π is not a homotopy equivalence, but is a quasi-isomorphism by theorem 3.2.11.

Definition 3.2.13 (Suspension)**.** For $M_\bullet$ a chain complex in Ch_R, we define

$$\Sigma M_\bullet := \mathsf{hcoker}(M_\bullet \to 0) \tag{3.2.13}$$

as the homotopy cokernel of the zero map $M_\bullet \to 0$, and we call this the **suspension** of $M_\bullet$.

Observe that if we take the cone $\mathsf{Cone}(M_\bullet \to 0)$ as a concrete model for the homotopy cokernel of $M_\bullet \to 0$, then we find that

$$\begin{aligned} (\Sigma M_\bullet)_n &= \mathsf{Cone}(M_\bullet \to 0)_n \\ &= M_{n-1} \oplus 0 \\ &\approx M_{n-1}, \end{aligned}$$

and that

$$\begin{aligned} d_n^{\Sigma M_\bullet}(a,b) &= \left(-d_{n-1}^M(a), 0\right) \\ &\approx -d_{n-1}^M(a), \qquad \text{canonical isomorphism.} \end{aligned}$$

Hence the nicest model to work with here is this isomorphic model with complexes shifted down by one degree and with differentials in degree n as $-d_{n-1}^M$. We denote this complex by $M_\bullet[1]$ and unless otherwise stated, make the identification $\Sigma M_\bullet = M_\bullet[1]$. One notes that for $k \geqslant 0$, $\Sigma^k M_\bullet$ is the complex which in degree n has the module M_{n-k} and differential $d_n^{\Sigma^k M_\bullet} = (-1)^k d_{n-k}^{M_\bullet}$. We denote this by $M_\bullet[k]$.

There is then a natural extension for $k \geqslant 0$, as $\Sigma^{-k} M_\bullet = M_\bullet[-k]$ defined as the complex with degree n module as M_{n+k} and differential $d_n^{\Sigma^{-k} M_\bullet} = (-1)^k d_{n+k}^{M_\bullet}$. We will call the process of taking $\Sigma^{-1} M_\bullet$ of a complex $M_\bullet$ **desuspension**. It follows that for any $k \in \mathbb{Z}$ we may write $\Sigma^k M_\bullet = M_\bullet[k]$. We will later show that there is a corresponding homotopical construction to the "ordinary kernel", the "homotopy kernel" $\mathsf{hker}(\,\cdot\,)$, such that $\Sigma^{-1} M_\bullet$ and $\mathsf{hker}(0 \to M_\bullet)$ are canonically isomorphic, i.e. the former is a *model* of the latter. We observe that

$$\mathsf{H}_n\left(\Sigma^k M_\bullet\right) = \mathsf{H}_{n-k}(M_\bullet), \tag{3.2.14}$$

since homology is insensitive to the sign-changes in front of the differentials and only care about shifts in degree.

Furthermore, it is (almost) immediate that Σ^k defines an equivalence of categories

$$\Sigma^k : \mathsf{Ch}_R \to \mathsf{Ch}_R.$$

if one just think of Σ as the *shift functor*. Perhaps more notable is that Σ^k is such that it both preserves and reflects homotopies, i.e.

$$f \simeq g \Leftrightarrow \Sigma^k f \simeq \Sigma^k g.$$

It is furthermore immediate that $\Sigma^k 0_\bullet \cong 0_\bullet$. In particular, we claim it follows that Σ^k preserves homotopy cokernel sequences, i.e. sequences on the form

$$M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f)$$

for some chain map $M \xrightarrow{f} N$ in Ch_R, which will become important in §3.4.

Remark 3.2.14. We reiterate that we will mostly make the identification of $\Sigma^i X$ with $X[i]$ such that

$$\left(\Sigma^i X\right)_\ell = (X[i])_\ell = X_{\ell-i}$$

and with differential

$$d_\ell^{\Sigma^i X} = d_\ell^{X[i]} = (-1)^i d_{\ell-i}^X,$$

in this thesis.

Theorem 3.2.15. *Let $M \xrightarrow{f} N$ be an arbitrary chain map in Ch_R. Then f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic.*

Proof. Take as a concrete model for $\mathsf{hcoker}(f)$, the cone $\mathsf{Cone}(f)$. We denote this cone by $C_\bullet$. Observe first that $C_\bullet$ is acyclic if and only if every *cycle* in C_n is a *boundary*, for each n. An element $x = (a, b) \in C_n = M_{n-1} \oplus N_n$ is a cycle precisely when $d_{n-1}^M(a) = 0$ and $d_n^N(b) + f_{n-1}(a) = 0 \Leftrightarrow d_n^N(b) = -f_{n-1}(a)$.

$\underline{\Rightarrow}$: Assume that f is a quasi-isomorphism, and let $(a, b) \in C_n$ be an arbitrary cycle. Then $f_{n-1}(a) = -d_n^N(b)$ is a boundary in N_{n-1}. It follows that

$$\begin{aligned} \mathsf{H}_{n-1}(f)([a]) &= [f_{n-1}(a)] \\ &= [d_n^N(-b)] \\ &= 0. \end{aligned}$$

By assumption, $\mathsf{H}_{n-1}(f)$ is injective, so $[a] = 0$, i.e. $a = d_n^M(a')$ for some $a' \in M_n$. Hence we have that

$$\begin{aligned} & d_n^N(b) + f_{n-1}(a) = 0 \\ \Leftrightarrow\ & d_n^N(b) + f_{n-1}\left(d_n^M(a')\right) = 0 \\ \Leftrightarrow\ & d_n^N(b) + d_n^N(f_n(a')) = 0, \qquad \text{since } f \text{ is a chain map} \\ \Leftrightarrow\ & d_n^N(b + f_n(a')) = 0 \\ \Rightarrow\ & b + f_n(a') \text{ is a cycle in } N_n. \end{aligned}$$

Since $\mathsf{H}_n(f)$ is surjective, there exists a cycle $y \in M_n$ such that $\mathsf{H}_n(f)([y]) = [f_n(y)] = [b + f_n(a')]$ in $\mathsf{H}_n(N)$, which means that there is a $z \in N_{n+1}$ so that

$$\begin{aligned} & d_{n+1}^N(z) = f_n(y) - (b + f_n(a')) \\ \Leftrightarrow\ & b = f_n(y - a') - d_{n+1}^N(z). \end{aligned}$$

Then we see that

$$\begin{aligned} d_{n+1}^C(-(y-a'),-z) &= \left(-d_n^M(y-a'), f_n(y-a') + d_{n+1}^N(-z)\right) \\ &= (a,b), \end{aligned}$$

so that (a,b) is a boundary. We have shown that every cycle is a boundary, hence $\mathsf{H}_n(C_\bullet) = 0$, i.e. $C_\bullet$ is acyclic.

$\underline{\Leftarrow}$: It is enough to show that $\mathsf{H}_n(f)$ is injective and surjective.

Injective: Let $[a]$ in $\mathsf{H}_n(M)$ such that $\mathsf{H}_n(f)[a] = 0$. Then $f_n(a)$ is a boundary in $\mathsf{H}_n(N)$, so there is some $b \in N_{n+1}$ such that $d_{n+1}^N(b) = f_n(a) \Leftrightarrow f_n(-a) + d_{n+1}^N(b) = 0$. Then

$$d_{n+1}^N(-a,b) = \left(d_n^M(a), f_n(-a) + d_{n+1}^N(b)\right) = (0,0),$$

using that a is a cycle in M_n. Since $C_\bullet$ is *acyclic* by assumption, it follows that every cycle is a boundary. Since $(-a,b)$ in $C_{n+1} = M_n \oplus N_{n+1}$ was a cycle, it is boundary, so there is an element $(x,y) \in C_{n+2} = M_{n+1} \oplus N_{n+2}$ such that

$$d_{n+2}^C(x,y) = (-a,b).$$

But this means precisely that $-d_{n+1}^M(x) = -a \Leftrightarrow d_{n+1}^M(x) = a$, i.e. a is a boundary, so that $[a] = 0$. Hence $\mathsf{H}_n(f)$ is *injective.*

Surjective: Let $[b] \in \mathsf{H}_n(N)$ be arbitrary, where b is *by definition* some cycle in N_n. Then $(0,b) \in C_n$ is a cycle, hence by acyclicity of $C_\bullet$ it is a boundary, i.e. there are $(x,y) \in C_{n+1}$ such that $d_{n+1}^C(x,y) = (0,b)$. This means that $-d_n^M(x) = 0$ and $f_n(x) + d_{n+1}^N(y) = b$.

Together, this gives that x is a cycle in M_n such that

$$\begin{aligned} \mathsf{H}_n(f)([x]) &= [f_n(x)] \\ &= [b - d_{n+1}^N(y)] \\ &= [b]. \end{aligned}$$

We conclude that $\mathsf{H}_n(f)$ is surjective.

It follows that $\mathsf{H}_n(f)$ is an isomorphism, and so f is a quasi-isomorphism. □

Theorem 3.2.16. *Given a chain map $M \xrightarrow{f} N$ in Ch_R, we have that f is a homotopy equivalence iff $\mathsf{hcoker}(f)$ is contractible.*

Remark 3.2.17. To give context to the proof below: Observe that whenever we have an R-module homomorphism $S : A \oplus B \to C \oplus D$ in Mod_R, then both $A \oplus B$ and $C \oplus D$ are **biproducts**, i.e. both *products* and *coproducts* in Mod_R, so there are associated product and coproduct maps $\iota_A, \iota_B, \iota_C, \iota_D$ and $\pi_A, \pi_B, \pi_C, \pi_D$. Let

$$\begin{cases} \alpha := \pi_C \circ S \circ \iota_A : A \to C \\ \beta := \pi_C \circ S \circ \iota_B : B \to C \\ \gamma := \pi_D \circ S \circ \iota_A : A \to D \\ \delta := \pi_D \circ S \circ \iota_B : B \to D \end{cases}. \tag{3.2.15}$$

Let $(a,b) \in A \oplus B$ be arbitrary. Then

$$\begin{aligned} S(a,b) &= S((a,0)+(0,b)) \\ &= S((a,0)) + S((0,b)) \\ &= S(\iota_A(a)) + S(\iota_B(b)) \\ &= (\alpha(a), \gamma(a)) + (\beta(b), \delta(b)) \\ &= (\alpha(a)+\beta(b), \gamma(a)+\delta(b)), \end{aligned}$$

where we in the next to last equality used that any $x \in C \oplus D$ is on the form $x = (\pi_C(x), \pi_D(x))$ together with how we defined the maps in 3.2.15. Hence we may represent any such map S on the form

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In particular, if we have the identity $\mathrm{id} : A \oplus B \to A \oplus B$ then one may check that we may represent id as

$$\mathrm{id} = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}.$$

Proof. We work with the *cone* $C := \mathsf{Cone}(f)$ of f as a model for $\mathsf{hcoker}(f)$.

$\underline{\Rightarrow}$: By hypothesis we are given the data of a chain map $g : N \to M$ together with chain homotopies h, k where $h : M \to M[-1]$ and $k : N \to N[-1]$ satisfies

$$d^M h + h d^M = g \circ f - \mathrm{id}_M \tag{3.2.16}$$

and

$$d^N k + k d^N = f \circ g - \mathrm{id}_N, \tag{3.2.17}$$

respectively. Define $A := (A_n)_{n\in\mathbb{Z}}$ with $A_n : M_{n-1} \oplus N_n \to M_n \oplus N_{n+1}$, explicitly by $A_n(a,b) := (h_{n-1}(a) + g_n(b), -k_n(b))$. Then we see that for $(a,b) \in M_{n-1} \oplus N_n$, we have

$$\begin{aligned} d^C_{n+1} A_n(a,b) &= d^C_{n+1}(h_{n-1}(a) + g_n(b), -k_n(b)) \\ &= \left(d^M_n(-h_{n-1}(a)) - d^M_n(g_n(b)), -d^N_{n+1}(k_n(b)) + f_n(h_{n-1}(a)) + f_n g_n(b)\right), \end{aligned}$$

and

$$\begin{aligned} A_{n-1}\left(d^C_n(a,b)\right) &= A_{n-1}\left(-d^M_{n-1}(a), d^N_n(b) + f_{n-1}(a)\right) \\ &= \left(-h_{n-2}\left(d^M_{n-1}(a)\right) + g_{n-1}\left(d^N_n(b)\right) + g_{n-1}f_{n-1}(a), -k_{n-1}\left(d^N_n(b)\right) - k_{n-1}(f_{n-1}(a))\right). \end{aligned}$$

Therefore, one finds that, by using that g is a chain map, together with 3.2.16 and 3.2.17,

$$(d^C_{n+1}A_n + A_{n-1}d^C_n)(a,b) = (a, b + f_n(h_{n-1}(a)) - k_{n-1}(f_{n-1}(a))).$$

Letting $\ell_n : M_{n-1} \to N_n$ be defined by $\ell_n(a) = f_n h_{n-1}(a) - k_{n-1} f_{n-1}(a)$, we see that the computation above gives that $d^C A + A d^C = P$ with $P_n(a,b) = (a, b + \ell_n(a))$. Now, since $P = d^C A + A d^C$, we have that

$$\begin{aligned} P_{n-1} \circ d^C_n &= \left(d^C_n h_{n-1} + h_{n-2} d^C_{n-1}\right) \circ d^C_n \\ &= d^C_n h_{n-1} d^C_n, \qquad \text{since } d^2 = 0, \end{aligned}$$

while

$$d_n^C \circ P_n = d_n^C \circ \left(d_{n+1}^C h_n + h_{n-1} d_n^C \right)$$
$$= d_n^C h_{n-1} d_n^C, \qquad \text{since } d^2 = 0.$$

It follows by the last two computations that $P : C \to C$ is a chain map. If we define $Q : C \to C$ by $Q_n(a,b) = (a, b - \ell_n(b))$ then it is straightforward to check that Q is degreewise a mutual inverse to P. Therefore, Q is also a chain map (to see this start from the equation $d_{n-1}^C P_n = P_n d_n^C$ and then first precompose with Q_n and then postcompose with Q_{n-1}).

Let $S : C \to C[-1]$ (i.e. as a family of homomorphisms $C_n \to C_{n+1}$) be defined by $S_n := A_n \circ Q_n$. We then see that for $(a,b) \in M_{n-1} \oplus N_n$ we have

$$S_n(a,b) = A_n(a, b - \ell_n(a))$$
$$= (h_{n-1}(a) + g_n(b) - g_n \ell_n(a), k_n(\ell_n(a)) - k_n(b)).$$

Then, we find that

$$\begin{aligned} d_{n+1}^C S_n + S_{n-1} d_n^C &= d_{n+1}^C A_n Q_n + A_{n-1} Q_{n-1} d_n^C \\ &= d_{n+1}^C A_n Q_n + A_{n-1} d_n^C Q_n, \qquad \text{since } Q \text{ is a chain map} \\ &= \left(d_{n+1}^C A_n + A_{n-1} d_n^C \right) \circ Q_n \\ &= P_n \circ Q_n, \qquad \text{since } d^C A + A d^C = P \\ &= \mathsf{id}_{C_n}, \qquad \text{since } Q \text{ is a mutual inverse of } P. \end{aligned}$$

We conclude that S is a witnessing null-homotopy $0 \simeq \mathsf{id}_C$, i.e., $C = \mathsf{Cone}(f)$ is contractible.

$\underline{\Leftarrow}$: Assume C is contractible. Then there exists a family of R-module homomorphisms $(S_n)_{n \in \mathbb{Z}}$ with $S_n : C_n = M_{n-1} \oplus N_n \to C_{n+1} = M_n \oplus N_{n+1}$ so that

$$d^C S + S d^C = \mathrm{id}_C .$$

By remark 3.2.17 we may represent S_n on the form

$$S_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

Let $(a,b) \in M_{n-1} \oplus N_n$ be arbitrary. We find that

$$\begin{aligned} d_{n+1}^C S_n(a,b) &= d^C(\alpha_n(a) + \beta_n(b), \gamma_n(a) + \delta_n(b)) \\ &= \left(-d_n^M(\alpha_n(a)) - d_n^M(\beta_n(b)), d_{n+1}^N(\gamma_n(a)) + d_{n+1}^N(\delta_n(b)) + f_n(\alpha_n(a)) + f_n(\beta_n(b)) \right). \end{aligned}$$

Next, we see that

$$\begin{aligned} S_{n-1} d_n^C(a,b) &= S_{n-1} \left(-d_{n-1}^M(a), d_n^N(b) + f_{n-1}(a) \right) \\ &= \Big(- \alpha_{n-1} \left(d_{n-1}^M(a) \right) + \beta_{n-1} \left(d_n^N(b) \right) + \beta_{n-1}(f_{n-1}(a)), \\ &\quad - \gamma_{n-1} \left(d_{n-1}^M(a) \right) + \delta_{n-1} \left(d_n^N(b) \right) + \delta_{n-1}(f_{n-1}(a)) \Big). \end{aligned}$$

Adding these together and using that this should equal (a, b), and comparing expressions, we see that

$$\begin{cases} -d_n^M \beta_n + \beta_{n-1} d_n^N = 0 \\ -d_n^M \alpha_n - \alpha_{n-1} d_{n-1}^M + \beta_{n-1} f_{n-1} = \mathrm{id}_{M_{n-1}} \\ d_{n+1}^N \delta_n + f_n \beta_n + \delta_{n-1} d_n^N = \mathrm{id}_{N_n} \end{cases}, \tag{3.2.18}$$

where we have left out one condition.

The first condition in 3.2.18 tells us that $\beta : N \to M$ is a chain map. The second condition tells us that $\beta f \simeq \mathrm{id}_M$ with chain homotopy $\alpha' := (-\alpha_n)_{n\in\mathbb{Z}}$, and the last condition tells us that $f\beta \simeq \mathrm{id}_N$ with chain-homotopy $\delta := (\delta_n)_{n\in\mathbb{Z}}$. We conclude that f is a homotopy equivalence. □

The statements in Theorems 3.2.15, 3.2.16 tells us that there are conditions on the homotopy cokernel of a chain map f that tells us when f is a quasi-isomorphism or a homotopy equivalence.

Theorem 3.2.18. *Given a morphism $M \xrightarrow{f} N$ in Ch_R, there is a homotopy equivalence*

$$\Sigma M \simeq \mathsf{hcoker}\left(N \xrightarrow{u} \mathsf{hcoker}(f)\right).$$

Proof. Take the concrete cone model $\mathsf{hcoker}(g) = \mathsf{Cone}(g)$ for each chain map g. Recall that the universal map $N \xrightarrow{u} \mathsf{hcoker}(f)$ is then on the form $b \overset{u}{\mapsto} (0, -b)$ in each degree.

We see that $\mathsf{hcoker}(u) = \mathsf{Cone}(u)$ is then such that

$$\begin{aligned} \mathsf{Cone}(u)_n &= N_{n-1} \oplus \mathsf{Cone}(f)_n \\ &\simeq N_{n-1} \oplus M_{n-1} \oplus N_n \end{aligned}$$

and we make the identification $\mathsf{Cone}(u)_n = N_{n-1} \oplus M_{n-1} \oplus N_n$, with differential

$$\begin{aligned} d_n^{\mathsf{Cone}(u)}(a, b, c) &= \left(-d_{n-1}^N(a), d_n^{\mathsf{Cone}(f)}(b, c) + u_{n-1}(a)\right) \\ &= \left(-d_{n-1}^N(a), \left(-d_{n-1}^M(b), d_n^N(c) + f_{n-1}(b)\right) + (0, -a)\right) \\ &= \left(-d_{n-1}^N(a), -d_{n-1}^M(b), d_n^N(c) + f_{n-1}(b) - a\right), \end{aligned}$$

for $(a, (b, c)) = (a, b, c)$ in $\mathsf{Cone}(u)_n$.

For ΣM we take its model as $M[1]$ with differential $a \mapsto -d_{n-1}^M(a)$ for $a \in (M[1])_n$.

Define

$$p_n : \mathsf{Cone}(u)_n \to (\Sigma M)_n, \quad (a, b, c) \mapsto b,$$

and

$$i_n : (\Sigma M)_n \to \mathsf{Cone}(u)_n, \quad x \mapsto (f_{n-1}(x), x, 0).$$

We check that p and i are chain maps.

p chain map: Consider $(a, b, c) \in \mathsf{Cone}(u)_n$. We check that the following square commutes,

$$\begin{array}{ccc} N_{n-1} \oplus M_{n-1} \oplus N_n & \xrightarrow{d_n^{\mathsf{Cone}(u)}} & N_{n-2} \oplus M_{n-2} \oplus N_{n-1} \\ \Big\downarrow {\scriptstyle p_n} & & \Big\downarrow {\scriptstyle p_{n-1}} \\ M_{n-1} & \xrightarrow[d_n^{\Sigma M}]{} & M_{n-2} \end{array} .$$

We have

$$\begin{aligned} p_{n-1} d_n^{\mathsf{Cone}(u)}(a, b, c) &= p_{n-1} \left(-d_{n-1}^N(a), -d_{n-1}^M(b), d_n^N(c) + f_{n-1}(b) - a \right) \\ &= -d_{n-1}^M(b) \\ &= -d_{n-1}^M p_n(a, b, c) \\ &= d_n^{\Sigma M} p_n(a, b, c). \end{aligned}$$

Hence p is indeed a chain map.

i chain map: Let $x \in (\Sigma M)_n = M_{n-1}$. We check that the following square commutes,

$$\begin{array}{ccc} M_{n-1} & \xrightarrow{d_n^{\Sigma M}} & M_{n-2} \\ \Big\downarrow {\scriptstyle i_n} & & \Big\downarrow {\scriptstyle i_{n-1}} \\ N_{n-1} \oplus M_{n-1} \oplus N_n & \xrightarrow[d_n^{\mathsf{Cone}(u)}]{} & N_{n-2} \oplus M_{n-2} \oplus N_{n-1} \end{array} .$$

We find that

$$\begin{aligned} d_n^{\mathsf{Cone}(u)} i_n(x) &= d_n^{\mathsf{Cone}(u)}(f_{n-1}(x), x, 0) \\ &= \left(-d_{n-1}^N(f_{n-1}(x)), -d_{n-1}^M(x), f_{n-1}(x) - f_{n-1}(x) \right) \\ &= \left(-f_{n-2} d_{n-1}^M(x), -d_{n-1}^M(x), 0 \right), \qquad \text{since } f \text{ is a chain map} \\ &= i_{n-1} \left(-d_{n-1}^M(x) \right) \\ &= i_{n-1} d_n^{\Sigma M}(x). \end{aligned}$$

Hence i is a chain map.

Define $h_n : \mathsf{Cone}(u)_n \to \mathsf{Cone}(u)_{n+1}, \quad h_n(a, b, c) = (-c, 0, 0)$. This is evidently a homomorphism in each degree. We find that

$$\begin{aligned} d_{n+1}^{\mathsf{Cone}(u)} h_n(a, b, c) &= d_{n+1}^{\mathsf{Cone}(u)}(-c, 0, 0) \\ &= \left(d_n^N(c), 0, c \right) \end{aligned}$$

and

$$h_{n-1} d_n^{\mathsf{Cone}(u)}(a, b, c) = \left(a - f_{n-1}(b) - d_n^N(c), 0, 0 \right)$$

so that

$$\begin{aligned}\left(d_{n+1}^{\mathsf{Cone}(u)} h_n + h_{n-1} d_n^{\mathsf{Cone}(u)}\right)(a,b,c) &= \left(d_n^N(c), 0, c\right) + \left(a - f_{n-1}(b) - d_n^N(c), 0, 0\right) \\ &= (a - f_{n-1}(b), 0, c) \\ &= (a,b,c) - (f_{n-1}(b), b, 0) \\ &= \left(\mathrm{id}_{\mathsf{Cone}(u)} - (i \circ p)\right)_n (a,b,c),\end{aligned}$$

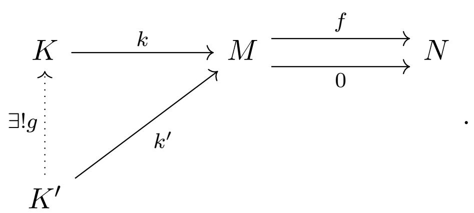
so that $i \circ p \simeq \mathrm{id}_{\mathsf{Cone}(u)}$. Clearly $p_n \circ i_n = \mathrm{id}_{(\Sigma M)_n}$, so that $p \circ i = \mathrm{id}_{\Sigma M}$. We conclude that $\Sigma M \simeq \mathsf{hcoker}(u)$.

□

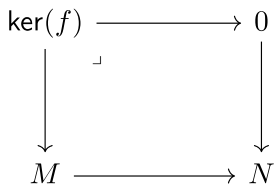
3.3 Homotopy kernels

We recall that in for example Mod_R (and *additive categories* more generally; cf. [Yek19, Def. 2.3.1]), given a morphism $M \xrightarrow{f} N$ of R-modules, there is an R-module $\mathsf{ker}(f)$ equipped with a morphism $\mathsf{ker}(f) \xrightarrow{k} M$, satisfying

1. $f \circ k = 0$.
2. If $k' : K' \to M$ is any other R-module homomorphism such that $f \circ k' = 0$, then there is a *unique* morphism $g : K' \to K$ such that $k' = k \circ g$. We may represent this with the following diagram,



One checks that the kernel $\mathsf{ker}(f)$ of an R-module homomorphism $M \xrightarrow{f} N$ is precisely the *pullback*



of the cospan $M \xrightarrow{f} N \leftarrow 0$.

We call a $A \xrightarrow{u} B \xrightarrow{v} C$ a **kernel sequence** if

(i) $v \circ u = 0$.

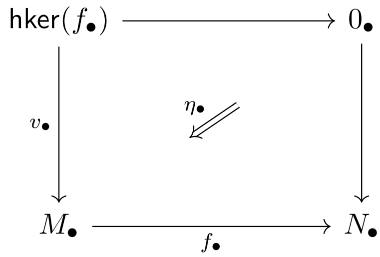
(ii) $(A, A \xrightarrow{u} B)$ is a kernel of v.

We now define the corresponding "homotopical" notion of a kernel, in Ch_R.

Definition 3.3.1 (Homotopy kernel)**.** Let $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ be a chain map in Ch_R. Then the **homotopy kernel** of $f_\bullet$ is a chain complex $\mathsf{hker}(f_\bullet)$ together with a universal chain map $\mathsf{hker}(f_\bullet) \xrightarrow{v_\bullet} M_\bullet$ such that for any chain complex $T_\bullet$, postcomposition with $v_\bullet$ and specified witnessing chain homotopy $0_\bullet \overset{\eta_\bullet}{\Rightarrow} f_\bullet \circ v_\bullet$ induces a natural bijection

$$\hom_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \approx \left\{ (g_\bullet, \nu_\bullet) \;\middle|\; \substack{\text{morphism } g_\bullet : T_\bullet \to M_\bullet \\ 0_\bullet \overset{\nu_\bullet}{\Rightarrow} f_\bullet \circ g_\bullet \text{ null-homotopy}} \right\}. \tag{3.3.1}$$

More explicitly, a homotopy kernel is the data of a *triple* $(\mathsf{hker}(f_\bullet), v_\bullet, \eta_\bullet)$ where the "postcomposition" $\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f_\bullet)) \ni \alpha_\bullet \mapsto (v_\bullet \circ \alpha_\bullet, \eta_\bullet \alpha_\bullet)$ determines the bijection 3.3.1 (cf. 3.2.3). Another way to phrase this, is that $\mathsf{hker}(f_\bullet)$ is the *homotopy pullback*



of the diagram $M_\bullet \xrightarrow{f_\bullet} N_\bullet \leftarrow 0_\bullet$.

Theorem 3.3.2. *With the model* $\mathsf{hcoker}(f_\bullet) = \mathsf{Cone}(f_\bullet)$ *of a chain map* $M_\bullet \xrightarrow{f_\bullet} N_\bullet$ *in* Ch_R*, we have that* $\Sigma^{-1}\mathsf{hocoker}(f_\bullet) \xrightarrow{\pi} M_\bullet$*, defined in degree n as the projection to the first argument M_n, is a homotopy kernel of $f_\bullet$, with witnessing chain homotopy* $h : 0_\bullet \simeq f_\bullet \circ \pi$ *defined in degree n by* $h_n(a, b) = -b$*.*

Proof. Let $K_\bullet := \mathsf{Cone}(f_\bullet)[-1]$, i.e. so that $K_n = M_n \oplus N_{n+1}$. First, it is easy enough to check that π is a chain map.

$0 \overset{h}{\simeq} f_\bullet \circ \pi$: Consider $h_n : K_n \to N_{n+1}$ on elements by $(a, b) \mapsto -b$. This is an evident family of homomorphisms for $n \in \mathbb{Z}$. Let $(m, x) \in K_n$. Then

$$\begin{aligned}
\left(d^N_{n+1} h_n + h_{n-1} d^K_n\right)(m, x) &= d^N_{n+1}(-x) + h_{n-1}\left(d^M_n(m), -d^N_{n+1}(x) - f_n(m)\right) \\
&= f_n(m) \\
&= (f_n \circ \pi)(m, x).
\end{aligned}$$

Hence $h := (h_n)_{n \in \mathbb{Z}}$ is a chain homotopy $0 \simeq f_\bullet \circ \pi$.

Universal property: Let $T_\bullet$ be any chain complex in Ch_R with a chain map $\alpha_\bullet : T_\bullet \to K_\bullet$. Since α_n maps into a (categorical) product, the universal property (in Mod_R) gives that $\alpha_n = (g_n, s_n)$ with $g_n : T_n \to M_n$ and $s_n : T_n \to N_{n+1}$. In particular, we see that $\pi \circ \alpha = g : T_\bullet \to M_\bullet$. Since α is a chain map, it follows that $d^K \alpha = \alpha d^T$, which in degree n is the same as

$$\left(d^M_n g_n, -d^N_{n+1} \circ s_n - f_n g_n\right) = \left(g_{n-1} d^T_n, s_{n-1} d^T_n\right).$$

Comparing first arguments gives that $g_\bullet := (g_n)_{n \in \mathbb{Z}}$ is a chain map. Comparing second arguments, we see that $s := (-s_n)_{n \in \mathbb{Z}} = (h_n \alpha_n)_{n \in \mathbb{Z}}$ is a chain homotopy $0_\bullet \simeq f_\bullet \circ g_\bullet$.

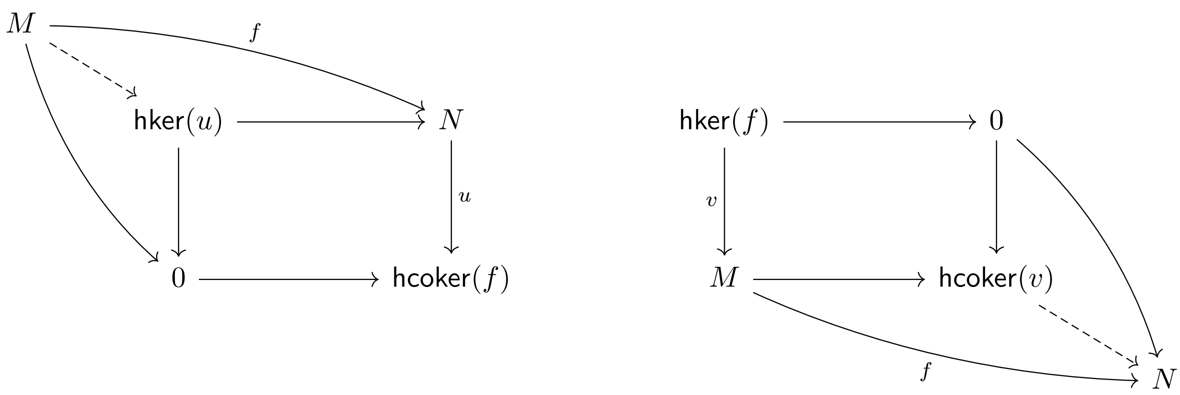
Now instead assume we have a chain map $g_\bullet : T_\bullet \to M_\bullet$ and a chain homotopy $0_\bullet \overset{s}{\simeq} f_\bullet \circ g_\bullet$. Let $\alpha := (\alpha_n)_{n \in \mathbb{Z}}$ with $\alpha_n(t) := (g_n(t), -s_n(t))$. Then

$$\begin{aligned}
d^K_n \alpha_n(t) &= \left(d^M_n g_n(t), d^N_{n+1} s_n(t) - f_n g_n(t)\right) \\
&= \left(g_{n-1} d^T_n(t), -s_{n-1} d^T_n(t)\right) \\
&= \alpha_{n-1} d^T_n(t),
\end{aligned}$$

so that α is a chain map $T_\bullet \to K_\bullet$. One checks that the two operations described above are mutual inverse operations, hence gives a bijection, where the first direction is precisely the defining operation by "postcomposition" with (π, h). The conclusion follows. $\square$

Remark 3.3.3. Note that since $\Sigma^{-1}\mathsf{hcoker}(f) \cong \mathsf{hker}(f)$ by theorem 3.3.2, and since $\Sigma^{-1}\mathsf{hcoker}(f)$ is acyclic iff $\mathsf{hcoker}(f)$ is acyclic, it follows by theorem 3.2.15 that $\mathsf{hker}(f)$ is acyclic iff f is a quasi-isomorphism.

Theorem 3.3.4. *For any morphism $M \xrightarrow{f} N$ in Ch_R, the dashed arrows below are homotopy equivalences.*



Proof. Recall definition 3.1.15. Let's work with models $\mathsf{hcoker}(f) = \mathsf{Cone}(f)$ and $\mathsf{hker}(f) = \mathsf{Cone}(f)[-1]$ for any morphism f in Ch_R. We saw in theorem 3.3.2 that projection to the first factor was the universal map for this model of the homotopy kernel, and we have seen that (cf. proof of 3.2.6) $u : N \to C_\bullet$ defined in degree n by $b \mapsto (0, -b)$ is the universal map of the homotopy cokernel model $C_\bullet$.

<u>$M \dashrightarrow \mathsf{hker}(u)$ is a homotopy equivalence</u>: We first observe that we have that

$$\begin{aligned}\mathsf{Cone}(u)_n &= N_{n-1} \oplus \mathsf{Cone}(f)_n \\ &= N_{n-1} \oplus M_{n-1} \oplus N_n.\end{aligned}$$

Hence

$$\begin{aligned}\mathsf{hker}(u)_n &= \left(\Sigma^{-1}\mathsf{Cone}(u)\right)_n \\ &= \mathsf{Cone}(u)_{n+1} \\ &= N_n \oplus M_n \oplus N_{n+1}.\end{aligned}$$

Since *desuspension* shifts the differential one degree upwards and puts a minus sign in front of it, we find that for $(x, a, b) \in N_n \oplus M_n \oplus N_{n+1}$ we have

$$d_n^{\mathsf{hker}(u)}(x, a, b) = \left(d_n^N(x), d_n^M(a), -d_{n+1}^N(b) - f_n(a) + x\right).$$

We need to find the induced map $M \dashrightarrow \mathsf{hker}(u)$. Assume we are given the chain map $M \xrightarrow{f} N$ and a null-homotopy $0 \overset{\mu}{\Rightarrow} u \circ f$. Observe that in degree n we have that $(u \circ f)_n(a) = (0, -f_n(a))$ for $a \in M_n$. By the proof of theorem 3.2.6 we see that $\gamma = (\gamma_n)_{n \in \mathbb{Z}}$ defined by $\gamma_n : M_n \to \mathsf{Cone}(f)_{n+1} = M_n \oplus N_{n+1}, \quad a \mapsto (-a, 0)$ is the requisite witnessing null-homotopy $0 \Rightarrow u \circ f$. By the proof of theorem 3.3.2 we then find that the dashed arrow $\alpha : M \dashrightarrow \mathsf{hker}(u)$ is on the form $\alpha_n = (f_n, -\gamma_n)$ so that

$$\begin{aligned}\alpha_n(a) &= (f_n(a), -(-a, 0)) \\ &= (f_n(a), a, 0).\end{aligned}$$

Define $\beta_n : \mathsf{hker}(u)_n = N_n \oplus M_n \oplus N_{n+1} \to M_n$ by $\beta_n(x, a, b) := a$. An easy check gives that β is a chain map.

It is immediate that $\beta \circ \alpha = \mathrm{id}_M$. Let $s := (s_n)_{n \in \mathbb{Z}}$ where $s_n : \mathsf{hker}(u)_n \to \mathsf{hker}(u)_{n+1}$ be defined by $s_n(x, a, b) = (b, 0, 0)$. Then we find that

$$d_{n+1}^{\mathsf{hker}(u)} s_n(x, a, b) = \left(d_{n+1}^N(b), 0, b\right)$$

and

$$s_{n-1} d_n^{\mathsf{hker}(u)}(x, a, b) = \left(-d_{n+1}^N(b) - f_n(a) + x, 0, 0\right).$$

Hence

$$\begin{aligned}\left(d^{\mathsf{ker}(u)} s + s d^{\mathsf{ker}(u)}\right)_n (x, a, b) &= (x - f_n(a), 0, b) \\ &= (\mathrm{id}_{\mathsf{hker}(u)} - (\alpha \circ \beta))_n(x, a, b).\end{aligned}$$

Therefore $\alpha \circ \beta \simeq \mathrm{id}_{\mathsf{hker}(u)}$. We conclude that $M \dashrightarrow \mathsf{hker}(u)$ is a homotopy equivalence.

<u>$\mathsf{hcoker}(v) \dashrightarrow N$ is a homotopy equivalence:</u> Let $K_\bullet := \mathsf{hker}(v) = \mathsf{Cone}(f)[-1]$, so that $K_n = M_n \oplus N_{n+1}$ and $d_n^K(a, b) = \left(d_n^M(a), -d_{n+1}^N(b) - f_n(a)\right)$ with universal map $v : K_\bullet \to M$ defined by $v_n(a, b) = a$, together with witnessing specified nullhomotopy

$$0 \overset{h}{\simeq} f \circ v, \qquad h_n(a, b) = -b, \qquad \text{(cf. proof of theorem 3.3.2)}.$$

Let $E_\bullet := \mathsf{hcoker}(v) = \mathsf{Cone}(v)$, so that

$$\begin{aligned}E_n &= K_{n-1} \oplus M_n \\ &= M_{n-1} \oplus N_n \oplus M_n\end{aligned}$$

and with differential

$$\begin{aligned}d_n^E(a, b, c) &= \left(-d_{n-1}^K(a, b), d_n^M(c) + v_{n-1}(a, b)\right) \\ &= \left((-d_{n-1}^M(a), d_n^N(b) + f_{n-1}(a)), d_n^M(c) + a\right) \\ &= \left(-d_{n-1}^M(a), d_n^N(b) + f_{n-1}(a), d_n^M(c) + a\right).\end{aligned}$$

Observe that the last equality above is really a canonical isomorphism.

We next investigate how $q : E_\bullet \dashrightarrow N$ is induced. From the proof of theorem 3.2.6 we see that $q_n(a, b, c) = -h_{n-1}(a, b) - f_n(c) = b - f_n(c)$.

To show that q is a homotopy equivalence, let $j : N \to E_\bullet$ be defined by $j_n(y) = (0, y, 0)$. It is straightforward to check that j defines a chain map. Then $q \circ j = \mathrm{id}_N$. On the other hand, if we let $\ell_n : E_n \to E_{n+1}$ be defined by $\ell_n(a, b, c) = (c, 0, 0)$, then

$$\begin{aligned}d_{n+1}^E \ell_n(a, b, c) &= d_{n+1}^E(c, 0, 0) \\ &= \left(-d_n^M(c), f_n(c), c\right),\end{aligned}$$

and

$$\begin{aligned}\ell_{n-1}d_n^E(a,b,c) &= \ell_{n-1}\left(-d_{n-1}^M(a), d_n^N(b)+f_{n-1}(a), d_n^M(c)+a\right)\\ &= \left(d_n^M(c)+a, 0, 0\right).\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\left(d^E\ell+\ell d^E\right)_n(a,b,c) &= (a, f_n(c), c)\\ &= (a,b,c)-(0, b-f_n(c), 0)\\ &= (a,b,c)-j_nq_n(a,b,c)\\ &= (\mathrm{id}_E - j\circ q)_n(a,b,c).\end{aligned}$$

It follows that $\mathrm{id}_E - jq = d^E\ell + \ell d^E$ so that $\mathrm{id}_E \simeq j\circ q$. We conclude that $q : E_\bullet \dashrightarrow N$ is a homotopy equivalence. $\square$

We say that two sequence $A \xrightarrow{a} B \xrightarrow{b} C$ and $A' \xrightarrow{a'} B' \xrightarrow{b'} C'$ in Ch_R are **homotopy equivalent** if there are vertical homotopy equivalences as indicated below such that the two squares commute *up to chain homotopy*

$$\begin{array}{ccccc} A & \xrightarrow{\quad a \quad} & B & \xrightarrow{\quad b \quad} & C \\ \simeq\big\downarrow & \circlearrowleft \text{ up to chain homotopy} & \simeq\big\downarrow & \circlearrowleft \text{ up to chain homotopy} & \big\downarrow\simeq \\ A' & \xrightarrow[\quad a' \quad]{} & B' & \xrightarrow[\quad b' \quad]{} & C' \end{array}.$$

The reason such sequences are interesting, is that when going to *homology*, the n^{th} homology functor H_n turns this into *actual* commuting squares in Mod_R, where homotopy equivalences becomes isomorphisms, i.e. we get an *isomorphism* of sequences in Mod_R (same notion as homotopy equivalent sequences but with homotopy equivalences $\simeq$ replaced by isomorphisms $\cong$ and occuring in Mod_R instead of Ch_R).

In particular, theorem 3.3.4 gives us, for *any* chain map $M \xrightarrow{f} N$, the following homotopy equivalent sequnces,

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{u} & \mathsf{hcoker}(f) \\ \simeq\big\downarrow & & \big\downarrow\mathsf{id} & & \big\downarrow\mathsf{id} \\ \mathsf{hker}(u) & \longrightarrow & N & \xrightarrow{u} & \mathsf{hcoker}(f) \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathsf{hker}(f) & \xrightarrow{v} & M & \longrightarrow & \mathsf{hcoker}(v) \\ \mathsf{id}\big\downarrow & & \big\downarrow\mathsf{id} & & \big\downarrow\simeq \\ \mathsf{hker}(f) & \longrightarrow & M & \xrightarrow[f]{} & N \end{array}.$$

Observe that the lower row in the left diagram is a homotopy kernel sequence and the upper row is a homotopy cokernel sequence, while the upper row in right diagram is a homotopy cokernel sequence and the lower row is a homotopy kernel sequence, so at the level of *homology* they are the same up to isomorphisms.

This is *not true* (exchanging homotopy equivalence for isomorphism) in for example Mod_R:

- Let $A \xrightarrow{u} B \xrightarrow{v} C$ be a kernel sequence in Mod_R. Then $v \circ u = 0$ and (A, u) is a kernel of v. That u is a kernel of v implies that u is a *monomorphism*, which is equivalent to u being injective (see Lemma 5.4.3). One may further show that u being a kernel of v means that $\mathsf{im}(u) = \mathsf{ker}(v)$. Hence we may augment the sequence to an exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C$. If v is a cokernel of u then this implies that v is *epic*, which by Lemma 5.3.3 tells us that v is surjective, so we may further augment the sequence to an exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$. This shows that if v is *not surjective*, then a kernel sequence can not be a cokernel sequence, e.g. take $0 \xrightarrow{u=0} \mathbb{Z} \xrightarrow{v:=\cdot 2} \mathbb{Z}$. Since $\mathsf{ker}(v) = 0 \Leftrightarrow v$ is injective it follows that $0 \xrightarrow{u} \mathbb{Z}$ is its kernel, so this is a kernel sequence. But $\mathsf{im}(v) = 2\mathbb{Z}$ so v is not surjective, hence this is not a cokernel sequence.
- On the other hand if $A \xrightarrow{u} B \xrightarrow{v} C$ is a cokernel sequence then $v \circ u = 0$ and (C, v) is a cokernel of u. Then one may show that $\mathsf{ker}(v) = \mathsf{im}(u)$ so it follows that we can augment this to an exact sequence $A \xrightarrow{u} B \xrightarrow{v} C \to 0$. If u is a kernel of v then u is monic hence injective so we may augment this further to a sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$. It follows that a cokernel sequence being a kernel sequence implies u is injective. Hence if we have a cokernel sequence on the form $A \xrightarrow{u} B \xrightarrow{v} C$ with u not injective then it can not be a kernel sequence. Take for example the sequence $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathsf{id}} \mathbb{Z}$. The composition $\mathsf{id} \circ 0 = 0$ is clearly zero and $\mathsf{coker}(0) = \mathsf{id}_{\mathbb{Z}}$ so this is a cokernel sequence, but $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$ is not injective, hence this is not a kernel sequence.

We summarize the definition of the homotopy kernel and cokernel given in §3.2 and §3.3, and how they relate to the corresponding kernel and cokernel notions in Mod_R. For the analogy between kernels in Mod_R and homotopy kernels in Ch_R, we have the following:

Mod_R	Ch_R
$\begin{array}{ccc} \mathsf{ker}(f) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ M & \xrightarrow{f} & N \end{array}$	$\begin{array}{ccc} \mathsf{hker}(f_\bullet) & \longrightarrow & 0 \\ \downarrow & \Swarrow & \downarrow \\ M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \end{array}$.
$\mathsf{hom}_{\mathsf{Mod}_R}(T, \mathsf{ker}(f)) \approx \{g : T \to M \mid f \circ g = 0\}$	$\mathsf{hom}_{\mathsf{Ch}_R}(T_\bullet, \mathsf{hker}(f)) \approx \{g_\bullet : T_\bullet \to M_\bullet \mid f_\bullet \circ g_\bullet \overset{\nu_\bullet}{\simeq} 0\}$.

For cokernels in Mod_R, respectively homotopy cokernels in Ch_R, we have the following relationships

Mod_R	Ch_R
$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathsf{coker}(f) \end{array}$	$\begin{array}{ccc} M_\bullet & \xrightarrow{f_\bullet} & N_\bullet \\ \downarrow & \Nearrow & \downarrow \\ 0 & \longrightarrow & \mathsf{hcoker}\,(f_\bullet) \end{array}$.
$\mathsf{hom}_{\mathsf{Mod}_R}(\mathsf{coker}(f), T)$ $\approx \{g : N \to T \mid g \circ f = 0\}$	$\mathsf{hom}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), T)$ $\approx \left\{g_\bullet : N_\bullet \to T_\bullet \mid g_\bullet \circ f_\bullet \overset{h_\bullet}{\simeq} 0\right\}$.

It is quite clear from the above comparisons that the definitions of homotopy kernel and homotopy cokernel in Ch_R are as close as you can get to the corresponding notions of kernel and cokernel in Mod_R if one replaces equalities with chain homotopies. Observe however that the bijections in the homotopical case involve a *choice* of witnessing nullhomotopy, and that one may emphasize this choice by decorating the arrow $\Longrightarrow$ in the homotopical pullback and pushout diagrams with the specified witness (although we don't do that in the above tables).

3.4 Exact sequences

Definition 3.4.1 (Exact complex)**.** We say that a chain complex $M_\bullet \in \mathsf{Ch}_R$ is **exact** in degree n if $\mathsf{H}_n(M_\bullet) = 0$.

Definition 3.4.2 (Short exact sequence)**.** A sequence of R-modules

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

in Mod_R such that f is injective, g surjective and $\mathsf{im}(f) = \mathsf{ker}(g)$, is called an **exact sequence**.

Remark 3.4.3. Observe that an exact sequence in Mod_R can also be seen as an acyclic chain-complex in Ch_R.

Theorem 3.4.4. *Let $M \xrightarrow{f} N$ be a chain map in Ch_R, the sequence*

$$\mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f))$$

is exact.

Proof. Let's choose the specific cone model $\mathsf{Cone}(f)$ for $\mathsf{hcoker}(f)$. Recall diagram 3.2.1. Since $\mathsf{H}_0(-)$ is functorial, since $\mathsf{H}_0(0) = 0$ and since $\mathsf{H}_0(-)$ identifies chain homotopic maps, it follows that $\mathsf{im}\,(\mathsf{H}_0(f)) \subseteq \mathsf{ker}\,(\mathsf{H}_0(u))$. It remains to show that $\mathsf{ker}\,(\mathsf{H}_0(u)) \subseteq \mathsf{im}\,(\mathsf{H}_0(f))$. Let $[b] \in \mathsf{ker}\,(\mathsf{H}_0(u))$ be such that

$$\mathsf{H}_0(u)[b] = 0. \tag{3.4.1}$$

Since $[b] \in \mathsf{H}_0(N)$ we have that $b \in N_0$ with $d_0^N(b) = 0$. With the cone model, 3.4.1 means that $[(0, -b)] = 0$. It follows that $(0, -b) \in C_0$ is a boundary, so there exists an element $(a, c) \in M_0 \oplus N_1$ such that

$$\begin{aligned} d_1^{\mathsf{Cone}(f)}(a, c) &= \left(-d_0^M(a), d_1^N(c) + f_0(a)\right) \\ &= (0, -b). \end{aligned}$$

It follows that $d_0^M(a) = 0$ so that a is a 0-cycle in M, and so that $f_0(a) + d_1^N(c) = -b$, i.e. so that $f_0(a)$ and $-b$ differ by a boundary, so that they become equal in $\mathsf{H}_0(N)$ (observe that if a is a 0-cycle then $-a$ is a 0-cycle in M so that $f_0(-a)$ is a 0-cycle in N_0 since f is a chain map). Hence

$$\begin{aligned} \mathsf{H}_0(f)[(-a)] &= [-f_0(a)] \\ &= [b], \end{aligned}$$

so that $[b] \in \mathsf{im}\,(\mathsf{H}_0(f))$. Therefore, $\mathsf{im}\,(\mathsf{H}_0(f)) = \mathsf{ker}\,(\mathsf{H}_0(u))$. Exactness follows. $\square$

Theorem 3.4.5. *Given a sequence of morphisms*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in Ch_R *that is exact in each dimension, there is an associated long exact sequence*

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\mathsf{H}_{n+1}(g)} & \mathsf{H}_{n+1}(N) & & & & & \\
& \downarrow \partial & & & & & \\
& \mathsf{H}_n(L) & \xrightarrow{\mathsf{H}_n(f)} & \mathsf{H}_n(M) & \xrightarrow{\mathsf{H}_n(g)} & \mathsf{H}_n(N) & \\
& & & & & \downarrow \partial & \\
& & & & & \mathsf{H}_{n-1}(L) & \xrightarrow[\mathsf{H}_{n-1}(f)]{} \cdots
\end{array} \tag{3.4.2}$$

Remark 3.4.6. We call the morphisms ∂ in diagram 3.4.2 **connecting homomorphisms**.

Proof. See [Hat01, pp. 116-117]. □

Starting with a morphism $M \xrightarrow{f} N$ of complexes in Ch_R, we have seen that we get an associated sequence of maps

$$\mathsf{hker}(f) \xrightarrow{v} M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f).$$

Now observe that we also have homotopy cokernel sequence

$$\mathsf{hker}(u) \to N \xrightarrow{u} \mathsf{hcoker}(f).$$

In theorem 3.3.2 we saw that $\Sigma^{-1}\mathsf{hcoker}(f)$ is a homotopy kernel of $f \rightsquigarrow \Sigma\left(\Sigma^{-1}\mathsf{hcoker}(f)\right) \cong \mathsf{hcoker}(f) \cong \Sigma\mathsf{hker}(f)$, and theorem 3.2.18 told us that $\Sigma M \simeq \mathsf{hcoker}(u)$. Furthermore, we recall that $\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$, so in particular $\mathsf{H}_0 \circ \Sigma^k \cong \mathsf{H}_{-k}$ for all $k \in \mathbb{Z}$. Then we observe the following:

(1) $M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f)$ is clearly a homotopy cokernel sequence.

(2) Consider the homotopy cokernel sequence $N \xrightarrow{u} \mathsf{hcoker}(f) \xrightarrow{r} \mathsf{hcoker}(u)$. By theorem 3.4.4 the associated sequence in 0^{th} homology

$$\mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f)) \xrightarrow{\mathsf{H}_0(r)} \mathsf{H}_0(\mathsf{hcoker}(u))$$

is exact. By 3.2.18 we have that $\mathsf{hcoker}(u) \simeq \Sigma M$ are homotopy equivalent. Taking homology, we see that $\mathsf{H}_0(\mathsf{hcoker}(u)) \approx \mathsf{H}_0(\Sigma M)$ are isomorphic. Therefore, the associated sequence

$$\mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f)) \xrightarrow{\beta} \mathsf{H}_0(\Sigma M)$$

is exact (with the appropriate map β). Recall that $\mathsf{H}_n \circ \Sigma^k \cong \mathsf{H}_{n-k}$. It follows that we may replace $\mathsf{H}_0(\Sigma M)$ by $\mathsf{H}_{-1}(M)$ and get an exact sequence

$$\mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f)) \xrightarrow{\alpha} \mathsf{H}_{-1}(M).$$

(3) Let $M \xrightarrow{q} \mathsf{hcoker}(v)$ and $\mathsf{hcoker}(v) \overset{\ell}{\dashrightarrow} N$ be as in the rightmost diagram in the statement of theorem 3.3.4. Then $\ell \circ q \simeq f$ are homotopic maps, and furthermore ℓ is a homotopy equivalence. Taking the 0^{th} homology, we find that

$$\begin{aligned} \mathsf{H}_0(\ell \circ q) &= \mathsf{H}_0(\ell) \circ \mathsf{H}_0(q) \\ &= \mathsf{H}_0(f), \end{aligned}$$

with $\mathsf{H}_0(\ell)$ an isomorphism. Therefore, exactness of

$$\mathsf{H}_0(\mathsf{hker}(f)) \xrightarrow{\mathsf{H}_0(v)} \mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(q)} \mathsf{H}_0(\mathsf{hcoker}(v))$$

is equivalent to exactness of

$$\mathsf{H}_0(\mathsf{hker}(f)) \xrightarrow{\mathsf{H}_0(v)} \mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N), \tag{3.4.3}$$

and the former sequence is induced from the homotopy cokernel sequence

$$\mathsf{hker}(f) \xrightarrow{v} M \xrightarrow{q} \mathsf{hcoker}(v),$$

hence is exact by theorem 3.4.4. It follows that the latter sequence (3.4.3) is exact.

Summarizing, from (1)-(3) we obtain the following exact sequences,

$$\begin{cases} \mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f)) \\ \mathsf{H}_0(N) \xrightarrow{\mathsf{H}_0(u)} \mathsf{H}_0(\mathsf{hcoker}(f)) \xrightarrow{\alpha} \mathsf{H}_{-1}(M) \\ \mathsf{H}_0(\mathsf{hker}(f)) \xrightarrow{\mathsf{H}_0(v)} \mathsf{H}_0(M) \xrightarrow{\mathsf{H}_0(f)} \mathsf{H}_0(N) \end{cases}.$$

Furthermore, we again recall that Σ^k preserves homotopy cokernel sequences, homotopy equivalences and chain homotopies. It follows that for each k, we get exact sequences

$$\begin{cases} \mathsf{H}_0\left(\Sigma^k M\right) \to \mathsf{H}_0\left(\Sigma^k N\right) \to \mathsf{H}_0\left(\Sigma^k \mathsf{hcoker}(f)\right) \\ \mathsf{H}_0\left(\Sigma^k N\right) \to \mathsf{H}_0\left(\Sigma^k \mathsf{hcoker}(f)\right) \to \mathsf{H}_0(\Sigma^{k+1} M) \\ \mathsf{H}_0\left(\Sigma^k \mathsf{hker}(f)\right) \to \mathsf{H}_0\left(\Sigma^k M\right) \to \mathsf{H}_0\left(\Sigma^k N\right) \end{cases},$$

$$\rightsquigarrow \begin{cases} \mathsf{H}_{-k}(M) \to \mathsf{H}_{-k}(N) \to \mathsf{H}_{-k}(\mathsf{hcoker}(f)) \\ \mathsf{H}_{-k}(N) \to \mathsf{H}_{-k}(\mathsf{hcoker}(f)) \to \mathsf{H}_{-k-1}(M) \\ \mathsf{H}_{-k}(\mathsf{hker}(f)) \to \mathsf{H}_{-k}(M) \to \mathsf{H}_{-k}(N) \end{cases} \quad \text{are exact sequences.}$$

We have a long sequence of maps

$$\begin{array}{l} \cdots \xrightarrow{\Sigma^{-1}v} \Sigma^{-1}M \xrightarrow{\Sigma^{-1}f} \Sigma^{-1}N \xrightarrow{\Sigma^{-1}u} \Sigma^{-1}\mathsf{hcoker}(f) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \cong \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathsf{hker}(f) \xrightarrow{v} M \xrightarrow{f} N \xrightarrow{u} \mathsf{hcoker}(f) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \cong \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Sigma\mathsf{hker}(f) \xrightarrow{\Sigma v} \Sigma M \xrightarrow{\Sigma f} \Sigma N \xrightarrow{\Sigma u} \cdots \end{array}.$$

Then taking 0^{th} homology, i.e. applying $\mathsf{H}_0(\,\cdot\,)$ to the above, we get (up to appropriate homotopies) the long exact sequence in homology (i.e. in Mod_R)

$$\begin{array}{ccccccccccc}
\cdots & \mathsf{H}_1(M) & \longrightarrow & \mathsf{H}_1(N) & \longrightarrow & \mathsf{H}_1(\mathsf{hcoker}(f)) & & & & & \\
 & & & & \overset{\partial}{\searrow} & \downarrow \cong & & & & & \\
 & & & & & \mathsf{H}_0(\mathsf{hker}(f)) & \longrightarrow & \mathsf{H}_0(M) & \longrightarrow & \mathsf{H}_0(N) & \longrightarrow \mathsf{H}_0(\mathsf{hcoker}(f)) \\
 & & & & & & & & & \overset{\partial}{\searrow} & \downarrow \cong \\
 & & & & & & & & & & \mathsf{H}_{-1}(\mathsf{hker}(f)) \longrightarrow \mathsf{H}_{-1}(M) \longrightarrow \mathsf{H}_{-1}(N) \longrightarrow \cdots
\end{array} \tag{3.4.4}$$

Observation: If we start with a short exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ of complexes in Ch_R, then since it is exact in each degree, f is levelwise injective and $\mathsf{coker}(f_n) \cong N_n$. Furthermore these levelwise isomorphisms assemble into an isomorphism of *complexes* $\mathsf{coker}(f) \cong N$. By theorem 3.2.11 the canonical map $\mathsf{hcoker}(f) \xrightarrow{\approx} \mathsf{coker}(f)$ is a quasi-isomorphism. Therefore it follows that N and $\mathsf{hcoker}(f)$ are quasi-isomorphic. Hence we have isomorphisms $\mathsf{H}_n(\mathsf{hcoker}(f)) \cong \mathsf{H}_n(N)$ in each degree n. Hence if we apply the general reasoning in 3.4.4 to the morphism $L \xrightarrow{f} M$ we get a long exact sequence

$$\cdots \to \mathsf{H}_1(L) \to \mathsf{H}_1(M) \to \mathsf{H}_1(\mathsf{hcoker}(f)) \xrightarrow{\partial} \mathsf{H}_0(M) \to \mathsf{H}_0(N) \to \cdots$$

which we may then replace with a long exact sequence

$$\cdots \to \mathsf{H}_1(L) \to \mathsf{H}_1(M) \to \mathsf{H}_1(N) \xrightarrow{\partial} \mathsf{H}_0(L) \to \mathsf{H}_0(M) \to \cdots\,,$$

that is, we get back the long exact sequence coming from theorem 3.4.5. The point of this observation, is that the general construction in 3.4.4 *recovers* the sequence we get from 3.4.5.

Chapter 4

Introducing the dg-category of complexes

4.1 Hom-complexes

Definition 4.1.1 (Hom-complexes)**.** Let $M, N \in \mathsf{Ch}_R$ be complexes. Then we define $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N) \in \mathsf{Ch}_{\Bbbk}$ in degree n as

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_n := \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}),$$

with differential

$$\prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \xrightarrow{d_n} \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1}),$$

defined explicitly as

$$\{f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})\}_{i\in\mathbb{Z}} \xmapsto{d_n} \left\{\left(d^N_{i+n} \circ f_i + (-1)^{n-1} f_{i-1} \circ d^M_i\right) \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n-1})\right\}_{i\in\mathbb{Z}}. \tag{4.1.1}$$

Remark 4.1.2. We may at times write $\underline{\mathsf{hom}}(M,N)$ instead of $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)$.

Theorem 4.1.3. $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)$ *is a complex in* $\mathsf{Ch}_{\Bbbk}$.

Proof. $\underline{\Bbbk\text{-module}}$: Observe that since R is a $\Bbbk$-algebra, $\Bbbk$ acts centrally, hence each $\mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n})$ is a $\Bbbk$-module with multiplication

$$\Bbbk \times \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}) \ni (\lambda, f_i) \mapsto \lambda \cdot f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i, N_{i+n}).$$

Taking the (direct) product over $i \in \mathbb{Z}$ is still internal to $\mathsf{Ch}_{\Bbbk}$, and the conclusion follows.

$\underline{d^2 = 0}$: Let $f := (f_i)_{i\in\mathbb{Z}}$. Then

$$\begin{aligned}
d_{n-1}d_n f &= d_{n-1}\left(d^N_{i+n}\circ f_i + (-1)^{n-1}\cdot f_{i-1}\circ d^M_i\right)\\
&= \underbrace{d^N_{i+n-1}d^N_{i+n}f_i}_{=0} + (-1)^{n-1}d^N_{i+n-1}f_{i-1}d^M_i\\
&+ (-1)^{n-2}d^N_{i+n-1}f_{i-1}d^M_i + \underbrace{(-1)^{2n-3}f_{i-2}\underbrace{d^M_{i-1}d^M_i}_{=0}}_{=0}\\
&= (-1)^{n-1}d^N_{i+n-1}f_{i-1}d^M_i + (-1)^{n-2}d^N_{i+n-1}f_{i-1}d^M_i\\
&= 0.
\end{aligned}$$

The conclusion follows. □

We observe the following:

(a) Since, by definition,

$$\mathsf{Z}_0(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)) = \mathsf{ker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_0 \xrightarrow{d_0} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_{-1}\right)$$

with

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_0 = \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i,N_i)$$

and an element in the kernel is then precisely an element $(f)_{i\in\mathbb{Z}}$ in the product, with $f_i : M_i \to N_i$, such that $d_0(f) = 0$. This means that

$$\begin{aligned}
(d_0 f)_i &= d^N_i \circ f_i - f_{i-1}d^M_i\\
&= 0\\
\Leftrightarrow d^N_i f_i &= f_{i-1}d^M_i, \qquad \forall i\in\mathbb{Z},
\end{aligned}$$

i.e. the maps f_i must assemble into a chain map. Since clearly $\mathsf{hom}_{\mathsf{Ch}_R}(M,N) \subset \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_i,N_i)$ (under the identification $f \mapsto (f_i)_{i\in\mathbb{Z}}$) and the above shows inclusion in the other direction, it follows that

$$\mathsf{Z}_0\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)\right) = \mathsf{hom}_{\mathsf{Ch}_R}(M,N).$$

(b) Since a nullhomotopy $0 \overset{h}{\Rightarrow} f$ is the data of homomorphisms $h : M \to N[-1]$ such that for all $i\in\mathbb{Z}$ we have $d^N_{i+1}h_i + h_{i-1}d^M_i$ we find that, since for any $h \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_1$ we have the formula

$$(d_1 h)_i = d^N_{i+1}h_i + h_{i-1}d^M_i.$$

for the 1st differential, this gives that f is nullhomotopic iff there is an element $h \in \underline{\mathsf{Hom}}_{\mathsf{Ch}_R}(M,N)_1$ such that $d_1(h) = f$.

Taken together, (a) and (b) shows that the induced map $\tilde{\Phi}$ below, from the $\Bbbk$-module isomorphism Φ, is itself a $\Bbbk$-module isomorphism,

$$
\begin{array}{ccc}
(f_i)_{i\in\mathbb{Z}} & \longmapsto & f \\
\mathsf{Z}_0(\underline{\mathsf{hom}}(M,N)) & \xrightarrow{\quad\Phi\quad} & \mathsf{hom}_{\mathsf{Ch}_R}(M,N) \\
\downarrow & & \downarrow \\
\mathsf{H}_0(\underline{\mathsf{hom}}(M,N)) & \overset{\tilde{\Phi}}{\dashrightarrow} & \mathsf{hom}_{\mathsf{Ch}_R}(M,N)/(\text{nullhomotopic maps})
\end{array}
\quad . \tag{4.1.2}
$$

Observe that $\tilde{\Phi}$ is defined by taking $[f]$ in homology to $[f]$ viewed as what we may call its *homotopy class*, where $f \sim g$ iff $f - g$ is nullhomotopic.

Before our next set of observations, to not confuse the reader unnecessarily, we refer back to remark 3.2.14.

We observe that

$$
\begin{aligned}
\left(\Sigma^i \underline{\mathsf{hom}}(M,N)\right)_j &= \mathsf{hom}(M,N)_{j-i} \\
&= \prod_{n\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_n, N_{n+j-i}) \\
&= \prod_{\ell\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}(M_{\ell+i}, N_{\ell+j}), \qquad \text{after reindexing with } \ell = n - i \\
&= \prod_{\ell\in\mathbb{Z}} \mathrm{hom}_{\mathsf{Mod}_R}\left(\left(\Sigma^{-i}M\right)_\ell, N_{\ell+j}\right) \\
&= \underline{\mathsf{hom}}\left(\Sigma^{-i}M, N\right)_j .
\end{aligned}
$$

Since iterated shifts may introduce sign changes in differentials one has to be more careful about differentials: From the above we define $\Phi : \left(\Sigma^i\underline{\mathsf{hom}}(M,N)\right) \to \underline{\mathsf{hom}}\left(\Sigma^{-i}M,N\right)_j$ explicitly by $(\Phi_j(f))_\ell = (-1)^{ij} f_{\ell+i}$ for $f \in \underline{\mathsf{hom}}\left(\Sigma^i M, N\right)_j$. It is straightforward to check that Φ_j is $\Bbbk$-linear. Let $\Psi_j : \underline{\mathsf{hom}}\left(\Sigma^{-i}M,N\right)_j \to \left(\Sigma^i\underline{\mathsf{hom}}(M,N)\right)_j$ be defined by $(\Psi_j(g))_n = (-1)^{ij} g_{n-i}$ with $g \in \underline{\mathsf{hom}}\left(\Sigma^{-1}M,N\right)_j$.

Then we see that

$$
\begin{aligned}
(\Psi_j(\Phi_j(f)))_n &= (-1)^{ij}(\Phi_j(f))_{n-i} \\
&= (-1)^{2ij} f_{(n-i)+i} \\
&= f_n
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_j(\Psi_j(g))_n &= (-1)^{ij}(\Psi_j(g))_{n+i} \\
&= (-1)^{2ij} g_{(n+i)-i} \\
&= g_n .
\end{aligned}
$$

Hence Ψ_j is a two-sided inverse to Φ_j so is also $\Bbbk$-linear. Then $\Phi := (\Phi_j)_{j\in\mathbb{Z}}$ has an inverse $\Psi := (\Psi_j)_{j\in\mathbb{Z}}$. It is then enough to show that Φ is a chain map, i.e. a morphism in $\mathsf{Ch}_{\Bbbk}$: We compute that

$$\begin{aligned}\left(d_j^{\underline{\mathsf{hom}}(\Sigma^{-i}M,N)}\left(\Phi_j(f)\right)\right)_\ell &= d^N_{j+\ell}\Phi_j(f)_\ell + (-1)^{j-1}(\Phi_j(f))_{\ell-1}d_\ell^{\Sigma^{-i}M} \\ &= (-1)^{ij}d^N_{j+\ell}f_{\ell+i} + (-1)^{(j-1)+ij}f_{\ell-1+i}(-1)^{-i}d^M_{i+\ell} \\ &= (-1)^{ij}d^N_{j+\ell}f_{\ell+i} + (-1)^{(j-1)+ij-i}f_{\ell-1+i}d^M_{i+\ell}.\end{aligned}$$

We also have

$$\begin{aligned}\left(\Phi_{j-1}\left(d_j^{\Sigma^i\underline{\mathsf{hom}}(M,N)}(f)\right)\right)_\ell &= (-1)^{i(j-1)}\left(d_j^{\Sigma^i\underline{\mathsf{hom}}(M,N)}(f)\right)_{\ell+i} \\ &= (-1)^{i(j-1)}\left((-1)^i d_{j-i}^{\underline{\mathsf{hom}}(M,N)}(f)\right)_{\ell+i} \\ &= (-1)^{i(j-1)}(-1)^i d^N_{(j-i)+\ell+i}f_{\ell+i} + (-1)^{j-i-1+ij-i+i}f_{\ell+i-1}d^M_{\ell+i} \\ &= (-1)^{ij}d^N_{j+\ell}f_{\ell+i} + (-1)^{j-i-1+ij}f_{\ell+i-1}d^M_{\ell+i}.\end{aligned}$$

Comparison gives that the two computation agrees. We conclude that

$$\Sigma^i\underline{\mathsf{hom}}(M,N) \cong \underline{\mathsf{hom}}\left(\Sigma^{-i}M,N\right) \tag{4.1.3}$$

in $\mathsf{Ch}_{\Bbbk}$.

Furthermore, we note that

$$\begin{aligned}\left(\Sigma^i\underline{\mathsf{hom}}\,(M,N)\right)_\ell &= \underline{\mathsf{hom}}(M,N)_{\ell-i} \\ &= \prod_{j\in\mathbb{Z}}\mathsf{hom}_{\mathsf{Mod}_R}(M_j,N_{j+\ell-i}) \\ &= \prod_{j\in\mathbb{Z}}\mathsf{hom}_{\mathsf{Mod}_R}(M_j,\left(\Sigma^iN\right)_{j+\ell}) \\ &= \underline{\mathsf{hom}}\left(M,\Sigma^iN\right)_\ell.\end{aligned}$$

The above computation motivates defining

$$\Lambda_\ell : \Sigma^i\underline{\mathsf{hom}}\,(M,N)_\ell \to \underline{\mathsf{hom}}\left(M,\Sigma^{-i}N\right)_\ell, \qquad f\mapsto f,$$

i.e. as the identity map in each degree. What remains to check is that Λ_ℓ "plays well" with the differential, i.e. is a chain map. We compute

$$\begin{aligned}\left(d_\ell^{\underline{\mathsf{hom}}(M,\Sigma^iN)}\Lambda_\ell(f)\right)_n &= \left(d_\ell^{\underline{\mathsf{hom}}(M,\Sigma^iN)}(f)\right)_n \\ &= d^{\Sigma^iN}_{\ell+n}f_n + (-1)^{\ell-1}f_{n-1}d^M_n \\ &= (-1)^i d^N_{\ell+n-i}f_n + (-1)^{\ell-1}f_{n-1}d^M_n.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left(\Lambda_{\ell-1}\left(d_\ell^{\Sigma^i\underline{\mathsf{hom}}(M,N)}(f)\right)\right)_n &= \left(d_\ell^{\Sigma^i\underline{\mathsf{hom}}(M,N)}(f)\right)_n \\ &= \left((-1)^i d_{\ell-i}^{\underline{\mathsf{hom}}(M,N)}(f)\right)_n \\ &= (-1)^i d^N_{\ell-i+n}f_n + (-1)^{\ell-i-1+i}f_{n-1}d^M_n.\end{aligned}$$

The two computations agree, and since Λ_ℓ is an isomorphism, it follows that

$$\Sigma^i \underline{\mathsf{hom}}(M,N) \cong \underline{\mathsf{hom}}(M, \Sigma^i N) \quad \text{in } \mathsf{Ch}_{\Bbbk}.$$

Taken together, we then have

$$\Sigma^i \underline{\mathsf{hom}}(M,N) \cong \underline{\mathsf{hom}}\left(\Sigma^{-i}M, N\right) \cong \underline{\mathsf{hom}}\left(M, \Sigma^i N\right) \quad \text{in } \mathsf{Ch}_{\Bbbk}. \tag{4.1.4}$$

Below, we introduce a notion of "homotopies between homotopies", and give a condition when such a higher homotopy exists.

Theorem 4.1.4. *Let $f, g : M \rightrightarrows N$ be chain maps, and let $h_0, h_1 : f \overset{\Rightarrow}{\Rightarrow} g$ be homotopies. Then there exists a notion of homotopy $h_0 \Rightarrow h_1$ such that such a homotopy exists iff $\mathsf{H}_1(\mathsf{hom}_{\mathsf{Ch}_R}(M,N)) = 0$.*

Proof. Let's define our notion of a homotopy $H : h_0 \Rightarrow h_1$ as an element $H \in \underline{\mathrm{hom}}_{\mathsf{Ch}_R}(M,N)_2$ such that $d_2(H) = h_1 - h_0$ where we note that in component maps, $H_i : M_i \to N_{i+2}$ and

$$\begin{aligned}(d_2 H)_i &= d^N_{i+2} H_i - H_{i-1} d^M_i \\ &= (h_1 - h_0).\end{aligned}$$

We observe that

$$\begin{aligned}d_1(h_1 - h_0) &= d_1(h_1) - d_1(h_0) \\ &= (g - f) - (g - f), \qquad \text{by (b) in our earlier remarks} \\ &= 0,\end{aligned}$$

i.e. so that $h_1 - h_0$ is a 1-cycle.

$\underline{\Leftarrow}$: If $\mathsf{H}_1(\underline{\mathsf{hom}}(M,N)) = 0$ then every 1-cycle is a 1-boundary, i.e. there is some $H \in \underline{\mathsf{hom}}(M,N)_2$ such that $d_2(H) = h_1 - h_0$. This shows one direction.

$\underline{\Rightarrow}$: Assume that there is an element $H \in \underline{\mathrm{hom}}_{\mathsf{Ch}_R}(M,N)_2$ such that $d_2(H) = h_1 - h_0$ for *every pair* of homotopies

$$h_0, h_1 : f \overset{\Rightarrow}{\Rightarrow} g$$

between chain maps $f, g : M \rightrightarrows N$. Let $z \in \mathsf{Z}_1(\underline{\mathrm{hom}}(M,N))$ so that $d_1(z) = 0$. Letting $f = g = 0 : M \to N$, then $h_0 : 0 \Rightarrow 0$ is a homotopy, and since $d_1(z) = 0 = g - f$ we have that $h_1 := z$ is a homotopy $0 \Rightarrow 0$. By hypothesis, there exists an element $H \in \underline{\mathsf{hom}}(M,N)_2$ such that $d_2(H) = z - 0 = z$. Hence every 1-cycle is a boundary, so that $\mathsf{H}_1(\underline{\mathsf{hom}}(M,N)) = 0$. □

4.1.1 Brief interlude on *enriched* category theory

Before taking the next step towards defining the (derived) ∞-category of R-modules, we need to present some further notions. For this, we turn to *enriched* category theory. For reference, see [Rie14, Chapter 3.1-3.2 of part I].

Definition 4.1.5 ($\mathcal{V}$-enriched category)**.** To define a **$\mathcal{V}$-enriched** category $\underline{\mathcal{D}}$, let $(\mathcal{V}, \times, \mathbb{1})$ be a **symmetric monoidal category** in the sense of [Mil25, Definition 2.1]. Then we say that the *data* of a $\mathcal{V}$-enriched category $\underline{\mathcal{D}}$ consists of

- a collection of objects $L, M, N \in \underline{\mathcal{D}}$.

- for each pair of objects L, M, there is a **hom-object** (cf. hom-complex in §4.1) $\underline{\mathcal{D}}(L, M)$.
- For each object $L \in \underline{\mathcal{D}}$ there is a a morphism $\iota_L : \mathbb{1} \to \underline{\mathcal{D}}(L, L)$.
- For each triple of objects $L, M, N \in \underline{\mathcal{D}}$ there is a morphism

$$\underline{\mathcal{D}}(M, N) \times \underline{\mathcal{D}}(L, M) \xrightarrow{\chi_{L,M,N}} \underline{\mathcal{D}}(L, N)$$

subject to the constraints that the following diagrams commute, for all objects $L, M, N, O \in \underline{\mathcal{D}}$:

$$\begin{array}{ccc} \underline{\mathcal{D}}(N,O) \times \underline{\mathcal{D}}(M,N) \times \underline{\mathcal{D}}(L,M) & \xrightarrow{\mathrm{id} \times \chi_{L,M,N}} & \underline{\mathcal{D}}(N,O) \times \underline{\mathcal{D}}(L,N) \\ \Big\downarrow {\scriptstyle \chi_{M,N,O} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \chi_{L,N,O}} \\ \underline{\mathcal{D}}(M,O) \times \underline{\mathcal{D}}(L,M) & \xrightarrow[\chi_{L,M,O}]{} & \underline{\mathcal{D}}(L,O) \end{array} , \tag{4.1.5}$$

$$\begin{array}{ccc} \underline{\mathcal{D}}(M,N) \times \mathbb{1} & \xrightarrow{\mathrm{id} \times \iota_M} & \underline{\mathcal{D}}(M,N) \times \underline{\mathcal{D}}(M,M) \\ & {\scriptstyle \cong} \searrow & \Big\downarrow {\scriptstyle \chi_{M,M,N}} \\ & & \underline{\mathcal{D}}(M,N) \end{array} \qquad \begin{array}{ccc} \mathbb{1} \times \underline{\mathcal{D}}(L,M) & \xrightarrow{\iota_M \times \mathrm{id}} & \underline{\mathcal{D}}(M,M) \times \underline{\mathcal{D}}(L,M) \\ & {\scriptstyle \cong} \searrow & \Big\downarrow {\scriptstyle \chi_{L,M,M}} \\ & & \underline{\mathcal{D}}(L,M) \end{array} \tag{4.1.6}$$

where the diagonal isomorphisms in the diagrams in 4.1.6 are the natural ones coming from the symmetric monoidal structure on $\mathcal{V}$.

4.2 The dg category of complexes

We now aim to define maps ι_M and $\chi_{L,M,N}$ as in 4.1.5 such that Ch_R becomes *enriched* over $(\mathsf{Ch}_{\Bbbk}, \otimes, \underset{\sim}{\Bbbk})$ (cf. theorem 2.5.4). For brevity we may write $\Bbbk$ instead of $\underset{\sim}{\Bbbk}$, as perhaps noted earlier.

Given $L, M, N \in \mathsf{Ch}_R$, we want to construct a map

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \otimes \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, M) \xrightarrow{\chi_{L,M,N}} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, N)$$

in $\mathsf{Ch}_{\Bbbk}$. It is enough to define a map in each degree ℓ

$$\big(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N) \otimes \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, M)\big)_\ell \xrightarrow{(\chi_{L,M,N})_\ell} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, N)_\ell$$

which is the same as defining a map

$$\bigoplus_{p+q=\ell} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_p \otimes_{\Bbbk} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, M)_q \xrightarrow{(\chi_{L,M,N})_\ell} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, N)_\ell \tag{4.2.1}$$

We first define it on pure tensors $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_p \otimes \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, M)_q \xrightarrow{(\chi_{L,M,N})_\ell} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, N)_\ell$, where we may denote the domain as $A_p \otimes B_q$ for brevity. Defining a $\Bbbk$-linear such map is the same as defining a bilinear map $A_p \times B_q$ by the universal property of the (relative) tensor product. If we let $(f,g) \xmapsto{\eta} f \circ g$ with

$$(f \circ g)_i = f_{i+q} \circ g_i. \tag{4.2.2}$$

An easy check (using that every R-linear map is $\Bbbk$-linear whenever R is a $\Bbbk$-algebra) shows that η is $\Bbbk$-bilinear. Hence we get an induced map $A_p \otimes B_q \to \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L,N)_\ell$ defined by $(f \otimes g) \mapsto (f \circ g)$ with $(f \circ g)_i = f_{i+q} \circ g_i$. By the *universal property of the coproduct* this uniquely determines a $\Bbbk$-linear map $(\chi_{L,M,N})_\ell$ as in 4.2.1. These maps then assemble into a family of maps $\chi_{L,M,N}$.

It remains to show that $(\chi_{L,M,N})$ is a *chain map* in $\mathsf{Ch}_\Bbbk$. By linearity it is enough to check this on pure tensors $f \otimes g \in A_p \otimes B_q$. Recall that $d(f \otimes g) = d(f) \otimes g + (-1)^p f \otimes d(g)$.

We then find that

$$\begin{aligned}(\chi_{L,M,N})_\ell(d(f \otimes g)) &= (\chi_{L,M,N})_\ell(d(f) \otimes g + (-1)^p f \otimes d(g)) \\ &= d(f) \circ g + (-1)^p f \circ d(g), \qquad \text{by linearity of } \chi. \end{aligned} \tag{4.2.3}$$

In degrees, with $f_i : M_i \to N_{i+p}$ and $g_i : L_i \to M_{i+q}$ we then see that

$$\begin{aligned}(d(f) \circ g + (-1)^p f \circ d(g))_i &= (d(f) \circ g)_i + ((-1)^p f \circ d(g))_i, \qquad \text{since } (u+v)_i = u_i + v_i \\ &= (d(f))_{i+q} \circ g_i + (-1)^p f_{i+q} \circ (d(g))_i, \qquad \text{by (4.2.2)} \\ &= d^N_{i+p+q} \circ f_{i+q} \circ g_i + (-1)^{p-1} f_{i+q-1} \circ d^M_{i+q} \circ g_i \\ &+ (-1)^p \left(f_{i+q-1} \circ d^M_{i+q} \circ g_i + (-1)^{q-1} f_{i+q-1} \circ g_{i-1} \circ d^L_i \right) \\ &= d^N_{i+p+q} \circ f_{i+q} \circ g_i + \cancel{(-1)^{p-1} f_{i+q-1} \circ d^M_{i+q} \circ g_i} \\ &+ \cancel{(-1)^p f_{i+q-1} \circ d^M_{i+q}} \circ g_i + (-1)^{p+q-1} f_{i+q-1} \circ g_{i-1} \circ d^L_i \\ &= d^N_{i+p+q} \circ f_{i+q} \circ g_i + (-1)^{p+q-1} f_{i+q-1} \circ g_{i-1} \circ d^L_i. \end{aligned} \tag{4.2.4}$$

On the other hand, we have that

$$\begin{aligned} d(\chi_{L,M,N}(f \otimes g))_i &= d(f \circ g)_i \\ &= d^N_{i+p+q} \circ (f \circ g)_i + (-1)^{p+q-1} (f \circ g)_{i-1} \circ d^L_i \\ &= d^N_{i+p+q} \circ f_{i+q} \circ g_i + (-1)^{p+q-1} f_{i+q-1} \circ g_{i-1} \circ d^L_i. \end{aligned} \tag{4.2.5}$$

Comparing 4.2.4 and 4.2.5 we see that they agree, since $i \in \mathbb{Z}$ was arbitrary we see that they agree in each degree, hence must be the same element. Since f and g were arbitrary it follows that $d\chi = \chi d$, i.e., χ is a chain map.

Furthermore, in accordance with definition 4.1.5 we define, for each $M \in \mathsf{Ch}_R$, a chain map (in $\mathsf{Ch}_\Bbbk$)

$$\Bbbk \xrightarrow{\iota_M} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M)$$

by $(\iota_M)_n = 0$ for $n \neq 0$ and

$$(\iota_M)_0(\lambda) = \lambda \,\mathrm{id}_M, \qquad \text{for all } \lambda \in \Bbbk.$$

To check that it is a chain map, we need to show that

$$d_n^{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,M)} \circ (\iota_M)_n = (\iota_M)_{n-1} \circ d_n^{\Bbbk}.$$

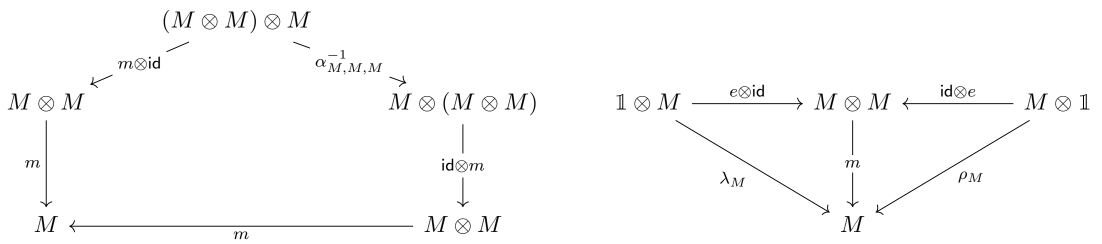
Since $d_n^{\Bbbk} = 0$ for all n, the right-hand side is always zero. Hence (since $(\iota_M)_n = 0$ for $n \neq 0$) it is enough to check that the left-hand side is zero for $n = 0$.

We find that for $\lambda \in \Bbbk$, we have (in degree i)

$$\begin{aligned}
\left(\left(d_0^{\underline{\hom}_{\mathsf{Ch}_R}(M,M)} \circ (\iota_M)_0\right)(\lambda)\right)_i &= \left(d_0^{\underline{\hom}_{\mathsf{Ch}_R}(M,M)}(\lambda\, \mathrm{id}_M)\right)_i \\
&= d_i^M \circ (\lambda\, \mathrm{id}_{M_i}) + (-1)^{-1} \lambda\, \mathrm{id}_{M_{i-1}} \circ d_i^M \\
&= d_i^M \circ \lambda\, \mathrm{id}_{M_i} - \lambda\, \mathrm{id}_{M_{i-1}} \circ d_i^M \\
&= 0,
\end{aligned}$$

since $\lambda\, \mathrm{id}_M$ is a chain map (this follows directly from id_M being a chain map). We conclude that ι_M is a chain map in $\mathsf{Ch}_{\Bbbk}$.

Definition 4.2.1 (Monoid in a monoidal category)**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. Then an object $M \in \mathsf{Ob}(\mathcal{C})$ is a **monoid** if there are morphisms $m : M \otimes M \to M$ and $e : \mathbb{1} \to M$ such that the following diagrams commute,



where α, λ, ρ are the associator, left unitor and right unitor, respectively, in $\mathcal{C}$.

Consider the triple $(\underline{\mathsf{hom}}_{\mathbf{K}_R}(M, M), \chi_{M,M,M}, \iota_M)$. Checking on homogeneous elements that the left diagram above commutes with respect to this triple, with $m = \chi_{M,M,M}$ and $e = \iota_M$, comes down to verifying that $f \circ (g \circ h) = (f \circ g) \circ h$ (i.e. enriched associativity), which is straightforward to check from definition of enriched composition. That the right diagram commutes follows essentially by definition of ι_M and how λ and ρ were defined in the proof (sketch) of theorem 2.5.4. Hence $\underline{\mathsf{hom}}_{\mathbf{K}_R}(M, M)$ with multiplication $\chi_{M,M,M}$ and unit ι_M is a *monoid* in the symmetric monoidal category $\left(\mathsf{Ch}_{\Bbbk}, \otimes, \underset{\sim}{\Bbbk}\right)$, or in this particular instance, we may call $\underline{\mathsf{hom}}_{\mathbf{K}_R}(M, M)$ a **dg-algebra** (or **dga**).

Theorem 4.2.2. *With the maps $\chi_{L,M,N}$ and ι_M above, the category Ch_R is a $\mathsf{Ch}_{\Bbbk}$-enriched in the sense of definition 4.1.5.*

Proof. We need to show that $\chi_{L,M,N}$ and ι_M participates in commutative diagrams as in 4.1.5 and 4.1.6 for all objects $L, M, N, O \in \mathsf{Ch}_R$.

Diagram 4.1.5 commutes: One may check this on pure tensors $h \otimes f \otimes g$. Then showing that the diagram commutes boils down to checking that $h \circ (f \circ g) = (h \circ f) \circ g$. We check this in degrees, with h in degree r, f in degree p and g in degree q.

We have that

$$\begin{aligned}
(h \circ (f \circ g))_i &= h_{i+p+q} \circ (f \circ g)_i, && \text{by (4.2.2)} \\
&= h_{i+p+q} \circ (f_{i+q} \circ g_i) \\
&= (h_{i+p+q} \circ f_{i+q}) \circ g_i, && \text{since composition is associative in } \mathsf{Mod}_R \\
&= (h \circ f)_{i+q} \circ g_i \\
&= ((h \circ f) \circ g)_i.
\end{aligned}$$

The conclusion follows.

The diagrams in 4.1.6 commute: We again check this on pure tensors $\lambda \otimes f$ with $f \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(L, M)_p$ in degree p and $\lambda \in \Bbbk$ (i.e., in degree zero). Let ℓ and r be the left respectively right unitors $\ell_A : \Bbbk \otimes A \approx A$ and $r_A : A \otimes \Bbbk \approx A$ defined in degree n as

$$\Bbbk \otimes A_n \approx A_n, \qquad \lambda \otimes a \xmapsto{(\ell_A)_n} \lambda a,$$

with $(r_A)_n$ defined similarly.

Since showing that two diagrams commute essentially boils down to the same computations, we will only do one, the rightmost one in 4.1.6. By going right-downwards in said diagram, we get

$$\lambda \otimes f \xrightarrow{\iota_M \otimes \mathrm{id}} (\lambda \,\mathrm{id}_M) \otimes f \xrightarrow{\chi_{L,M,M}} \lambda \,\mathrm{id}_M \circ f,$$

which in degree $i \in \mathbb{Z}$ gives

$$\begin{aligned}(\lambda \,\mathrm{id}_M \circ f)_i &= (\lambda \,\mathrm{id}_M)_{i+p} \circ f_i \\ &= \lambda \,\mathrm{id}_{M_{i+p}} \circ f_i \\ &= \lambda f_i \end{aligned} \tag{4.2.6}$$

while going diagonally downwards directly via $\ell_{\underline{\mathrm{hom}}_{\mathsf{Ch}_R}(L,M)}$ gives

$$\lambda \otimes f \xmapsto{\ell_{\underline{\mathrm{hom}}_{\mathsf{Ch}_R}(L,M)}} \lambda f,$$

which by definition of scalar multiplication means that $\lambda f = (\lambda f_i)_{i \in \mathbb{Z}}$, which agrees with 4.2.6. The conclusion follows. $\square$

From now on, we denote this $(\mathsf{Ch}_\Bbbk, \otimes, \underset{\sim}{\Bbbk})$-enriched category Ch_R with the structure maps $\chi_{L,M,N}$ and ι_M, as $\mathbf{K}_R$ with $\mathsf{hom}_{\mathbf{K}_R}(M, N) := \underline{\mathrm{hom}}_{\mathsf{Ch}_R}(M, N)$, which we will call the **dg-category of complexes of R-modules**. Observe that the hom-complexes in this category assemble the data of chain maps $f, g : M \rightrightarrows N$, homotopies between chain maps, and homotopies between homotopies into a single object. For a general element $H \in \mathsf{hom}_{\mathbf{K}_R}(M, N)_n$ observe that $d(H) = h_1 - h_0$ means that h_1 and h_0 become equal in homology (since they differ by a boundary), one may call this relationship for $n \gg 0$ (and $n \ll 0$) *keeping track of higher (and lower) homotopies.*

Theorem 4.2.3. *The following statements are equivalent.*

(a) A complex $M \in \mathbf{K}_R$ is contractible.

(b) $\mathsf{hom}_{\mathbf{K}_R}(M, M) \in \mathsf{Ch}_\Bbbk$ is acyclic.

(c) The complex $\mathsf{hom}_{\mathbf{K}_R}(M, M)$ is such that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M, M)) = 0$.

(d) The element $[\mathrm{id}_M] \in \mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M, M))$ is zero, i.e. $[\mathrm{id}_M] = 0$.

Proof. (a) $\Rightarrow$ (b): That M is contractible means that there is a nullhomotopy $s : M \to M[-1]$, i.e. so that

$$d^M_{i+1} s_i + s_{i-1} d^M_i = \mathrm{id}_{M_i}, \quad \forall i \in \mathbb{Z}.$$

This is the same as (by definition of the 1st differential of the hom-complex) $d_1(s) = \mathrm{id}_M$.

Let $z \in \mathsf{Z}_n(\mathsf{hom}_{\mathbf{K}_R}(M,M))$ be an arbitrary n-cycle. Then, by definition, $d_n(z) = 0$. By theorem 4.2.2 there is a chain map (composition)

$$\mathsf{hom}_{\mathbf{K}_R}(M,M) \otimes \mathsf{hom}_{\mathbf{K}_R}(M,M) \xrightarrow{\chi_{M,M,M}} \mathsf{hom}_{\mathbf{K}_R}(M,M).$$

Consider the element

$$s \otimes z \in \mathsf{hom}_{\mathbf{K}_R}(M,M)_1 \otimes \mathsf{hom}_{\mathbf{K}_R}(M,M)_n \subset \left(\mathsf{hom}_{\mathbf{K}_R}(M,M) \otimes \mathsf{hom}_{\mathbf{K}_R}(M,M)\right)_{n+1}.$$

Since $\chi_{M,M,M}$ is a *chain map*, we see that

$$\begin{aligned}
d_{n+1}((\chi_{M,M,M})_{n+1}(s \otimes z)) &= (\chi_{M,M,M})_n \, d_{n+1}(s \otimes z) \\
\Leftrightarrow d_{n+1}(s \circ z) &= (\chi_{M,M,M})_n \left(\underbrace{d_1(s)}_{=\mathrm{id}_M} \otimes z - \underbrace{s \otimes \underbrace{d_n(z)}_{=0}}_{=0} \right) \\
\Leftrightarrow d_{n+1}(s \circ z) &= (\chi_{M,M,M})_n (\mathrm{id}_M \otimes z) \\
\Leftrightarrow d_{n+1}(s \circ z) &= \mathrm{id}_M \circ z \\
\Leftrightarrow d_{n+1}(s \circ z) &= z,
\end{aligned}$$

where we in the last step used that $(\mathrm{id}_M \circ z)_i = \mathrm{id}_{M_{n+i}} \circ z_i = z_i$, i.e. $\mathrm{id}_M \circ z = z$, by (4.2.2). Hence z is a boundary, and so it follows that $z \in \mathsf{B}_n(\mathsf{hom}_{\mathbf{K}_R}(M,M))$. Since n and z was arbitrary, it follows that

$$\mathsf{H}_n(\mathsf{hom}_{\mathbf{K}_R}(M,M)) = 0$$

for all $n \in \mathbb{Z}$, i.e. $\mathsf{hom}_{\mathbf{K}_R}(M,M)$ is acyclic.

(b) $\Rightarrow$ (c): Follows directly from definition, since if $\mathsf{hom}_{\mathbf{K}_R}(M,M)$ is acyclic, then all its homology groups vanish, so in particular $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M,M)) = 0$.

(c) $\Rightarrow$ (d): After recalling that $\mathsf{Z}_0(\mathsf{hom}_{\mathbf{K}_R}(M,M)) = \mathsf{hom}_{\mathsf{Ch}_R}(M,M) \ni \mathrm{id}_M$, the conclusion is immediate from the hypothesis that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(M,M)) = 0$.

(d) $\Rightarrow$ (a): By hypothesis, id_M is the boundary of some element

$$s \in \mathsf{hom}_{\mathbf{K}_R}(M,M)_1 = \prod_{i \in \mathbb{Z}} \mathrm{hom}_{\mathsf{Mod}_R}(M_i, M_{i+1}).$$

This means that $d_1(s) = \mathrm{id}_M$, or written in components, this is

$$\begin{aligned}
d^M_{i+1} \circ s_i + (-1)^{1-1} s_{i-1} \circ d^M_i &= \mathrm{id}_{M_i}, \quad \forall i \in \mathbb{Z} \\
\Leftrightarrow d^M_{i+1} \circ s_i + s_{i-1} \circ d^M_i &= \mathrm{id}_{M_i}, \quad \forall i \in \mathbb{Z}.
\end{aligned}$$

That is, s is a contraction (nullhomotopy) of its identity map, i.e. M is contractible. □

Something special happens whenever we have a homotopy equivalence $M \to N$ in $\mathbf{K}_R$:

Theorem 4.2.4. *Let $f : M \to N$ be a homotopy equivalence in $\mathbf{K}_R$. Then there are natural transformations*

$$\mathsf{hom}_{\mathbf{K}_R}(-, M) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(-, N) \qquad \textit{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(M, -) \longleftarrow \mathsf{hom}_{\mathbf{K}_R}(N, -)$$

which are natural equivalences, i.e. for each fixed complex $L \in \mathbf{K}_R$, there are homotopy equivalences

$$\mathsf{hom}_{\mathbf{K}_R}(L, M) \simeq \mathsf{hom}_{\mathbf{K}_R}(L, N) \qquad \textit{and} \qquad \mathsf{hom}_{\mathbf{K}_R}(M, L) \simeq \mathsf{hom}_{\mathbf{K}_R}(N, L).$$

Remark 4.2.5. Observe that when we say *natural*, we mean that they are natural with respect to all *chain maps*, i.e. with respect $\mathsf{Z}_0(\mathsf{hom}_{\mathbf{K}_R}(M, N)) = \mathsf{hom}_{\mathsf{Ch}_R}(M, N)$. In fact, one can probably upgrade this statement (perhaps with appropriate tweaks to the conditions) to a stronger statement about *enriched* naturality; see [Rie14, Def. 3.5.8, part I].

Proof. By assumption, there is a chain map $g : N \to M$ such that $f \circ g \simeq \mathrm{id}_N$ and $g \circ f \simeq \mathrm{id}_M$.

<u>$\mathsf{hom}_{\mathbf{K}_R}(-, M) \longrightarrow \mathsf{hom}_{\mathbf{K}_R}(-, N)$</u>: For each object $L \in \mathbf{K}_R$, define

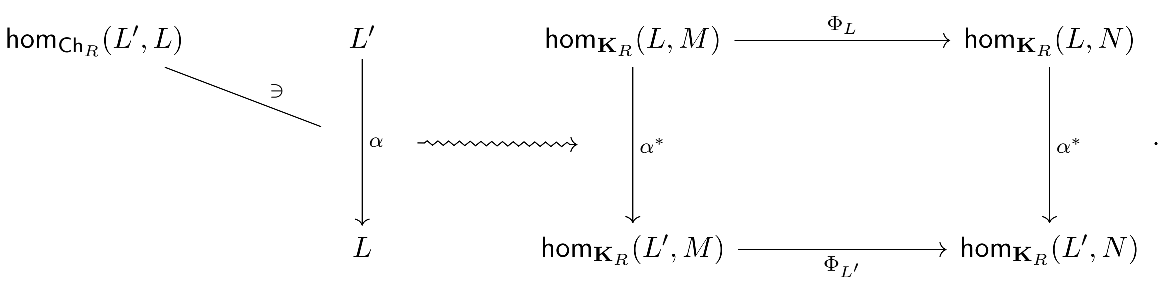
$$(\Phi_L)_n : \mathsf{hom}_{\mathbf{K}_R}(L, M)_n \to \mathsf{hom}_{\mathbf{K}_R}(L, N)_n, \qquad u \xmapsto{\chi_{L,M,N}(f \otimes (\,\cdot\,))} f \circ u,$$

for elements $u \in \mathsf{hom}_{\mathbf{K}_R}(L, M)_n$ in degree n, with, recall, $(f \circ u)_i = f_{i+n} \circ u_i : L_i \to N_{i+n}$.

Since f is a chain map, we have that $d_0(f) = 0$. Therefore, it follows that

$$\begin{aligned} d_n((\Phi_L)_n(u)) &= d_n((\chi_{L,M,N})_n(f \otimes u)) \\ &= (\chi_{L,M,N})_{n-1}(d_n(f \otimes u)), \qquad \text{since } \chi_{L,M,N} \text{ is a chain map} \\ &= (\chi_{L,M,N})_{n-1}\left(\underbrace{\underbrace{d_0(f)}_{=0} \otimes u}_{=0} + f \otimes d_n(u) \right) \\ &= (\chi_{L,M,N})_{n-1}(f \otimes d_n(u)) \\ &= (\Phi_L)_{n-1}(d_n(u)). \end{aligned}$$

Hence $\Phi_L := ((\Phi_L)_n)_{n \in \mathbb{Z}}$ assemble into a chain map. It remains to show that Φ_L is *natural*: Let $\alpha : L' \to L$ be an arbitrary chain map. Naturality amounts to checking that the following square below to the right commmutes,



An easy check, using that α is a 0-cycle in $\mathbf{K}_R$, together with how the differentials of the hom-complexes are defined, gives that α^* is in fact a chain map in $\mathsf{Ch}_{\Bbbk}$. Let $u \in \mathsf{hom}_{\mathbf{K}_R}(L, M)_n$ for arbitrary $n \in \mathbb{Z}$. Then we see that

$$\alpha^*(\Phi_L(u)) = (f \circ u) \circ \alpha.$$

In degree i, this is

$$\begin{aligned} ((f \circ u) \circ \alpha)_i &= (f \circ u)_i \circ \alpha_i \\ &= f_{i+n} \circ u_i \circ \alpha_i. \end{aligned} \tag{4.2.7}$$

Next we compute

$$\Phi_{L'}(\alpha^*(u)) = f \circ (u \circ \alpha).$$

In degree i, this is

$$(f \circ (u \circ \alpha))_i = f_{i+n} \circ u_i \circ \alpha_i.$$

Comparison with 4.2.7 shows that they are equal in each degree i, i.e. defines the same chain map.

Let

$$\Psi_L : \mathsf{hom}_{\mathbf{K}_R}(L, N) \to \mathsf{hom}_{\mathbf{K}_R}(L, M), \qquad v \mapsto g \circ v,$$

with recall g the homotopy inverse for f. Essentially the same computations as above shows that $\Psi_L = (\Psi_L)_{n \in \mathbb{Z}}$ is a chain map for each object $L \in \mathbf{K}_R$. Let $h \in \mathsf{hom}_{\mathbf{K}_R}(M, M)_1$ be a witnessing homotopy $g \circ f \simeq \mathrm{id}_M$, i.e. so that

$$d(h) = g \circ f - \mathrm{id}_M . \tag{4.2.8}$$

Let $A := \mathsf{hom}_{\mathbf{K}_R}(L, M)$. Consider the map $h_* : A \to A[-1]$ defined by postcomposition, i.e. $h_*(u) = h \circ u$ for $u \in A_n$ with $(h \circ u)_i = h_{n+i} \circ u_i : L_i \to M_{n+i+1}$.

Since h has degree one, and since recall that composition here is really defined in terms of the tensor product and χ (see 4.2.3), it follows that

$$d(h \circ u) = d(h) \circ u - h \circ d(u).$$

Then we see that

$$\begin{aligned} d(h_*(u)) + h_*(d(u)) &= d(h \circ u) + h \circ d(u) \\ &= d(h) \circ u - h \circ d(u) + h \circ d(u) \\ &= d(h) \circ u \\ &= ((g \circ f) - \mathrm{id}_M) \circ u \\ &= (g \circ f) \circ u - \mathrm{id}_M \circ u \\ &= \Psi_L(\Phi_L(u)) - \mathrm{id}_M \circ u \\ &= \Psi_L(\Phi_L(u)) - (\mathrm{id}_M)_* (u) \\ \Rightarrow dh_* + h_* d &= \Psi_L \Phi_L - \mathrm{id}_A, \end{aligned}$$

since $(\mathrm{id}_M)_* = \mathrm{id}_A$. One may check that h_* is $\Bbbk$-linear, hence h_* is a chain homotopy $\Psi_L \circ \Phi_L \simeq \mathrm{id}_A$. To show that $\Phi_L \circ \Psi_L \simeq \mathrm{id}_B$ with $B = \mathsf{hom}_{\mathbf{K}_R}(L, N)$, use a witnessing chain homotopy $k \in \mathsf{hom}_{\mathbf{K}_R}(N, N)_1$.

$\underline{\mathsf{hom}_{\mathbf{K}_R}(M,-) \longleftarrow \mathsf{hom}_{\mathbf{K}_R}(N,-)}$: Analogous to the computations above with appropriate changes (e.g. precompose with $f : M \to N$ instead of postcomposing when defining the natural transformations). Perhaps the main difference here is that if $d(h) = g \circ f - \mathsf{id}_M$, then for $v \in \mathsf{hom}_{\mathbf{K}_R}(M,L)_n$ we define $H(v) = (-1)^n v \circ h$ to get the "right" cancellation.

□

4.3 (Important) properties of the dg-category of complexes $\mathbf{K}_R$

4.3.1 Failure of preserving quasi-isomorphisms for dg-hom functors

We will see in §5.1 that "ordinary" hom and ⊗-functors obtained from Mod_R do not in general respect (i.e. *preserve*) both kernels and cokernels. We aim to repair this "failure" in two steps, by first going from Mod_R to Ch_R, and then from "ordinary" kernels and cokernels to the *homotopy* kernels and cokernels we introduced in §3.

Consider the complex $M_\bullet$ in example 3.1.17. Since $M_\bullet$ is acyclic it follows that $0 \to M_\bullet$ is a quasi-isomorphism. However, upon applying the functor $\mathsf{hom}_{\mathbf{K}_{\mathbb{Z}/4}}(M_\bullet, -)$ to this we get a map

$$\mathsf{hom}_{\mathbf{K}_{\mathbb{Z}/4}}(M_\bullet, 0) \cong 0 \to \mathsf{hom}_{\mathbf{K}_{\mathbb{Z}/4}}(M_\bullet, M_\bullet).$$

Since $M_\bullet$ it is not contractible, it follows by theorem 4.2.3 that $\mathsf{hom}_{\mathbf{K}_{\mathbb{Z}/4}}(M_\bullet, M_\bullet)$ is not acyclic which means that the induced map above can not be a quasi-isomorphism, hence we can not expect that $\mathsf{hom}_{\mathbf{K}_R}(M,-)$ preserves quasi-isomorphisms, in general. Essentially the same reasoning applies to $\mathsf{hom}_{\mathbf{K}_R}(-,M)$ with the 3.1.17 as a counter-example.

The failure of the functors above to *preserve quasi-isomorphisms* motivates the introduction of *projective* and *injective resolutions*, which are introduced in §5.

4.3.2 Tensor-hom adjunctions for Ch_R

Let R be a $\Bbbk$-algebra (everything being commutative unital), let $T \in \mathsf{Ch}_\Bbbk$ and let $M \in \mathsf{Ch}_R$. Then we may form the complex

$$T \otimes_\Bbbk M \in \mathsf{Ch}_R.$$

That this in fact lives in Ch_R follows from the corresponding notions in Mod_R and $\mathsf{Mod}_\Bbbk$ (cf. remark 2.2.2) and how the differential is defined on the tensor-product.

If we let $N \in \mathsf{Ch}_R$, then also $N \in \mathsf{Ch}_\Bbbk$, and we may then consider

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T,N),$$

with

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T,N)_n = \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_\Bbbk}(T_i, N_{i+n}), \qquad \forall n \in \mathbb{Z}.$$

We saw in §2.4 that $\hom_\Bbbk(T_i, N_{i+n})$ has a natural (right) R-module structure. It follows that so does the (direct) product above. Therefore $\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T,N)$ is an R-module in each degree. Furthermore, an straightforward (degree-wise) check of the dg-differential shows that ít is R-linear. It follows that $\underline{\mathsf{hom}}_{\mathsf{Ch}_\Bbbk}(T,N) \in \mathsf{Ch}_R$.

Theorem 4.3.1. *For any $T \in \mathsf{Ch}_{\Bbbk}$ and $M, N \in \mathsf{Ch}_R$, we have natural isomorphisms*

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) &\approx \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}\left(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)\right) \\ &\approx \underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N)\right),\end{aligned}$$

in $\mathsf{Ch}_{\Bbbk}$.

Remark 4.3.2. Compare with theorem 2.2.1.

Remark 4.3.3. The proof below is almost complete (as far as we can tell) in terms of formality. We are skipping some steps in the computations involving how the functors $A(-,-,-), B(-,-,-)$ and $C(-,-,-)$ (defined in the proof) act on triplets of morphisms when it comes to showing that they are honest chain maps, but we have convinced ourselves that in fact defining them as we do *does give us chain maps*, and we at least try to give a plausible argument for why this is the case.

Proof. Let

$$\begin{cases} A := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \\ B := \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}\left(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)\right) \\ C := \underline{\mathsf{hom}}_{\mathsf{Ch}_R}\left(M, \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, N)\right) \end{cases}. \tag{4.3.1}$$

$\underline{A \approx B}$: Let $n \in \mathbb{Z}$, and $F \in A_n$, i.e. so that F is a family of R-linear maps

$$F_s : (T \otimes_{\Bbbk} M)_s \to N_{n+s}$$

where

$$(T \otimes_{\Bbbk} M)_s = \bigoplus_{p+q=s} T_p \otimes_{\Bbbk} M_q,$$

so consider pure tensors $t \otimes m$ with $t \in T_p$ and $m \in M_q$. Define $\Phi_n : A_n \to B_n$ by

$$(\Phi_n(F))_p(t)_q(m) := F_{p+q}(t \otimes m),$$

and define

$$\Psi_n : B_n \to A_n$$

with $\varphi \in B_n$ as

$$(\Psi_n(\varphi))_s(t \otimes m) := \varphi_p(t)_q(m), \qquad p + q = s$$

on the direct summand $T_p \otimes_{\Bbbk} M_q \subset \bigoplus_{p+q=s} T_p \otimes_{\Bbbk} M_q = (T \otimes_{\Bbbk} M)_s$. (By the universal property of the coproduct this gives a unique map in each degree s).

Since A_n and B_n are $\Bbbk$-modules, and how element-wise evaluation is defined, we see that Φ_n and Ψ_n are $\Bbbk$-linear. We claim that $\Phi := (\Phi_n)_{n \in \mathbb{Z}}$ is a *chain map*, i.e. that

$$d_n^B \Phi_n(F) = \Phi_{n-1} d_n^A(F), \qquad \forall F \in A_n, \forall n \in \mathbb{Z}.$$

Both sides above are elements of

$$\begin{aligned} B_{n-1} &= \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}\left(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)\right)_{n-1} \\ &= \prod_{p \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}\left(T_p, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N)_{p+n-1}\right) \\ &= \prod_{p \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_{\Bbbk}}\left(T_p, \prod_{q \in \mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}\left(M_q, N_{p+n+q-1}\right)\right). \end{aligned}$$

Fix $p, q \in \mathbb{Z}$ and elements $t \in T_p$ and $m \in M_q$. Then we find that evaluation at p gives,

$$\left(d_n^B \Phi_n(F)\right)_p = d_{p+n}^{\mathsf{hom}_{\mathsf{Ch}_R}(M,N)} \circ (\Phi_n(F))_p + (-1)^{n-1}(\Phi_n(F))_{p-1} \circ d_p^T, \quad \text{by (4.1.1)}.$$

Evaluation at t then gives

$$\left(d_n^B \Phi_n(F)\right)_p (t) = d_{p+n}^{\mathsf{hom}_{\mathsf{Ch}_R}(M,N)} \circ (\Phi_n(F))_p(t) + (-1)^{n-1}(\Phi_n(F))_{p-1} \circ d_p^T(t)$$

Evaluating at q and then $m \in M_q$ then gives,

$$\begin{aligned}
\left(d_n^B \Phi_n(F)\right)_p (t)_q(m) &= \left(d_{p+n}^{\mathsf{hom}_{\mathsf{Ch}_R}(M,N)} \circ (\Phi_n(F))_p(t)\right)_q (m) \\
&+ (-1)^{n-1} \left((\Phi_n(F))_{p-1} \circ d_p^T(t)\right)_q (m) \\
&= d_{p+q+n}^N \circ (\Phi_n(F))_p(t)_q(m) + (-1)^{p+n-1}(\Phi_n(F))_p(t)_{q-1} \circ d_q^M(m) \\
&+ (-1)^{n-1} \left((\Phi_n(F))_{p-1} \circ d_p^T(t)\right)_q (m) \\
&= d_{p+q+n}^N \circ F_{p+q}(t \otimes m) + (-1)^{p+n-1} F_{p+q-1}\left(t \otimes d_q^M(m)\right) \\
&+ (-1)^{n-1} F_{p+q-1}\left(d_p^T(t) \otimes m\right). \qquad (4.3.2)
\end{aligned}$$

We also have that

$$\begin{aligned}
\left(\Phi_{n-1} d_n^A(F)\right)_p (t)_q(m) &= \left(d_n^A(F)\right)_{p+q} (t \otimes m) \\
&= d_{p+q+n}^N \circ F_{p+q}(t \otimes m) + (-1)^{n-1} F_{p+q-1} \circ d_{p+q}^{T \otimes_{\Bbbk} M}(t \otimes m) \\
&= d_{p+q+n}^N \circ F_{p+q}(t \otimes m) \\
&+ (-1)^{n-1}\left(F_{p+q-1}\left(d_p^T(t) \otimes m + (-1)^p t \otimes d_q^M(m)\right)\right) \\
&= d_{p+q+n}^N \circ F_{p+q}(t \otimes m) + (-1)^{n-1} F_{p+q-1}\left(d_p^T(t) \otimes m\right) \\
&+ (-1)^{p+n-1} F_{p+q-1}\left(t \otimes d_q^M(m)\right).
\end{aligned}$$

Comparison with (4.3.2) gives that they agree for all p, for all $t \in T_p$, for all q and for all $m \in M_q$. Hence they must be the same map, i.e. Φ is a chain map.

To show that $\Psi = (\Psi_n)_{n \in \mathbb{Z}}$ is a chain map, we first show that Ψ_n and Φ_n are mutual inverses, for all $n \in \mathbb{Z}$.

Let $F \in A_n$ be arbitrary, let $p, q \in \mathbb{Z}$, $t \in T_p$ and $m \in M_q$ be arbitrary. Then

$$\begin{aligned}
(\Psi_n(\Phi_n(F)))_{p+q}(t \otimes m) &= (\Phi_n(F))_p(t)_q(m) \\
&= F_{p+q}(t \otimes m).
\end{aligned}$$

Since $F \in A_n$ was arbitrary, it follows that $\Psi_n \circ \Phi_n = \mathsf{id}_{A_n}$.

On the other hand, let $\varphi \in B_n$ be arbitrary and let $p, q \in \mathbb{Z}$ be arbitrary with $t \in T_p$ and $m \in M_q$, then

$$\begin{aligned}
(\Phi_n(\Psi_n(\varphi)))_p (t)_q(m) &= (\Psi_n(\varphi))_{p+q}(t \otimes m) \\
&= \varphi_p(t)_q(m).
\end{aligned}$$

We conclude that $\Phi_n \circ \Psi_n = \mathsf{id}_{B_n}$. To see that Ψ is a chain map, observe that we already showed that Φ is a chain map. Then

$$\begin{aligned}
\Psi_{n-1} \circ d_n^B &= \Psi_{n-1} \circ d_B^n \circ \mathsf{id}_{B_n} \\
&= \Psi_{n-1} \circ d_B^n \circ \Phi_n \circ \Psi_n \\
&= \Psi_{n-1} \circ \Phi_{n-1} \circ d_n^A \circ \Psi_n, \qquad \text{since } \Phi \text{ is a chain map} \\
&= \mathsf{id} \circ d_n^A \circ \Psi_n \\
&= d_n^A \circ \Psi_n.
\end{aligned}$$

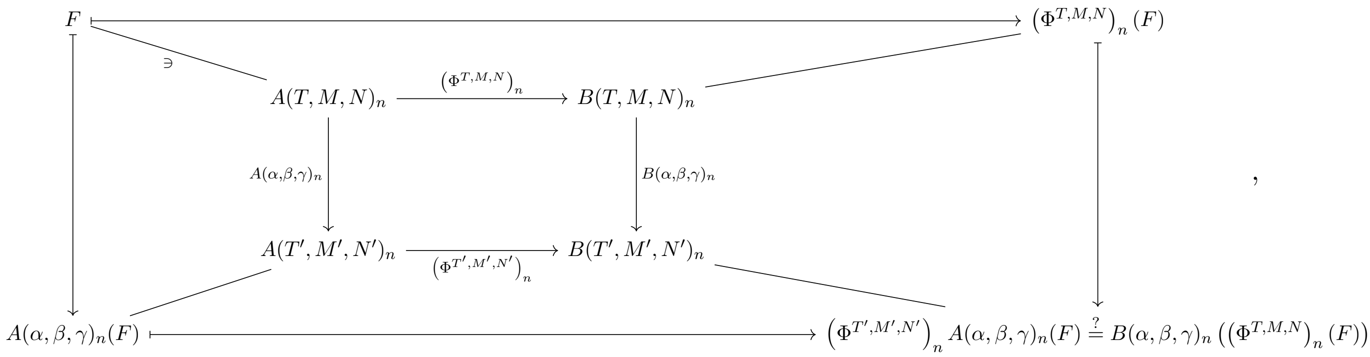
Hence Ψ is indeed a chain map. It remains to show *naturality* of Φ. The constructions above was for fixed T, M, N, so we should spell out that Φ is really $\Phi^{T,M,N} : \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T \otimes_{\Bbbk} M, N) \to \underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(T, \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, N))$, dependent on $T \in \mathsf{Ch}_{\Bbbk}$ and $M, N \in \mathsf{Ch}_R$, defined in degree n as above. We may instead write this as

$$\Phi^{T,M,N} : A(T, M, N) \to B(T, M, N),$$

borrowing notation from (4.3.1). Naturality then amounts to the assertion that for every triple of chain maps

$$\begin{cases} \alpha : T' \to T \\ \beta : M' \to M \\ \gamma : N \to N' \end{cases}, \tag{4.3.3}$$

the following diagram commutes in each degree n,



,

where for $t' \in T'_p$ and $m' \in M'_q$, $B(\alpha, \beta, \gamma)_n(\varphi)$ for $\varphi \in B(T, M, N)_n$ is defined as (suppressing degrees with respect to *enriched composition* χ)

$$\begin{aligned}
(B(\alpha, \beta, \gamma)_n(\varphi))_p (t')_q(m') &:= \chi(\mathscr{E} \otimes \chi(\varphi \otimes \alpha))_p(t')_q(m') \\
&= \gamma_{p+q+n} \circ \varphi_p(\alpha_p(t'))_q(\beta_q(m')),
\end{aligned}$$

where $\mathscr{E}(h) := \chi(\gamma \otimes \chi(h \otimes \beta))$ and

$$\begin{aligned}
A(\alpha, \beta, \gamma)_n(F) &= \chi(\gamma \otimes \chi(F \otimes (\alpha \otimes \beta)) \\
&= \gamma \circ (F \circ (\alpha \otimes \beta)) \\
\Rightarrow ((A(\alpha, \beta, \gamma))_n(F))_s &:= \gamma_{n+s} \circ F_s \circ (\alpha \otimes \beta)_s.
\end{aligned}$$

One should here try to convince oneself that $A(\alpha, \beta, \gamma)$ and $B(\alpha, \beta, \gamma)$ are chain maps, since they are defined in terms of χ which we recall is a chain map, together with appropriate degree considerations (cf. with how we treat $C(-, -, -)$ in the case $B \approx C$).

This means that on pure tensors $t' \otimes m'$ with $t' \in T'_p$ and $m' \in M'_q$ where $p + q = s$, we have (cf. (2.5.3)),

$$(A(\alpha, \beta, \gamma)_n(F))_s (t' \otimes m') = \gamma_{p+q+n} \circ F_{p+q} \circ (\alpha_p(t') \otimes \beta_q(m')).$$

We then compute that for arbitrary $F \in A(T, M, N)_n$ and $t' \in T'_p, m' \in M'_q$ we have that

$$\begin{aligned}\left(B(\alpha, \beta, \gamma) \circ \left(\Phi^{T,M,N}(F)\right)\right)_p (t')_q(m') &= \gamma_{p+q+n} \circ \left(\Phi^{T,M,N}(F)\right)_p (\alpha_p(t'))_q(\beta_q(m')) \\ &= \gamma_{p+q+n} \circ F_{p+q} \circ \left(\alpha_p(t') \otimes \beta_q(m')\right), \qquad (4.3.4)\end{aligned}$$

while

$$\begin{aligned}\left(\left(\Phi^{T',M',N'}\right)_n \circ A(\alpha, \beta, \gamma)_n(F)\right)_p (t')_q(m') &= (A(\alpha, \beta, \gamma)_n(F))_{p+q} (t' \otimes m') \\ &= \gamma_{p+q+n} \circ F_{p+q} \circ (\alpha_p(t') \otimes \beta_q(m')) \qquad (4.3.5)\end{aligned}$$

Comparing 4.3.4 and 4.3.5 we see that they agree, and the conclusion follows.

$\underline{B \approx C}$: Define $\Theta_n : B_n \to C_n$ by $(\Theta_n(\varphi))_q(m)_p(t) := (-1)^{pq}(\varphi_p(t))_q(m)$ for $\varphi \in B_n$, and let $\Omega_n : C_n \to B_n$ be defined by

$$(\Omega_n(\psi))_p(t)_q(m) := (-1)^{pq}\psi_q(m)_p(t),$$

for $\psi \in C_n$.

We have

$$\begin{aligned}(\Omega_n(\Theta_n(\varphi)))_p (t)_q(m) &= (-1)^{pq}(\Theta_n(\varphi))_q(m)_p(t) \\ &= (-1)^{2pq}\varphi_p(t)_q(m) \\ &= \varphi_p(t)_q(m)\end{aligned}$$

and

$$\begin{aligned}(\Theta_n(\Omega_n(\psi)))_q(m)_p(t) &= (-1)^{pq}(\Omega_n(\psi))_p(t)_q(m) \\ &= (-1)^{2pq}\psi_q(m)_p(t) \\ &= \psi_q(m)_p(t).\end{aligned}$$

Hence $\Theta := (\Theta_n)_{n\in\mathbb{Z}}$ and $\Omega := (\Omega_n)_{n\in\mathbb{Z}}$ are degree-wise inverses. It is then enough (cf. the $A \approx B$ case) to show that Θ is a chain map.

We have that,

$$\left(d_n^C \Theta_n(\varphi)\right)_q = d_{q+n}^{\mathsf{hom}(T,N)} \circ \Theta_n(\varphi)_q + (-1)^{n-1}\Theta_n(\varphi)_{q-1} \circ d_q^M.$$

Evaluation at $m \in M_q$ and $t \in T_p$ gives

$$\begin{aligned}\left(d_n^C \Theta_n(\varphi)\right)_q (m)_p(t) &= d_{q+n}^{\mathsf{hom}(T,N)} \circ \Theta_n(\varphi)_q(m)_p(t) + (-1)^{n-1}\Theta_n(\varphi)_{q-1} \left(d_q^M(m)\right)_p (t) \\ &= d_{p+q+n}^N(\Theta_n(\varphi))_q(m)_p(t) + (-1)^{q+n-1}\Theta_n(\varphi)_q(m)_{p-1} \left(d_p^T(t)\right) \\ &+ (-1)^{n-1}\Theta_n(\varphi)_{q-1} \left(d_q^M(m)\right)_p (t) \\ &= (-1)^{pq} d_{p+q+n}^N \varphi_p(t)_q(m) + (-1)^{q+n-1+(p-1)q}\varphi_{p-1} \left(d_p^T(t)\right)_q (m) \\ &+ (-1)^{n-1+p(q-1)}\varphi_p(t)_{q-1} \left(d_q^M(m)\right) \\ &= (-1)^{pq} d_{p+q+n}^N \varphi_p(t)_q(m) + (-1)^{n-1+pq}\varphi_{p-1} \left(d_p^T(t)\right)_q (m) \\ &+ (-1)^{n-1+pq+p}\varphi_p(t)_{q-1} \left(d_q^M(m)\right).\end{aligned}$$

Then we compute that

$$
\begin{aligned}
\left(\Theta_{n-1}\left(d_n^B(\varphi)\right)\right)_q(m)_p(t) &= (-1)^{pq}\left(\left(d_n^B(\varphi)\right)_p(t)_q(m)\right. \\
&= (-1)^{pq}\left(d_{p+n}^{\mathsf{hom}(M,N)}\varphi_p(t) + (-1)^{n-1}\varphi_{p-1}\left(d_p^T(t)\right)\right)_q(m) \\
&= (-1)^{pq} d_{p+n+q}^N \varphi_p(t)_q(m) + (-1)^{p+n-1+pq}\varphi_p(t)_{q-1}\left(d_q^M(m)\right) \\
&\quad + (-1)^{pq+n-1}\varphi_{p-1}\left(d_p^T(t)\right)_q(m).
\end{aligned}
$$

Comparison gives that the two computations above agree.

Naturality of Θ amounts to the assertion that the following diagram commutes,

$$
\begin{array}{ccc}
B(T,M,N) & \xrightarrow{\Theta^{T,M,N}} & C(T,M,N) \\
\Big\downarrow{\scriptstyle B(\alpha,\beta,\gamma)} & & \Big\downarrow{\scriptstyle C(\alpha,\beta,\gamma)} \\
B(T',M',N') & \xrightarrow[\Theta^{T',M',N'}]{} & C(T',M',N')
\end{array},
$$

with α, β, γ as in 4.3.3. Then $C(\alpha,\beta,\gamma)_n$ is defined on $\psi \in C(T,M,N)_n$ as first precomposing with β then applying ψ, then precomposing with α and post-composing with γ, and $B(\alpha,\beta,\gamma)_n$ is defined as in the case $A \approx B$. To be a bit more precise of how $C(\alpha,\beta,\gamma)_n$ is defined: It is really the composition (in the *enriched sense*, i.e. via the maps χ), as (suppressing degrees)

$$\psi \overset{E_\beta}{\longmapsto} \chi(\psi \otimes \beta) \overset{L_{\mathscr{D}}}{\longmapsto} \chi(\mathscr{D} \otimes \chi(\psi \otimes \beta))$$

where $\mathscr{D} := \chi(\gamma \otimes \chi((\,\cdot\,) \otimes \alpha))$. One may check that $\mathscr{D}$ is a chain map, i.e. it really lives in degree zero of the appropriate dg hom-complex. Succintly we may write

$$C(\alpha,\beta,\gamma) := \chi(\mathscr{D} \otimes \chi((\,\cdot\,) \otimes \beta)).$$

Now one may show that E_β is a chain map since β is a chain map hence a degree zero cycle together with $d\chi = \chi d$. We claim that one may check that $d\mathscr{D} = 0$ (the differential here being the differential in the appropriate dg hom-complex), so it follows that $L_{\mathscr{D}}$ is a chain map, hence $C(\alpha,\beta,\gamma)$ is a chain map as a composition of chain maps.

We then compute that, with $\varphi \in B(T,M,N)_n, t' \in T_p$ and $m' \in M'_q$,

$$
\begin{aligned}
\left(\Theta_n^{T',M',N'} B(\alpha,\beta,\gamma)_n(\varphi)\right)_q(m')_p(t') &= (-1)^{pq}(B(\alpha,\beta,\gamma)_n(\varphi))_p(t')_q(m') \\
&= (-1)^{pq}\gamma_{p+q+n}(\varphi_p(\alpha_p(t')))_q(\beta_q(m')).
\end{aligned}
$$

On the otherhand, we have that

$$
\begin{aligned}
\left(C(\alpha,\beta,\gamma)_n \Theta_n^{T,M,N}(\varphi)\right)_q(m')_p(t') &= \left(\mathscr{D}_{q+n}\left(\Theta_n^{T,M,N}(\varphi)_q(\beta_q(m'))\right)\right)_p(t') \\
&= \gamma_{p+q+n}(\Theta_n(\varphi)_q(\beta_q(m')))_p(\alpha_p(t')) \\
&= (-1)^{pq}\gamma_{p+q+n}(\varphi_p(\alpha_p(t'))_q(\beta_q(m')).
\end{aligned}
$$

Comparison gives that the two computations agree. □

Theorem 4.3.4. *Let $M, N, T \in \mathsf{Ch}_R$ and let $M \xrightarrow{f} N$ be a morphism in Ch_R. Then there natural isomorphisms*

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, \mathsf{hker}(f)) \approx \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, M) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T,f)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(T, N)\right),$$

and

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(\mathsf{hcoker}(f), T,) \approx \mathsf{hker}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(N, T) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,T)} \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M, T)\right).$$

Theorem 4.3.5. *The functor $\mathsf{Ch}_{\Bbbk}^{\mathsf{op}} \times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_{\Bbbk}}(-,-)} \mathsf{Ch}_R$ preserves homotopy co/kernel sequences.*

Theorem 4.3.6. *The functor $\mathsf{Ch}_{\Bbbk} \times \mathsf{Ch}_R \xrightarrow{(\,\cdot\,)\otimes(\,\cdot\,)} \mathsf{Ch}_R$ preserves homotopy co/kernel sequences.*

Chapter 5

Projective and injective resolutions

5.1 Why resolutions

The original motivation for homological algebra was the fact that many naturally arising functors in Mod_R are not *exact*, that is, do *not* preserve exact sequences. We provide some quite straightforward examples below. In the language of *additive* functors $F : \mathcal{C} \to \mathcal{D}$ between *abelian* categories $\mathcal{C}, \mathcal{D}$ one would say that F is **left exact** if it *commutes with kernels*, meaning that whenever we have a morphism $\phi : M \to N$ in $\mathcal{C}$ with kernel (K, k), then $(F(K), F(k))$ is a kernel of $F(\phi) : F(M) \to F(N)$.

With the same notation as in the previous paragraph, we say that F is **left exact** if whenever (C, c) is a cokernel of a morphism $\phi : M \to N$ then $(F(C), F(c))$ is a cokernel of $F(\phi)$. Such a functor F is called **exact** if it is both left and right exact.

This more categorical way of defining exactness generalizes the more intuitive meaning of what we think left and right exact means in the context of *modules*, to arbitrary (additive) functors between abelian categories (cf. [Yek19, Prop 2.5.10]).

Example 5.1.1. Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \overset{\pi}{\twoheadrightarrow} \mathbb{Z}/2\mathbb{Z} \to 0. \tag{5.1.1}$$

Tensoring by $(\,\cdot\,) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ we obtain the sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\cdot 2)\otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi \otimes \mathrm{id}} \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to 0.$$

Up to canonical isomorphism (contraction), this is the same as the sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$ is not injective, this is not an *exact* sequence.

By symmetry, tensoring the exact sequence 5.1.1 by $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} (\,\cdot\,)$ again gives a sequence which is not exact.

Example 5.1.1 shows that, in general, the bi-functor

$$\mathsf{Mod}_{\mathbb{k}} \times \mathsf{Mod}_R \xrightarrow{(\,\cdot\,)\otimes_{\mathbb{k}}(\,\cdot\,)} \mathsf{Mod}_R$$

is not exact in either variable. However, the above functor preserves cokernels in each variable separately.

Example 5.1.2. Again, consider the exact sequence 5.1.1. Upon applying the (contravariant) functor $\mathsf{hom}_{\mathbb{Z}}(-,\mathbb{Z})$ we get the sequence

$$0 \to \mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \xrightarrow{\pi^*} \mathsf{hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{(\cdot 2)^*} \mathsf{hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \to 0.$$

Since $(\cdot 2)^*$ is not surjective (e.g. it misses the identity $\mathrm{id}_{\mathbb{Z}} \in \mathsf{hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$), this is not exact.

Example 5.1.3. Consider again the exact sequence 5.1.1. We apply the (covariant) functor $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2,-)$ to the sequence, giving us, the sequence

$$0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$$

since $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) = 0$ (there can be no non-trivial homomorphisms since $\mathbb{Z}/2$ has *torsion*) while $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2)$ consists of the zero morphism and the identity. Since $0 \to \mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2)$ is not surjective, this is not an exact sequence.

With respect to the bi-functor

$$\mathsf{Mod}_R^{\mathsf{op}} \times \mathsf{Mod}_R \xrightarrow{\mathsf{hom}_{\mathsf{Mod}_R}(-,-)} \mathsf{Mod}_R,$$

examples 5.1.2 and 5.1.3 shows that it is not exact in either variable.

Example 5.1.4. Consider again the sequence 5.1.1, but now considered as a complex E in $\mathsf{Ch}_{\mathbb{Z}}$ with $E_0 = \mathbb{Z}/2$. Let $B = \underset{\sim}{\mathbb{Z}/2\mathbb{Z}}$. Then we see that

$$\begin{aligned}(E \otimes B)_\ell &= \bigoplus_{m+n=\ell} E_m \otimes_{\mathbb{Z}} B_n \\ &= E_\ell \otimes B_0, \qquad \text{since } B_n = 0 \text{ for } n \neq 0.\end{aligned}$$

It follows that $(E \otimes B)_\ell = 0$ for $\ell \neq 0,1,2$ and

$$\begin{aligned}(E \otimes B)_2 &= E_2 \otimes B_0 \\ &= \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \\ &\approx \mathbb{Z}/2,\end{aligned}$$

$$\begin{aligned}(E \otimes B)_1 &= E_1 \otimes_{\mathbb{Z}} B_0 \\ &= \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \\ &\approx \mathbb{Z}/2\end{aligned}$$

and

$$\begin{aligned}(E \otimes B)_0 &= E_0 \otimes_{\mathbb{Z}} B_0 \\ &= \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \\ &\approx \mathbb{Z}/\mathsf{gcd}(2,2)\mathbb{Z} \\ &= \mathbb{Z}/2\mathbb{Z}\end{aligned}$$

Since the differential for B is zero, we find that on pure tensors $(a\otimes b) \in E_i \otimes B_j \subset (E\otimes B)_\ell$, we have

$$\begin{aligned} d_\ell^{E\otimes B}(a\otimes b) &= d_i^E(a)\otimes b + \underbrace{(-1)^i a\otimes d_j^B(b)}_{=0} \\ &= d_i^E(a)\otimes b. \end{aligned}$$

By linearity (and since B_0 is the only non-zero term) it follows that $d_\ell^{E\otimes B} = d_\ell^E \otimes \mathrm{id}_{B_0}$ for all $\ell \in \mathbb{Z}$. We note that $d_2^E = (\cdot 2)$, and so $d_2^{E\otimes B} = d_2^E \otimes \mathrm{id}_{B_0} = (\cdot 2)\otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} : \mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ becomes the zero map since any multiple of two is killed in the tensor. Hence $\mathsf{Z}_2(E\otimes B) = \mathbb{Z}/2\mathbb{Z}$ while $\mathsf{B}_3(E\otimes B) = 0$ since $d_3^E = 0$. Therefore $\mathsf{H}_2(E\otimes B) \approx \mathbb{Z}/2\mathbb{Z}$, hence $E\otimes B$ is *not acyclic.*

By symmetry and concentration in degree zero (so there are no introduction of sign-changes in the differential) it follows that also $B\otimes E$ has the same homology, hence is not acyclic.

The above example shows that the bifunctor

$$\mathsf{Ch}_{\Bbbk}\otimes \mathsf{Ch}_R \xrightarrow{(\,\cdot\,)\otimes(\,\cdot\,)} \mathsf{Ch}_R,$$

in general does not preserve *acyclic* complexes in either variable, separately.

We may consider $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)$ as a bifunctor $\mathsf{Ch}_R^{\mathsf{op}}\times \mathsf{Ch}_R \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)} \mathsf{Ch}_{\Bbbk}$ as follows: How it acts on objects is clear. On morphisms, if $f : M' \to M$ and $g : N \to N'$ then we define

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,g) : \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N) \to \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M',N')$$

explicitly degreewise as, whenever $\phi = (\phi_i)_{i\in\mathbb{Z}} \in \underline{\mathsf{hom}}_{\mathsf{Ch}_R}(M,N)_n$ then this is sent to $(g_{i+n}\circ \phi_i \circ f_i)_{i\in\mathbb{Z}}$. We may formulate this in terms of *enriched composition*, i.e as

$$\begin{aligned} (\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(f,g)(\phi))_i &= \chi(g\otimes \chi(\phi\otimes f))_i \\ &= (g\circ \chi(\phi\otimes f))_i \\ &= g_{i+n}\circ(\phi\circ f)_i, \qquad \text{since } \phi \text{ lives in degree } n \text{ and } f \text{ in degree zero} \\ &= g_{i+n}\circ \phi_i \circ f_i \in \mathsf{hom}_{\mathsf{Mod}_R}(M_i', N_{i+n}'). \end{aligned}$$

To check that it is a chain map, we sketch the intuition (without worrying too much about writing out where everything lives and suppressing any degrees),

$$\begin{aligned} d\chi(g\otimes\chi(\phi\otimes f)) &= \chi(d(g\otimes\chi(\phi\otimes f))) \\ &= \chi\left(\underbrace{\underbrace{dg}_{=0}\otimes\chi(\phi\otimes f)}_{=0} + (-1)^0 g\otimes d\chi(\phi\otimes f)\right) \\ &= \chi(g\otimes\chi d(\phi\otimes f)) \\ &= \chi\left(g\otimes\chi\left(d\phi\otimes f + \underbrace{(-1)^n\phi\otimes\underbrace{df}_{=0}}_{=0}\right)\right) \\ &= \chi(g\otimes\chi(d\phi\otimes f)), \end{aligned}$$

where we have repeatedly used that $d\chi = \chi d$ together with the definition of the differential of a tensor product, as well as the fact that $d(f) = d(g) = 0$ since they are chain maps.

Below, we will provide examples showing that this hom-functor does in general not preserve acyclic complexes in either variable.

Example 5.1.5. We again consider the complexes E and B as in example 5.1.4. We then observe that

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,E)_n &= \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(B_i, E_{i+n}) \\ &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(B_0, E_n), \qquad \text{up to a canonical identification.}\end{aligned}$$

Since $E_n = 0$ for $n \notin \{0,1,2\}$ the only interesting things happen in degree $0,1,2$. From the above computation, we find that

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,E)_2 &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2,\mathbb{Z}) \\ &\cong 0,\end{aligned}$$

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,E)_1 &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2,\mathbb{Z}) \\ &\cong 0,\end{aligned}$$

and

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,E)_0 &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2,\mathbb{Z}/2) \\ &\cong \mathbb{Z}/2.\end{aligned}$$

Therefore the zero differential d_0 is up to isomorphism the map $d_0 : \mathbb{Z}/2 \to 0$ so has kernel $\mathbb{Z}/2$, while d_1 is up to isomorphism a map $0 \to \mathbb{Z}/2$ so is zero, hence the zeroth homology is $\mathsf{H}_0\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,E)\right) \cong \mathbb{Z}/2$, hence not acyclic. Thus, $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}(B,-)$ does not preserve acyclic complexes (since E was acyclic).

To show that $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)$ in general does not preserve acyclic complexes in the first variable, we instead fix $\underset{\sim}{\mathbb{Z}}$ in the second variable, and consider $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)$. We then find that

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_n &= \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}\left(E_i, \left(\underset{\sim}{\mathbb{Z}}\right)_{n+i}\right) \\ &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(E_{-n}, \mathbb{Z}).\end{aligned}$$

The only interesting stuff happens when $n = -2, -1, 0$. We compute that

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_{-2} &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(E_2, \mathbb{Z}) \\ &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z},\end{aligned}$$

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_{-1} &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(E_1, \mathbb{Z}) \\ &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z},\end{aligned}$$

and

$$\begin{aligned}\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_0 &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(E_0, \mathbb{Z}) \\ &= \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2, \mathbb{Z}) \\ &\cong 0.\end{aligned}$$

We then note that, up to isomorphism, $d_{-2} : \mathbb{Z} \to 0$, so $\mathsf{ker}(d_{-2}) \cong \mathbb{Z}$. We need to put a little more effort into understanding

$$d_{-1} : \underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_{-1} \to \underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)_{-2}.$$

By a relatively straightforward computation (one just have to keep track of degrees) one finds that (up to isomorphisms) the differential is multiplication by 2, hence has image $2\mathbb{Z}$. Morover, the same isomorphism takes $\mathsf{ker}(d_{-2})$ to $\mathbb{Z}$ and $\mathsf{im}(d_{-1})$ to $2\mathbb{Z}$, by the map $\psi \mapsto \psi(1)$. Hence by the first isomorphism theorem, Φ is an isomorphism as below,

$$\begin{array}{ccc} \psi & \longmapsto & \psi(1) + 2\mathbb{Z} \\ \Cap & & \Cap \\ \mathsf{ker}(d_{-2}) = \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & \overset{\exists!\Phi}{\nearrow} & \\ \mathsf{H}_{-2}\left(\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)\right) = \mathsf{ker}(d_{-2})/\mathsf{im}(d_{-1}) & & \end{array}.$$

We conclude that $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(E, \underset{\sim}{\mathbb{Z}}\right)$ is not acyclic. Since E is acyclic, $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)$ does in general not preserve acyclic complexes in the first variable.

To show that it is not preserve acyclic complexes in the second variable: Consider $\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(\underset{\sim\sim}{\mathbb{Z}/2}, E\right)$ with E as above. Since E is an extension to a complex in $\mathsf{Ch}_{\mathbb{Z}}$ from a short exact sequence in $\mathsf{Mod}_{\mathbb{Z}}$, it is exact at each degree, i.e. it is an acyclic complex. By similar computation as above, we find that

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}}}\left(\underset{\sim\sim}{\mathbb{Z}/2}, E\right)_n \cong \mathsf{hom}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2, E_n).$$

This is clearly zero whenever $n \notin \{0,1,2\}$. Since $E_0 = \mathbb{Z}/2$ and $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ [1] so that the resulting complex in degree 0 is up to isomorphism $\mathbb{Z}/2$. For $n = 1, 2$ we have that $E_n = \mathbb{Z}$, and since $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = 0$ the resulting complex looks like

$$\cdots \longrightarrow 0 \longrightarrow \underset{\sim\sim}{\mathbb{Z}/2} \overset{d}{\longrightarrow} 0 \longrightarrow \cdots.$$

By definition of the differential we see that $d = d_0^E = 0$, hence it has kernel $\mathbb{Z}/2$ and so its zeroth homology is $\mathbb{Z}/2 \neq 0$. Therefore, this is not an acyclic complex.

We conclude that $\underline{\mathsf{hom}}_{\mathsf{Ch}_R}(-,-)$ does not in general preserve acyclic complexes in the second variable.

[1] To see this: we may use that $\mathsf{hom}_R(R/I, M) \cong \mathsf{Ann}_I(M)$ generally, with M and R-module, I ideal of R and $\mathsf{Ann}_I(M) = \{m \in M \mid mi = 0, \forall i \in I\}$.

5.2 Introducing projective and injective complexes

Definition 5.2.1 ($\mathbf{A}_R$)**.** We denote by $\mathbf{A}_R$ the *full* dg-subcategory of $\mathbf{K}_R$ consisting of all *acyclic* complexes (i.e. the acyclic objects in Ch_R).

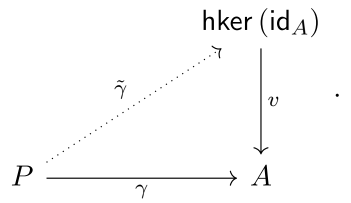
Definition 5.2.2 (Projective complex)**.** We say that an object $P \in \mathbf{K}_R$ is **projective** if it holds that for every acyclic complex $A \in \mathbf{A}_R$, we have that $\mathsf{hom}_{\mathbf{K}_R}(P, A) \in \mathsf{Ch}_{\Bbbk}$ is acyclic.

We observe that

$$\begin{aligned} \mathsf{H}_n\left(\mathsf{hom}_{\mathbf{K}_R}(P, A)\right) &\cong \mathsf{H}_0\left(\Sigma^{-n}\,\mathsf{hom}_{\mathbf{K}_R}(P, A)\right) \\ &\cong \mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}(P, \Sigma^{-n}A)\right), \qquad \text{by 4.1.4.} \end{aligned}$$

Since A is acyclic iff $\Sigma^{-n}A$ is acyclic for all n, one may equivalently define an object $P \in \mathbf{K}_R$ to be projective if $\mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}(P, A\right) = 0$ for every acyclic complex $A \in \mathbf{A}_R$. But this latter definition means precisely that every chain map $P \xrightarrow{f} A$ is nullhomotopic. [2]. So an object P is projective precisely when all chain maps out of P into an acyclic complex are nullhomotopic. We write $\mathbf{P}_R \subseteq \mathbf{K}_R$ for the (full) dg-subcategory of all projective complexes.

Theorem 5.2.3. *We have that $P \in \mathbf{P}_R$ iff for every acyclic complex $A \in \mathbf{K}_R$ and every chain map $P \xrightarrow{\gamma} A$ in Ch_R, there exists a chain map $\tilde{\gamma}$ such that the diagram below commutes,*



Proof. Recall from theorem 3.3.2 that $\Sigma^{-1}\mathsf{hcoker}(f) = \mathsf{Cone}(f)[-1]$ is a model for $\mathsf{hker}(f)$. Let $K := \mathsf{Cone}(\mathsf{id}_A)[-1]$ so that $K_n = A_n \oplus A_{n+1}$, with differential $d_n^K(a,b) = \left(d_n^A(a), -d_{n+1}^A(b) - a\right)$, and with $v = \pi_1$ the projection to the first factor.

$\underline{\Rightarrow}$: If P is projective then there is a nullhomotopy $0 \overset{s}{\simeq} \gamma$. Define $\tilde{\gamma} : P \to K$ by $\tilde{\gamma}_n(x) = (\gamma_n(x), -s_n(x))$. It is clear that $v_n \circ \tilde{\gamma}_n = \gamma_n$. We check that $\tilde{\gamma}$ is a chain map. We find that

$$\begin{aligned} d_n^K \tilde{\gamma}_n(x) &= d_n^K(\gamma_n(x), -s_n(x)) \\ &= \left(d_n^A(\gamma_n(x)), d_{n+1}^A(s_n(x)) - \gamma_n(x)\right), \\ &= \left(d_n^A(\gamma_n(x)), -s_{n-1}\left(d_n^P(x)\right)\right), \qquad \text{since } d^A s + s d^P = \gamma \end{aligned}$$

while

$$\begin{aligned} \tilde{\gamma}_{n-1} d_n^P(x) &= \left(\gamma_{n-1}\left(d_n^P(x)\right), -s_{n-1}\left(d_n^P(x)\right)\right) \\ &= \left(d_n^A(\gamma_n(x)), -s_{n-1}\left(d_n^P(x)\right)\right), \qquad \text{since } \gamma \text{ is a chain map.} \end{aligned}$$

The conclusion follows.

[2] Recall that $\mathsf{Z}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A)) = \mathsf{hom}_{\mathsf{Ch}_R}(P, A)$ and $\mathsf{B}_0(\mathsf{hom}_{\mathbf{K}_R}(P, A))$ are maps which in degree i looks like $d(h)_i = d_{i+1}h_i + h_{i-1}d_i$ for $h \in \underline{\mathsf{hom}}(P, A)_1$

$\Leftarrow$: Take any chain map $P \xrightarrow{\gamma} A$ where A is acyclic. Then there is a lift $\tilde{\gamma}$ so that $v \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}$ is a chain map. We may write $\tilde{\gamma}_n$ on the form $\tilde{\gamma}_n(x) = (\gamma_n(x), t_n(x))$ (cf. remark 3.2.17) since $v_n \circ \tilde{\gamma}_n = \gamma_n$ forces the first summand to equal γ_n. Since this is a chain map, we find that for $x \in P_n$, we have

$$\begin{aligned} d_n^K \tilde{\gamma}_n(x) + \tilde{\gamma}_{n-1} d_n^P(x) &= 0 \\ \Leftrightarrow d_n^K(\gamma_n(x), t_n(x)) + \left(\gamma_{n-1} d_n^P(x), t_{n-1} d_n^P(x)\right) &= 0 \\ \Leftrightarrow \left(d_n^A \gamma_n(x), -d_{n+1}^A t_n(x) - \gamma_n(x)\right) + \left(\gamma_{n-1} d_n^P(x), t_{n-1} d_n^P(x)\right) &= 0. \end{aligned}$$

Comparing the second factors (and rearranging) we see that $\gamma_n(x) = -d_{n+1}^A t_n(x) - t_{n-1} d_n^P(x)$ for all $x \in P_n$. If we let $s_n := -t_n$ for all $n \in \mathbb{Z}$ then $s := (s_n)_{n \in \mathbb{Z}}$ is a nullhomotopy $0 \overset{s}{\simeq} \gamma$. □

Remark 5.2.4. We observe here that v is a levelwise surjective quasi-isomorphism: By theorem 3.3.2 we have $\mathsf{hker}(\mathrm{id}_A) \cong \Sigma^{-1}\mathsf{hcoker}(\mathrm{id}_A)$. Since id_A is a quasi-isomorphism it follows from theorem 3.2.15 that $\mathsf{hcoker}(\mathrm{id}_A)$ is acyclic. Since $\mathsf{H}_n \circ \Sigma^{-1} \cong \mathsf{H}_{n+1}$ it follows that $\Sigma^{-1}\mathsf{hcoker}(\mathrm{id}_A)$ is acyclic, and hence $\mathsf{hker}(\mathrm{id}_A)$ is acyclic. Since A is acyclic, it follows that the induced map $\mathsf{H}_n(v) : \mathsf{H}_n(\mathsf{hker}(\mathrm{id}_A) \to \mathsf{H}_n(A)$ is the zero map in each degree n, so that v is a quasi-isomorphism. Since the concrete instance in theorem 3.3.2 of v is levelwise-surjective. If we use the characterization of the homotopy kernel as a *representing object*, then it follows from yoneda lemma that any other instance of the homotopy kernel must have a corresponding map w that is levelwise surjective and a unique isomorphism α of homotope kernels such that we have a factorization $w \circ \alpha = \pi$ with π as in 3.3.2. Degree-wise, w is then forced to be surjective since π and α are.

Definition 5.2.5 (Injective complex)**.** We say that an object $I \in \mathbf{K}_R$ is **injective** if for every acyclic complex $A \in \mathbf{A}_R$ it holds that $\mathsf{hom}_{\mathbf{K}_R}(A, I) \in \mathsf{Ch}_{\Bbbk}$ is acyclic.

Since

$$\begin{aligned} \mathsf{H}_n\left(\mathsf{hom}_{\mathbf{K}_R}(A, I)\right) &\cong \mathsf{H}_0\left(\Sigma^{-n}\,\mathsf{hom}_{\mathbf{K}_R}(A, I)\right) \\ &\cong \mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}(\Sigma^n A, I)\right), \qquad \text{by 4.1.4.} \end{aligned}$$

An equivalent condition for I being projective is that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(A, I)) = 0$ for all acyclic complexes $A \in \mathbf{A}_R$. This in turn means precisely that I is injective if every chain map $A \to I$ from an acyclic complex A, admits a nullhomotopy. We write $\mathbf{I}_R \subseteq \mathbf{K}_R$ for the full dg-subcategory of injective complexes.

Theorem 5.2.6. *We have that $I \in \mathbf{I}_R$ iff the diagram below can be completed as indicated to a commuting diagram,*

$$\begin{array}{ccc} A & \xrightarrow{\beta} & I \\ {\scriptstyle u}\big\downarrow & \overset{\tilde{\beta}}{\nearrow} & \\ \mathsf{hcoker}(\mathrm{id}_A) & & \end{array} .$$

Remark 5.2.7. The proof of this theorem is somewhat similar (in spirit) to the proof of (the in some sense dual) theorem 5.2.3 so we won't write out the details.

Proof. Take the concrete cone model $\mathsf{Cone}(\mathsf{id}_A) =: K$ as our homotopy cokernel, so that $K_n = A_{n-1} \oplus A_n$ and with differential $d_n^K(a,b) = \left(d_{n-1}^A(a), d_n^A(b) + a\right)$ and with universal map $b \overset{u_n}{\longmapsto} (0, -b)$.

$\underline{\Rightarrow}$: Let s be a witnessing nullhomotopy $0 \overset{s}{\simeq} \beta$ and define $\tilde{\beta} : K \to I$ by $\tilde{\beta}_n : K_n \to I_n$ by $(a,b) \mapsto -s_{n-1}(a) - \beta_n(b)$. Then check (using that s is a nullhomotopy) that this is in fact a chain map.

$\underline{\Leftarrow}$: Let $s = (s_n)$ with $s_n : A_n \to I_{n+1}$ be defined by $s_n(a) = -\tilde{\beta}_{n+1}(a, 0)$. Then use that $\tilde{\beta}$ is a chain map to show that s is a witnessing nullhomotopy of β. □

We are mostly interested in quasi-isomorphisms (perhaps one of the main reasons, if not *the main reason*, is that we want to create categories by *localizing* at quasi-isomorphisms). To that end, we proceed to the next theorem.

Theorem 5.2.8. *All quasi-isomorphisms between projective (respectively injective) complexes are homotopy equivalences.*

Proof. Projective case: Let $f : P \to Q$ be a morphism in $\mathbf{P}_R$ so that P, Q are projective. By theorem 3.2.16 it is enough to show that $\mathsf{Cone}(f) =: K$ is contractible. By theorem 3.2.15, K is acyclic. If we can show that K is projective, then $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(K, K)) = 0$, so by theorem 4.2.3 it would follow that K is contractible and hence that f is a homotopy equivalence.

We observe that by the fact that $\mathsf{H}_n \circ \Sigma^k = \mathsf{H}_{n-k}$ and by 4.1.4 we have that ΣP is projective.

Let $u : Q \to K$ be the universal map, and assume we have a map $K \xrightarrow{\gamma} A$ with $A \in \mathbf{A}_R$. Then $\gamma \circ u : Q \to A$ must be nullhomotopic (since Q is projective and A is acyclic). Let $0 \overset{r}{\simeq} \gamma \circ u$ be a witnessing nullhomotopy, i.e. so that $\gamma \circ u = d^A r + r d^Q$. Define $R : K \to A[-1]$ as $R_n(a,b) = -r_n(b)$ so that $Ru = r$. Then if we let $\tilde{\gamma}' := \gamma - (d^A R + R d^K)$ we see that $\tilde{\gamma}' u = 0$. It follows (by construction) that

$$\tilde{\gamma}_n'(a,b) = \tilde{\gamma}_n'(a,0).$$

We may then define $\zeta : \Sigma P \to A$ by $\zeta_n : P_{n-1} \to A_n$ as $\zeta_n(a) := \tilde{\gamma}_n'(a,0)$. Observe that $p : K \to \Sigma P$ defined in degree n by $p_n(a,b) = a$ is a chain map since shifting the differential of P by Σ makes the signs agree. We check that ζ is a chain map: $\tilde{\gamma}'$ is the difference of two chain maps γ and $d^A R + R d^K$, so it is a chain map. We then see that

$$\begin{aligned}
d_n^A \zeta_n(a) &= d_n^A \tilde{\gamma}_n'(a,0) \\
&= \tilde{\gamma}_{n-1}' d_n^K(a,0) \\
&= \tilde{\gamma}_{n-1}' \left(-d_{n-1}^P(a), f_{n-1}(a)\right) \\
&= \tilde{\gamma}_{n-1}' \left(-d_{n-1}^P(a), 0\right), \qquad \text{since } \tilde{\gamma}' \text{ kills the second factor} \\
&= \zeta_{n-1}\left(-d_{n-1}^P(a)\right) \\
&= \zeta_{n-1}\left(d_n^{\Sigma P}(a)\right), \qquad \text{by definition of } d^{\Sigma P}.
\end{aligned}$$

Hence ζ is indeed a chain map. Since ΣP is projective and A acyclic, ζ nullhomotopic. Since $\tilde{\gamma}' = \zeta \circ p$ it follows that $\tilde{\gamma}'$ is nullhomotopic (use that p is a chain map). Therefore,

$$\gamma = \tilde{\gamma}' + (d^A R + R d^K),$$

is nullhomotopic (since $d^A R + R d^K$ is nullhomotopic, for which R is the witness, and since the addition of two nullhomotopic maps is nullhomotopic).

Injective case: Let $f : I \to J$ be quasi-isomorphism between injective complexes $I, J \in \mathbf{I}_R$. Letting $K = \mathsf{Cone}(f)$, it is acyclic. It is again enough to show that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(K, K)) = 0$ (i.e. that K is injective) since then K is contractible from which it would follow that f is a homotopy equivalence.

The proof of this is analogous to the projective case so we just write out a sketch:

(a) Let $\gamma : A \to K$ be a chain map from an acyclic complex $A \in \mathbf{A}_R$, and observe that if I is injective then ΣI is injective. Let $p : K \to \Sigma I$ be the chain map $p_n(a, b) = a$. Then $p \circ \gamma : A \to \Sigma I$ is a chain map into an injective complex from an acyclic one so it is nullhomotopic. Let s be a witnessing nullhomotopy.

(b) Use s to define a maps $S_n : A_n \to I_n \oplus J_{n+1} = K_{n+1}$ by $S_n(x) = (s_n(x), 0)$. Then the first component in $(d^K S + S d^A)$ is $p \circ \gamma$, so that $p \circ \underbrace{\left(\gamma - \left(d^K S + S d^A\right)\right)}_{:=\tilde{\gamma}} = 0$, i.e. $p \circ \tilde{\gamma} = 0$. This gives a map $\zeta : A \to J$ which is a chain map since $\tilde{\gamma}$ is a chain map. Then since A is acyclic and J is injective, ζ is nullhomotopic.

(c) With $u : J \to K$ the universal map we find that with $\eta := -\zeta$ we have $\tilde{\gamma} = u \circ \eta$. Since η is nullhomotopic and u is a chain map it follows that $\tilde{\gamma}$ is nullhomotopic.

(d) Proceed as in the previous case to show that γ is nullhomotopic.

□

Theorem 5.2.9. *If $P \in \mathbf{P}_R$ is projective and $P \simeq Q$ is a homotopy equivalence, then Q is projective. Similarly, if $I \in \mathbf{I}_R$ is injective and $I \simeq J$ is a homotopy equivalence, then J is injective.*

Proof. Projective case: Let $\alpha : P \leftrightarrows Q : \beta$ be witnessing chain maps for $P \simeq Q$. Let $A \in \mathbf{A}_R$, and let $\gamma : Q \to A$ be a chain map. Since P is projective, the composite chain map $\gamma \circ \beta : P \to A$ is nullhomotopic. Then $\gamma \circ \beta \circ \alpha$ is also nullhomotopic. Since then γ can be written as a difference of two nullhomotopc maps $(\gamma \circ \beta \circ \alpha) - ((\gamma \circ \beta \circ \alpha) - \gamma)$ it follows that γ is nullhomotopic.

Injective case: Similar enough that we don't write it out. □

Theorem 5.2.10. *If $P \in \mathbf{P}_\Bbbk$ and $Q \in \mathbf{P}_R$ then $P \otimes_\Bbbk Q \in \mathbf{P}_R$.*

Proof. By the (dg-)tensor-hom adjunction 4.3.4 we have that

$$\mathsf{hom}_{\mathbf{K}_R}(P \otimes_\Bbbk Q, A) \approx \mathsf{hom}_{\mathbf{K}_\Bbbk}(P, \mathsf{hom}_{\mathbf{K}_R}(Q, A)).$$

If A is acyclic then since Q is projective we find that $\mathsf{hom}_{\mathbf{K}_R}(Q, A)$ is acyclic. It follows that $\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_\Bbbk}(P, \mathsf{hom}_{\mathbf{K}_R}(Q, A))) = 0$, so also

$$\mathsf{H}_0(\mathsf{hom}_{\mathbf{K}_R}(P \otimes_\Bbbk Q, A)) = 0.$$

Hence $P \otimes_\Bbbk Q$ is projective. □

Theorem 5.2.11. *For projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R$, the functors*

$$\mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P,-)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(Q,-)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

Proof. $\underline{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P,-) \text{ preserves quasi-isomorphisms}}$: Let $M \xrightarrow{f} N$ be a quasi-isomorphism and let $\zeta := \mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P, f)$. By theorem 3.2.15 f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic. But $\mathsf{hker}(f) \cong \Sigma^{-1}\mathsf{hcoker}(f)$ and $\Sigma^{-1}\mathsf{hcoker}(f)$ is acyclic iff $\mathsf{hcoker}(f)$ is acyclic. Hence by hypothesis, $\mathsf{hker}(f)$ is acyclic. By theorem 4.3.4 we find that

$$\mathsf{hker}(\zeta) \cong \mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P, \mathsf{hker}(f)).$$

Since P is projective, the right-hand side is acyclic. Hence $\mathsf{hker}(\zeta)$ is acyclic from which it follows that ζ is a quasi-isomorphism.

$\underline{\mathsf{hom}_{\mathbf{K}_R}(Q,-) \text{ preserves quasi-isomorphisms}}$: Essentially identical reasoning: By theorem 4.3.5 we find that, with $\zeta := \mathsf{hom}_{\mathbf{K}_R}(Q, f)$ for $M \xrightarrow{f} N$ quasi-isomorphism, we have that

$$\mathsf{hom}_{\mathbf{K}_R}(Q, \mathsf{hker}(f)) \cong \mathsf{hker}(\zeta).$$

Since Q is projective and $\mathsf{hker}(f)$ is acyclic, $\mathsf{hker}(\zeta)$ is acyclic, i.e. ζ is a quasi-isomorphism. □

Theorem 5.2.12. *If $I \in \mathbf{I}_{\Bbbk} \cap \mathbf{I}_R$, then the functors*

$$K_{\Bbbk}^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(-,I)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_R^{\mathsf{op}} \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(-,I)} \mathbf{K}_{\Bbbk}$$

preserve quasi-isomorphisms.

Proof. Similar in spirit to the proof of theorem 5.2.11 but uses the second isomorphism in 4.3.4 instead and the corresponding isomorphism in 4.3.5, where we use that f is a quasi-isomorphism iff $\mathsf{hcoker}(f)$ is acyclic. □

Remark 5.2.13. The hypothesis in theorem 5.2.12 may be weakened to only require I in $\mathbf{I}_{\Bbbk}$ for the first functor $\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(-, I)$ and $I \in \mathbf{I}_R$ for the second functor.

Theorem 5.2.14. *We have that $\mathbf{P}_R \cap \mathbf{A}_R = \mathbf{I}_R \cap \mathbf{A}_R$, and that this is precisely the contractible complexes.*

Proof. Since $0 \to A$ and $A \to 0$ is always nullhomotopic, the zero complex $0_\bullet$ is both projective and injective. If $A \in \mathbf{P}_R \cap \mathbf{A}_R$ then $A \to 0$ is a quasi-isomorphism (both are acyclic). By theorem 5.2.8 $A \simeq 0$, so that A is contractible.

Similarly, if $A \in \mathbf{I}_R \cap \mathbf{A}_R$ then $0 \to A$ is a quasi-isomorphism between injective complexes, and so it follows that $A \simeq 0$, i.e. A is contractible.

On the other hand if A is contractible, then $A \simeq 0$. Since H_n is homotopy-invariant it follows that $\mathsf{H}_n(A) = 0$ for all n, i.e. A is acyclic. Since 0 is both projective and object, it follows by theorem 5.2.9 that A is projective and injective. Therefore we have $A \in \mathbf{P}_R \cap \mathbf{A}_R$ and $\mathbf{I}_R \cap \mathbf{A}_R$. □

5.2.1 Short interlude on Derived functors and Resolutions

The interest we have in projective and injective complexes is not so much derived from studying them in isolation, but rather from the fact that one may show that every complex $M \in \mathsf{Ch}_R$ admits a **projective resolution** $P \xrightarrow[p]{\approx} M$ and an **injective resolution** $M \xrightarrow[q]{\approx} I$, meaning that $P \in \mathbf{P}_R$ is projective, $I \in \mathbf{I}_R$ and p and q are quasi-isomorphisms. The fact that they are quasi-isomorphisms means that they are all isomorphic in any localization of Ch_R at the quasi-isomorphisms.

Below we write out the "naive" functors and their "correct" (or "derived") versions.

$$\begin{aligned}(\,\cdot\,) \otimes_{\Bbbk} M &\rightsquigarrow (\,\cdot\,) \otimes_{\Bbbk} P,\\ \mathsf{hom}_{\mathbf{K}_R}(M,-) &\rightsquigarrow \mathsf{hom}_{\mathbf{K}_R}(P,-),\\ \mathsf{hom}_{\mathbf{K}_R}(-,M) &\rightsquigarrow \mathsf{hom}_{\mathbf{K}_R}(-,I).\end{aligned}$$

One may ask, why are these the "correct" ones? One answer may be that they *preserve* all quasi-isomorphisms (under nice conditions; see remark 5.2.19) . We will see later that $(\,\cdot\,) \otimes_{\Bbbk} P$ preserves quasi-isomorphisms (see 5.2.17); for why the other two functors preserve quasi-isomorphisms, recall theorems 5.2.11,5.2.12.

Theorem 5.2.15. *Let $M \in \mathbf{K}_R$ be such that $\mathsf{hom}_{\mathbf{K}_R}(P,M)$ is acyclic for every projective complex $P \in \mathbf{P}_R$. Then M is acyclic.*

Proof. Observe that $\underset{\sim}{R}$ is projective (cf. with Theorems 5.3.4, 5.3.9). Furthermore, we have

$$\begin{aligned}\mathsf{hom}_{\mathbf{K}_R}(\underset{\sim}{R},M)_n &= \prod_{i\in\mathbb{Z}} \mathsf{hom}_{\mathsf{Mod}_R}\left(\left(\underset{\sim}{R}\right)_i, M_{n+i}\right)\\ &= \mathsf{hom}_{\mathsf{Mod}_R}(R,M_n)\\ &\cong M_n, \qquad \text{by } f_0 \overset{\varphi_n}{\longmapsto} f_0(1).\end{aligned}$$

Furthermore, it is quite straightforward to check that $\varphi = (\varphi)_{n\in\mathbb{Z}} : \mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{R},M\right) \to M$ is a chain map. Since φ is an isomorphism and $\mathsf{hom}_{\mathbf{K}_R}$ is acyclic it follows that M is acyclic. □

Theorem 5.2.16. *Assume $M \in \mathbf{K}_R$ is such that $\mathsf{hom}_{\mathbf{K}_R}(M,I)$ is acyclic for every injective complex $I \in \mathbf{I}_R$. Then M is acyclic.*

Proof. By our earlier remarks, we may assume the existence of an *injective resolution* $M \xrightarrow[q]{\approx} I$. Fix I and consider

$$\mathsf{hom}_{\mathbf{K}_R}(I,I) \xrightarrow{\zeta := \mathsf{hom}_{\mathbf{K}_R}(q,I)} \mathsf{hom}_{\mathbf{K}_R}(M,I).$$

By theorem 5.2.12 (together with remark 5.2.13) we find that $\mathsf{hom}_{\mathbf{K}_R}(q,I)$ is a quasi-isomorphism. Since the codomain of ζ is acyclic, $\mathsf{hom}_{\mathbf{K}_R}(I,I)$ is acyclic. By theorem 4.2.3 it follows that I is contractible. Since contractible implies acyclic, I is acyclic. [3] Since $M \xrightarrow{q} I$ is a quasi-isomorphism, M has the same homology as I, i.e. M is acyclic. □

[3] Sketch: Take $x \in \mathsf{B}_n(C)$ for complex C, then we have there is a witnessing contraction s such that $\mathsf{id}_C = ds + sd$ so $x = ds$ (since $sdx = 0$ by the fact that x is a cycle). Hence x is a boundary.

Theorem 5.2.17. *For projective complexes $P \in \mathbf{P}_{\Bbbk}$ and $Q \in \mathbf{P}_R \cap \mathbf{P}_{\Bbbk}$, the functors*

$$\mathbf{K}_R \xrightarrow{P \otimes_{\Bbbk} (\,\cdot\,)} \mathbf{K}_R \qquad \text{and} \qquad \mathbf{K}_{\Bbbk} \xrightarrow{(\,\cdot\,) \otimes_{\Bbbk} Q} \mathbf{K}_R$$

preserve quasi-isomorphisms.

Proof. $\underline{P \otimes_{\Bbbk} (\,\cdot\,)}$ preserves quasi-isomorphisms: Let $M \xrightarrow{f} N$ be a quasi-isomorphism in Ch_R, we want to show that $P \otimes_{\Bbbk} M \xrightarrow{P \otimes_{\Bbbk} f} P \otimes_{\Bbbk} N$ is a quasi-isomorphism. Since f is a quasi-isomorphism, $\mathsf{hcoker}(f)$ is acyclic (by theorem 3.2.15). By theorem 4.3.6 we have

$$\mathsf{hcoker}(P \otimes f) \cong P \otimes \mathsf{hcoker}(f).$$

Lemma 5.2.18. *If $A \in \mathbf{A}_R$ is acyclic then upon tensoring with a projective complex $P \in \mathbf{P}_{\Bbbk}$, we get an acyclic complex $P \otimes_{\Bbbk} A$.*

Proof. By theorem 5.2.16 it is enough to show that for every injective complex $I \in \mathbf{I}_R$ we have that $\mathsf{hom}_{\mathbf{K}_R}(P \otimes_{\Bbbk} A, I)$ is acyclic.

By the dg-tensor hom adjunction 4.3.1 we have

$$\mathsf{hom}_{\mathbf{K}_R}(P \otimes_{\Bbbk} A, I) \approx \mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P, \mathsf{hom}_{\mathbf{K}_R}(A, I)).$$

Since A is acyclic and I is injective, we have that $\mathsf{hom}_{\mathbf{K}_R}(A, I)$ is acyclic. Since $P \in \mathbf{K}_{\Bbbk}$ is projective it follows that $\mathsf{hom}_{\mathbf{K}_{\Bbbk}}(P, \mathsf{hom}_{\mathbf{K}_R}(A, I))$ is acyclic so $\mathsf{hom}_{\mathbf{K}_R}(P \otimes_{\Bbbk} A, I)$ is acyclic. The conclusion follows. ☐

By the lemma, $P \otimes \mathsf{hcoker}(f)$ is acyclic, hence $\mathsf{hcoker}(P \otimes f)$ is acyclic, so by another application of 3.2.15 it follows that $P \otimes_{\Bbbk} f$ is a quasi-isomorphism.

$\underline{(\,\cdot\,) \otimes_{\Bbbk} Q}$ preserves quasi-isomorphisms: Let $L \xrightarrow{g} L'$ be a quasi-isomorphism in $\mathsf{Ch}_{\Bbbk}$. Then by theorem 4.3.6 we have

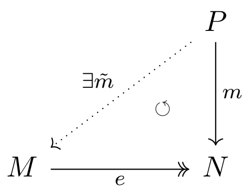
$$\mathsf{hcoker}(g \otimes_{\Bbbk} Q) \cong \mathsf{hcoker}(g) \otimes_{\Bbbk} Q,$$

where we note that $\mathsf{hcoker}(g)$ is acyclic. By a twist of lemma 5.2.18 utilizing the isomorphism realized by the *symmetrizer* σ coming from the symmetric monoidal structure on Ch_R (recall theorem 2.5.4), i.e. so that $A \otimes_{\Bbbk} Q \underset{\approx}{\xrightarrow{\sigma_{A,Q}}} Q \otimes_{\Bbbk} A$ is acyclic if A is acyclic, it follows (proceed as in the first case above) that $\mathsf{hcoker}(g) \otimes_{\Bbbk} Q$ is acyclic, so that $\mathsf{hcoker}(g \otimes_{\Bbbk} Q)$ is acyclic which implies that $g \otimes_{\Bbbk} Q$ is a quasi-isomorphism. ☐

Remark 5.2.19. Observe that "correcting" $(\,\cdot\,) \otimes_{\Bbbk} M$ with $(\,\cdot\,) \otimes_{\Bbbk} P$ with P belonging to $\mathbf{P}_R$ and $P \xrightarrow{\approx} M$ a projective resolution *in* Ch_R, it is *not the case* that we can conclude that $(\,\cdot\,) \otimes_{\Bbbk} P$ preserves quasi-isomorphisms by theorem 5.2.17, unless we also know that P is in $\mathbf{P}_{\Bbbk}$.

5.3 Bounded below (or "bounded to the right") projective resolutions

Definition 5.3.1 (Projective object)**.** Let $\mathcal{A}$ be an abelian category, and let $P \in \mathsf{Ob}(\mathcal{A})$. Then we call P a **projective object** if for every morphism $m : P \to N$ and every epimorphism $e : M \twoheadrightarrow N$ there exists a morphism $\tilde{m} : P \to \tilde{m}$ such that $e \circ \tilde{m} = m$, or in diagrammatic form as,



(5.3.1)

Example 5.3.2. Consider $\mathcal{A} = \mathsf{Mod}_R$. Let P be a projective object in Mod_R. Assume we have a morphism $m : P \to N$ and $e : M \to N$

Lemma 5.3.3. *$e : M \to N$ is an epimorphism in Mod_R iff it is surjective.*

Proof. $\underline{\Rightarrow}$: Consider the morphism $\pi : N \twoheadrightarrow N/\mathsf{im}(e)$, and let $g : 0 \to N/\mathsf{im}(e)$ be the corresponding zero map. Then

$$\pi(e(x)) = g(e(x)), \qquad \forall x \in M.$$

Since e is epi, it has the right-cancellation property, so it follows that $\pi = g$, which means that $\mathsf{im}(e) = N$, that is, e is surjective. ☐

$\underline{\Leftarrow}$: This direction is clear by working with elements.

It follows by the lemma that a projective object (which we call a **projective module**) is precisely a module such that any diagram on the form as in 5.3.1 with e surjective, can be "completed" with $\tilde{m}$ to a commutative diagram.

We recount some well-known (see e.g. [DF04, Chapter 10.5]) theorems about projective *modules.*

Theorem 5.3.4. *A module $P \in \mathsf{Mod}_R$ is projective iff there is some module $Q \in \mathsf{Mod}_R$ such that*

$$P \oplus Q \cong \bigoplus_{i \in I} R$$

for some index set I.

Remark 5.3.5. In particular, *free* modules are projective.

Theorem 5.3.6. *Every projective $\mathbb{Z}$-module is free.*

Proof. We will just provide a *sketch*: Observe that if P is a projective $\mathbb{Z}$-module then P is isomorphic to a submodule P' of a free $\mathbb{Z}$-module, which is the same as a free abelian group, under which P' becomes a subgroup of a free abelian group. But every subgroup of a free abelian group is free, and so the conclusion follows. ☐

Theorem 5.3.7. *Every projective $\mathbb{Z}/n$-module is free iff $n = 1$ or $n = p^k$ for some prime p and $k \geqslant 1$.*

Proof. $\Rightarrow$: We prove the contrapositive: assume $n = ab$ has at least two distinct prime factors $a, b > 1$ with $\mathsf{gcd}(a, b) = 1$. By the *chinese remainder theorem* we have a ring-isomorphism

$$\mathbb{Z}/n \overset{\varphi}{\cong} \mathbb{Z}/a \oplus \mathbb{Z}/b =: F.$$

By *restriction of scalars* we may view the right-hand side as a $\mathbb{Z}/n$-module. Then clearly $P := \mathbb{Z}/a \oplus 0 \cong \mathbb{Z}/a$ is a direct summand of the free $\mathbb{Z}/n$-module F, hence is a projective $\mathbb{Z}/n$-module. Suppose P was free, i.e. $\bigoplus_{s\in S} \mathbb{Z}/n \overset{\psi}{\cong} P$ with $S \neq \varnothing$ (since $|P| > 1$ as a set). Consider the element $(0, 1) \in F$ and let $\varepsilon := \varphi^{-1}((0, 1))$. Then (by using the $\mathbb{Z}/n$-module structure coming from restriction of scalars) we have, for any $(u, 0) \in P$,

$$\begin{aligned} \varepsilon \cdot (u, 0) &= \varphi(\varphi^{-1}(0, 1)) \cdot (u, 0) \\ &= (0, 1) \cdot (u, 0) \\ &= (0, 0), \\ \Rightarrow \varepsilon P &= 0. \end{aligned}$$

Then note that by $\mathbb{Z}/n$-linearity of ψ, we have that

$$0 = \psi(\varepsilon x)$$

for all $x \in \bigoplus_{s\in S} \mathbb{Z}/n$. By injectivity of ψ, it follows that $\varepsilon x = 0$ for all x.

Take any coordinate vector e_s in $\bigoplus_{s\in S} \mathbb{Z}/n$. Then $(\varepsilon \cdot e_s)_s = \varepsilon \neq 0$, hence $\varepsilon \cdot e_s \neq 0$, contradiction!

$\Leftarrow$: If $n = 1$, then $\mathbb{Z}/(1) = 0$ is the zero ring, with $0 = 1$. It follows from the module-axioms that any 0-module M is such that $M = 1 \cdot M = 0 \cdot M = 0$, so there is (up to unique isomorphism) only one module, and $M \cong 0^{\bigoplus \varnothing}$ is free.

If not then $n = p^k$ for some prime p and $k \geqslant 1$. Then $\mathbb{Z}/p^k\mathbb{Z}$ is a *local* ring with unique maximal ideal (p).

Then

$$\left(\mathbb{Z}/p^k\mathbb{Z}\right)/(p)$$

is a field with p, elements i.e. $\left(\mathbb{Z}/p^k\mathbb{Z}\right)/(p) \cong \mathbb{Z}/p$.

Let $R = \mathbb{Z}/p^k\mathbb{Z}$, and let P be a projective R-module. Then $P/(p)P$ is naturally a $R/(p) \cong \mathbb{F}_p$-module, i.e. a $\mathbb{Z}/p$-vector space. Let $(\overline{x}_i)_{i\in I}$ be a basis for $P/(p)P$ as an $\mathbb{Z}/p$-vector space (assuming Zorn's lemma) and "lift" them to elements $(x_i)_{i\in I} \subset P$ that projects down to the basis in the quotient module under the canonical projection map $\pi : P \twoheadrightarrow P/(p)P$. Define

$$R^{\oplus I} \xrightarrow{\varphi} P, e_i \mapsto x_i,$$

and extend by R-linearity. Then it is clear that

$$\theta := \pi \circ \varphi : R^{\oplus I} \twoheadrightarrow P/(p)P$$

is a surjective R-linear map. Consider $C := \mathsf{coker}(\varphi) = P/\mathsf{im}(\varphi)$. We claim that $C = (p)C$, where

$$(p)C := \left\{ \sum_i r_i c_i \;\middle|\; r_i \in (p), c_j \in C \right\}.$$

Since C is an R-module, it follows that $(p)C \subset C$, so it is enough to show that $C \subset (p)C$: Since (p) is principal, any element $r_i \in (p)$ is on the form $r_i = pa_i$ for $a_i \in R$, hence every element $\sum_i r_i c_j \in (p)C$ can be written on the form

$$\begin{aligned}\sum_i r_i c_i &= p \underbrace{\left(\sum_i a_i c_i\right)}_{\in C} \\ &= pc \\ &= p[y_1],\end{aligned}$$

where we used that C is an R-module and that $P \xrightarrow{\pi'} C$ is surjective, so that there is some $y_1 \in P$ that projects to $[y_1] = c$. Take an arbitrary element $[y] \in C$, then there is some $y \in P$ that is sent to $[y]$ under π'. Since θ is surjective, there is some $z \in R^{\oplus I}$ such that $\theta(z) = \pi(y)$, i.e. $\phi(\varphi(z)) = \pi(y)$. Therefore, $y - \varphi(z) \in \mathsf{ker}(\pi) = (p)P$. Hence, since $p(P) = \{py \ : \ y \in P\}$ (by similar reasoning as for $(p)C$) there is some $y_1 \in P$ such that

$$\begin{aligned} y - \varphi(z) &= py_1 \\ \Leftrightarrow y &= \varphi(z) + py_1 \\ \Rightarrow \pi'(y) &= [y] \\ &= [py_1] \\ &= p[y_1]. \end{aligned}$$

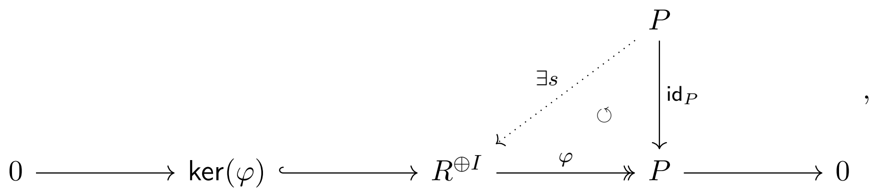
Since $[y_1] \in C$ it follows that $[y] = p[y_1] \in (p)C$. We conclude that $C = (p)C$. Therefore, by iterated application of this identity, we have

$$C = (p)C = (p)^2 C = \ldots = (p)^k C.$$

We note that $(p)^k = (p^k) = (0)$ in R, and so $C = (p)^k C = 0$. By definition of C it follows that $\mathsf{im}(\varphi) = P$, i.e. φ is surjective. Therefore, we have a short exact sequence

$$0 \to \mathsf{ker}(\varphi) \hookrightarrow R^{\oplus I} \xrightarrow{\varphi} P \to 0.$$

Since P is projective and φ is surjective, we get the existence of a map γ furnishing us with a commutative triangle above the sequence, as below,



so that $\varphi \circ s = \mathsf{id}_P$, i.e., s is a **section**. If we can show that $\mathsf{ker}(\varphi) = 0$, then we may conclude that P is free as an R-module.

We claim that $\mathsf{ker}(\pi \circ \varphi) = (p)R^{\oplus I}$. If $(r_i)_{i \in I} \in (p)R^{\oplus I}$ then $r_i \in (p)$ and so

$$\begin{aligned}(\pi \circ \varphi)((r_i)_{i\in I}) &= \pi\left(\sum_i r_i x_i\right) \\ &= \sum_i r_i \cdot \overline{x}_i \\ &= 0, \quad \text{since } r_i \cdot \overline{x}_i = \overline{r_i x_i} \text{ with } r_i x_i \in (p)P \text{ since } r_i \in (p).\end{aligned}$$

On the other hand, if $(r_i)_{i\in I} \in \mathsf{ker}(\pi \circ \varphi)$ then

$$\begin{aligned}(\pi \circ \varphi)((r_i)_{i\in I}) &= \sum_i r_i \cdot \overline{x}_i \\ &= \sum_i (r_i + (p))\overline{x}_i.\end{aligned}$$

We claim that $\mathsf{ker}(\varphi) = (p)\mathsf{ker}(\varphi)$. It is (cf. $C = (p)C$ case) enough to show that $\mathsf{ker}(\varphi) \subset (p)\mathsf{ker}(\varphi)$. To this end, let $x \in \mathsf{ker}(\varphi)$. Since $\varphi(x) = 0$ it follows that $x \in \mathsf{ker}(\pi \circ \varphi) = (p)R^{\oplus I}$, i.e., so that $x = py$ with $y \in R^{\oplus I}$. From the section $s : P \to R^{\oplus I}$ there is a naturally occuring map

$$\rho : R^{\oplus I} \to \mathsf{ker}(\varphi), \qquad z \mapsto z - s(\varphi(z)).$$

In particular, $\rho(y) = y - s(\varphi(y)) \in \mathsf{ker}(\varphi)$. Since $x = py \in \mathsf{ker}(\varphi)$ we have that

$$\begin{aligned}x &= x - s(\varphi(x)) \\ &= py - s(\varphi(py)) \\ &= py - ps(\varphi(y)) \\ &= p(y - s(\varphi(y))) \\ &= p \underbrace{\rho(y)}_{\in \mathsf{ker}(\varphi)},\end{aligned}$$

so that $x \in (p)\mathsf{ker}(\varphi)$. It follows that $\mathsf{ker}(\varphi) \subset (p)\mathsf{ker}(\varphi)$. Therefore, we have that

$$\mathsf{ker}(\varphi) = (p)\mathsf{ker}(\varphi) = (p)^2\mathsf{ker}(\varphi) = \ldots = (p)^k\mathsf{ker}(\varphi) = 0.$$

Hence $R^{\oplus I} \xrightarrow{\varphi} P$ is an isomorphism, i.e. P is a free R-module. □

Theorem 5.3.8. *An R-module $M \in \mathsf{Mod}_R$ is projective iff $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective.*

Proof. $\underline{\Rightarrow}$: Let A be acyclic. Observe that $\mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)_n = \mathsf{hom}_R(M, A_n)$ for any $n \in \mathbb{Z}$ with differential $\left(d_n^A\right)_*$ since e.g. $d_0^M = 0$ (recall the formula 4.1.1; also note that $f_{i-1} = 0$ for $i \neq 1$ whenever $f = (f_i)_{i\in\mathbb{Z}} \in \mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)_n$). Since A is acyclic, we have that

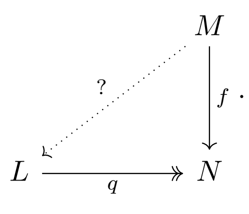
$$\cdots \xrightarrow{d_{n+2}^A} A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \xrightarrow{d_{n-1}^A} \cdots$$

is exact. Since M is projective, $\mathsf{hom}_R(M, -)$ is exact, so that

$$\cdots \xrightarrow{(d_{n+2}^A)_*} \mathsf{hom}_R(M, A_{n+1}) \xrightarrow{(d_{n+1}^A)_*} \mathsf{hom}_R(M, A_n) \xrightarrow{(d_n^A)_*} \mathsf{hom}_R(M, A_{n-1}) \xrightarrow{(d_{n-1}^A)_*} \cdots,$$

is exact. The conclusion follows.

$\underline{\Leftarrow}$: Assume $\underset{\sim}{M} \in \mathsf{Ch}_R$ is projective. Assume we want to complete the following diagram to a commutative triangle as below, with q surjective,



Consider the acyclic complex

$$A := \cdots \to 0 \to \mathsf{ker}(q) \hookrightarrow L \xrightarrow{q} \underset{\sim}{N} \to 0 \to \cdots .$$

Then $f : M \to N$ defines a chain map $\underset{\sim}{f}$ as below

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathsf{ker}(q) & \hookrightarrow & L & \underset{q}{\twoheadrightarrow} & N & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}.$$

Since the complex A is acyclic and $\underset{\sim}{M}$ is projective, it follows that

$$\begin{aligned}
\mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)\right) &= \mathsf{Z}_0\left(\mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)\right) \Big/ \mathsf{B}_0\left(\mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)\right) \\
&= \mathsf{hom}_{\mathsf{Ch}_R}\left(\underset{\sim}{M}, A\right) \Big/ \mathsf{B}_0\left(\mathsf{hom}_{\mathbf{K}_R}\left(\underset{\sim}{M}, A\right)\right) \\
&= 0.
\end{aligned}$$

Since $\underset{\sim}{f} \in \mathsf{hom}_{\mathsf{Ch}_R}\left(\underset{\sim}{M}, A\right)$ it follows that there is some $h \in \mathsf{hom}_{\mathsf{Ch}_R}\left(\underset{\sim}{M}, A\right)_1 = \mathsf{hom}_R(M, A_1) = \mathsf{hom}_R(M, L)$ such that $d_1(h) = \underset{\sim}{f}$. In degree zero, this says that

$$\begin{aligned}
d_1^A \circ h_0 - h_{-1} \circ d_0^M &= d_1^A \circ h_0 \\
&= (\underset{\sim}{f})_0 \\
&= f.
\end{aligned}$$

But $d_1^A = q$, i.e. $q \circ h_0 = f$, with $M \xrightarrow{h_0} L$ an R-module homomorphism, and the conclusion follows. □

Theorem 5.3.9. *Let $P \in \mathbf{K}_R$ be a bounded below complex* [4] *such that $P_n \in \mathsf{Mod}_R$ is projective for all n. Then $P \in \mathbf{P}_R$ is projective.*

Proof. Let $N \in \mathbb{Z}$ such that $P_n = 0$ for $n < N$, let $A \in \mathbf{A}_R$ be acyclic and let $f : P \to A$ be a chain map. Since f is a chain map we have $d_N^A f_N = f_{N-1} d_N^P = 0$ since the codomain of d_N^P is $P_{N-1} = 0$. Hence $\mathsf{im}(f_N) \subset \mathsf{Z}_N(A)$ which is equal to $\mathsf{B}_N(A) = \mathsf{im}\left(d_{N+1}^A\right)$ since A is *acyclic*. Therefore, if we corestrict the differential d_{N+1}^A to $\overline{d}_{N+1}^A$ and f_N to $\overline{f}_N$ with codomain $\mathsf{Z}_N(A)$, we get the existence of a homomorphism $h_N : P_N \to A_{N+1}$ making the following diagram commute

$$\begin{array}{ccc}
 & & P_N \\
 & \swarrow{\scriptstyle \exists h_N} & \downarrow{\scriptstyle \overline{f}_N} \\
A_{N+1} & \underset{\overline{d}_{N+1}^A}{\twoheadrightarrow} & \mathsf{Z}_N(A)
\end{array},$$

i.e. $\overline{d}_{N+1}^A h_N = \overline{f}_N$ so that with the inclusion $\iota : \mathsf{Z}_N(A) \hookrightarrow A_N$ applied to the left of both sides of the equality we get $d_{N+1}^A h_N = f_N$. If we let $h_n : P_n \to A_{n+1}$ be the zero-morphism

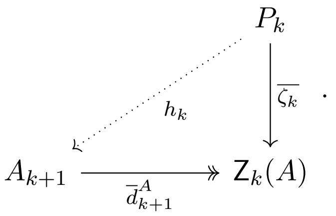
[4] That is, so that there is an $N \in \mathbb{Z}$ so that $P_n = 0$ for all $n < N$.

for $n < N$ then we see that for $n \leqslant N$ we have $d^A_{n+1}h_n + h_{n-1}d^P_n = f_n$. Assume inductively that we have constructed a family of homomorphisms $h := (h_n)_{n<k}$ up to some $k \in \mathbb{Z}$ as above, i.e. so that $d^A h + h d^P = f$ with $h_n : P_n \to A_{n+1}$.

Let $\zeta_k := f_k - h_{k-1}d^P_k : P_k \to A_k$. Then

$$\begin{aligned} d^A_k \zeta_k &= d^A_k f_k - d^A_k h_{k-1} d^P_k \\ &= f_{k-1}d^P_k - d^A_k h_{k-1} d^P_k, \qquad \text{since } f \text{ is a chain map} \\ &= \left(f_{k-1} - d^A_k h_{k-1}\right) \circ d^P_k \\ &= \left(\cancel{d^A_k h_{k-1}} + h_{k-2}d^P_{k-1} - \cancel{d^A_k h_{k-1}}\right) \circ d^P_k, \qquad \text{by the inductive assumption} \\ &= h_{k-2} \circ \underbrace{d^P_{k-1} \circ d^P_k}_{=0} \\ &= 0. \end{aligned}$$

Hence $\mathsf{im}\,(\zeta_k) \subset \mathsf{Z}_k(A)$. By corestriction of the differential d^A_{k+1} and ζ_k as before we by projectivity of P_k get the existence of a homomorphism h_k such that the following diagram commutes,



That is, so that

$$\begin{aligned} \overline{d}^A_{k+1} h_k &= \overline{\zeta}_k \\ \Rightarrow d^A_{k+1} h_k &= \zeta_k, \qquad \text{by postcomposing both sides with } \mathsf{Z}_k(A) \hookrightarrow A_k \\ \Leftrightarrow d^A_{k+1} h_k &= f_k - h_{k-1}d^P_k, \qquad \text{by definition of } \zeta_k \\ \Leftrightarrow d^A_{k+1} h_k + h_{k-1}d^P_k &= f_k. \end{aligned}$$

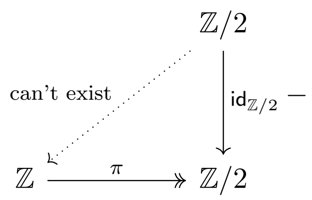
The existence of a family $h := (h_n)_{n\in\mathbb{Z}}$ such that $d^A h + h d^P = f$ then follows by induction. But this means precisely that f is nullhomotopic. Since f was an arbitrary chain map out of P into an acyclic complex A, it follows by definition that $P \in \mathbf{P}_R$. ☐

Below we give an example of a projective complex that is not *level*-wise projective.

Example 5.3.10. Consider the complex (over $R = \mathbb{Z}$)

$$P := \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\mathsf{id}_{\mathbb{Z}/2}} \underset{\sim\sim}{\mathbb{Z}/2} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

Then we may define a nullhomotopy of the identity id_P by letting $h_0 = \mathsf{id}_{\mathbb{Z}/2}$ and $h_n = 0$ for $n \neq 0$. Therefore $P \simeq 0$. Since 0 is projective, it follows by theorem 5.2.9 that $P \in \mathbf{P}_{\mathbb{Z}}$ is a projective complex. But P is *not level-wise* projective in $\mathsf{Ch}_{\mathbb{Z}}$, which can be seen by contemplating that no such map as indicated below can exist,



The reason for this is that $\mathsf{hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = 0$ (since $\mathbb{Z}$ has no torsion-elements!) but we would need such a morphism $s : \mathbb{Z}/2 \to \mathbb{Z}$ to satisfy $\pi \circ s = \mathsf{id}_{\mathbb{Z}/2}$. In particular, $(\pi \circ s)(1) = 1$ but $s(1) = 0$ so $(\pi \circ s)(1) = 0$ (since π is a homomorphism), contradiction!

Below, in example 5.3.13 we show how one may construct a projective resolution for every bounded below complex $M \in \mathbf{K}_R$. First we introduce a notion and a theorem.

Definition 5.3.11 (Enough projectives)**.** We say that an abelian category $\mathcal{A}$ has **enough projectives** if for every object $M \in \mathsf{Ob}(\mathcal{A})$ there is a projective object (recall definition 5.3.1) P and an epimorphism $P \twoheadrightarrow M$.

Theorem 5.3.12. Mod_R *has enough projectives.*

Proof. Let M be an R-module and consider the free R-module $P = \bigoplus_{m \in M} R$. Let e_m be the element in P with 1_R in argument m and zero otherwise. Then define a map $\{e_m : m \in M\} \to M$ by $e_m \mapsto m$. This uniquely extends to an R-linear map $p : P \to M$ which sends $p(e_m) = m$, so that p is surjective. By lemma 5.3.3 p is an epimorphism. By theorem 5.3.4 and remark 5.3.5 P is projective. The conclusion follows. □

Example 5.3.13. Let $M \in \mathbf{K}_R$ be bounded below, so that there is some $N \in \mathbb{Z}$ so that $M_n = 0$ for $n < N$. By theorem 5.3.12, M_N admits a surjection $p_N : P_N \twoheadrightarrow M_N$ from a projective module P_N. Let $P_n = 0, p_n : P_n \to M_n$ be the zero morphism, and $d_n^P = 0$ for $n < N$. Assume by induction that for $j < n$ we have constructed objects P_j, differentials d_j^P and R-module homomorphisms $p_j : P_j \to M_j$ so that the chain-map condition $d_j^M p_j = p_{j-1} d_j^P$ holds. Let

$$E_n := \left\{(m, z) \in M_n \oplus \mathsf{ker}\left(d_{n-1}^P\right) \; : \; d_n^M(m) = p_{n-1}(z)\right\}.$$

Since E_n is an R-module (straightforward to check) it admits a surjection $q_n : P_n \twoheadrightarrow E_n$ from a projective module P_n. Let $p_n := \pi_{M_n} \circ q_n$ be the post-composition of q_n with the projection to the first factor, and let $d_n^P := \iota \circ \pi_{\mathsf{ker}\left(d_{n-1}^P\right)} \circ q$ be the projection to the second factor together with the inclusion into P_{n-1}. Then by definition, the image of d_n^P is in $\mathsf{ker}\left(d_{n-1}^P\right)$ so that $d_{n-1}^P d_n^P = 0$. Now take any element $y \in P_n$, with $q_n(y) = (m, z)$. Then by definition

$$\begin{aligned} p_n(y) &= \pi_{M_n}(q_n(y)) \\ &= \pi_{M_n}(m, z) \\ &= m \in M_n, \end{aligned}$$

and

$$\begin{aligned} d_n^P(y) &= \iota\left(\pi_{\mathsf{ker}\left(d_{n-1}^P\right)}(q_n(y))\right) \\ &= \iota\left(\pi_{\mathsf{ker}\left(d_{n-1}^P\right)}(m, z)\right) \\ &= z \in P_{n-1}. \end{aligned}$$

Since $(m, z) \in E_n$ we have that $d_n^M(m) = p_{n-1}(z)$. Therefore, it follows that

$$\begin{aligned} d_n^M(p_n(y)) &= d_n^M(m) \\ &= p_{n-1}(z) \\ &= p_{n-1}\left(d_n^P(y)\right), \end{aligned}$$

i.e. $d_n^M p_n = p_{n-1} d_n^P$, and the chain-map condition is fulfilled up to degree n. By induction (inductively repeating this construction), we get a chain map $p : P \to M$ from a bounded-below complex P such that P_n is projective in each degree (note that $P_n = 0$ is projective for $n < N$). By construction, the p_n are surjective for $n \geqslant N$.

Since $P \in \mathbf{K}_R$ is bounded-below and $P_n \in \mathsf{Mod}_R$ is projective, $P \in \mathbf{P}_R$ is a projective complex by theorem 5.3.9. It remains to show that $P \xrightarrow{p} M$ is a quasi-isomorphism, i.e. that $\mathsf{H}_n(p) : \mathsf{H}_n(P) \to \mathsf{H}_n(M)$ is an isomorphism for all n. This is already clear for $n < N$ since $M_n = P_n = 0$.

$\mathsf{H}_n(p)$ surjective: Let $[m] \in \mathsf{H}_n(M)$ with $m \in \mathsf{Z}_n(M)$. If $n = N$ then $p_N : P_N \to M_N$ is surjective so there is some $x \in P_n$ such that $p_N(x) = m$. Since $P_{N-1} = 0$ we have that $d_N^P(x) = 0$ so that $x \in \mathsf{Z}_N(P)$. But then $\mathsf{H}_N([x]) = [p_n(x)] = [m]$.

If $n > N$ and $m \in \mathsf{Z}_n(M)$ then $d_n^M(m) = p_{n-1}(0)$ so that $(m, 0) \in E_n$. Since $q_n : P_n \to E_n$ is surjective, there is some $y \in P_n$ such that $q_n(y) = (m, 0)$, but then by construction, $p_n(y) = \pi_{M_n} \circ q_n(y) = \pi_{M_n}(m, 0) = m$. Furthermore we have

$$\begin{aligned} d_n^P(y) &= \iota \circ \pi_{\mathsf{ker}\left(d_{n-1}^P\right)} \circ q_n(y) \\ &= \iota \circ \pi_{\mathsf{ker}\left(d_{n-1}^P\right)}(m, 0) \\ &= 0 \in P_{n-1}, \end{aligned}$$

so that $y \in \mathsf{Z}_n(P)$. Therefore, $\mathsf{H}_n(p)([y]) = [p_n(y)] = [m]$.

$\mathsf{H}_n(p)$ injective: For $n < N$ there is nothing to show since the induced map in homology is the zero map, hence injective. So let $n \geqslant N$. If we let $[z] \in \mathsf{H}_n(P)$ with $z \in \mathsf{Z}_n(P)$ be such that $\mathsf{H}_n(p)([z]) = [p_n(z)] = 0$ then by definition, $p_n(z) = d_{n+1}^M(m)$ for some $m \in M_{n+1}$. Then (m, z) belongs to E_{n+1}. Since $q_{n+1} : P_{n+1} \to E_{n+1}$ is surjective let $y \in P_{n+1}$ be such that $q_{n+1}(y) = (m, z)$. Then we see that

$$\begin{aligned} d_{n+1}^P(y) &= \iota \circ \pi_{\mathsf{ker}(d_n^P)} \circ q_{n+1}(y) \\ &= \iota \circ \pi_{\mathsf{ker}(d_n^P)}(m, z) \\ &= z \in P_n, \end{aligned}$$

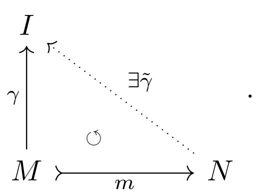
i.e. $z \in \mathsf{B}_n(P)$, so that $[z] = 0$.

We conclude that $\mathsf{H}_n(p)$ is an isomorphism in each degree n, i.e. p is a quasi-isomorphism. Thus, $P \xrightarrow[p]{\approx} M$ as constructed is a projective resolution.

5.4 Bounded above (or "bounded to the left") injective resolutions

We now introduce the *dual* notion of an object being projective.

Definition 5.4.1 (Injective object)**.** Let $\mathcal{A}$ be an abelian category, and let $I \in \mathsf{Ob}(\mathcal{A})$ be such that for every morphism $\gamma : M \to I$ and every monomorphism $m : M \rightarrowtail N$, there is a morphism $\tilde{\gamma} : N \to I$ such that $\tilde{\gamma} \circ m = \gamma$, or diagrammatically, as



Then we call I an **injective object**.

Example 5.4.2. Let R be a commutative unital ring, and consider an object $I \in \mathcal{A} = \mathsf{Mod}_R$, that has the properties described in definition 5.4.1.

Lemma 5.4.3. *For a morphism $M \xrightarrow{f} N$ in Mod_R, we have that f is a monomorphism iff f is injective.*

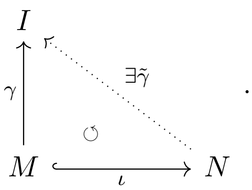
Proof. $\underline{\Rightarrow}$: Suppose $m \in M$ such that $f(m) = 0$, and consider the maps $g, h : R \rightrightarrows M$ with $g(r) = rm$ for all $r \in R$ (induced by the action $R \times M \to M$ by fixing m) and h the zero-morphism. Then for all $r \in R$, we have

$$\begin{aligned}(f \circ g)(r) &= f(rm) \\ &= rf(m) \\ &= 0 \\ &= (f \circ h)(r).\end{aligned}$$

Since f is a monomorphism, it follows that $g = h$. In particular, this means that $m = g(1) = h(1) = 0$. It follows that f is injective.

$\underline{\Leftarrow}$: If $h, g : X \rightrightarrows M$ are R-module homomorphisms such that $f \circ h = f \circ g$ then for all $x \in X$ we have $f(h(x)) = f(g(x)) \Rightarrow h(x) = g(x)$. It follows that $h = g$. $\square$

This means that an *injective object* in Mod_R is a module I for which the diagram can be completed as below, for injections ι and R-module homomorphisms γ,



We call such a module an **injective module**.

Definition 5.4.4 (Enough injectives)**.** We say that an abelian category $\mathcal{A}$ has **enough injectives** if every object $X \in \mathsf{Ob}(\mathcal{A})$ admits a monomorphism $X \rightarrowtail I$ to an injective object I.

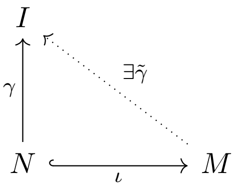
For the rest of the section, unless otherwise specified, we use ring/ rings for *commutative* ring/rings. This is more a choice of convenience, and there should be not be any major changes in the statements or proofs if one wants to state or prove the corresponding non-commutative ring statement.

We aim to show below that for any unital ring $\Bbbk$, $\mathsf{Mod}_{\Bbbk}$ has *enough injectives.*

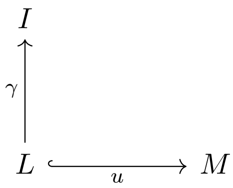
Proposition 5.4.5 (Baer's criterion)**.** *Let R be a ring and let I be an R-module. If every R-module homomorphism $\mathfrak{r} \to I$ from an ideal $\mathfrak{r}$ to R extends to an R-module homomorphism $R \to I$, then I is an injective R-module.*

Proof. It is enough to show that for an R-module M, $N \subseteq M$ a submodule and $\gamma : N \to I$ R-module homomorphism there is an extension to $\tilde{\gamma} : M \to I$ of γ for the following reason:

If we can prove this statement, then any diagram can be completed as below,



Now consider a diagram on the following form,

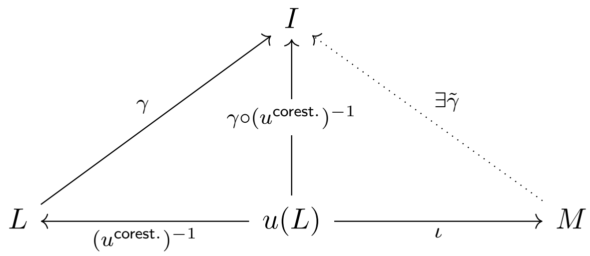


(5.4.1)

with u injective. We then have the existence of $\tilde{\gamma}$ such that (where $u^{\text{corest.}}$ is the *corestriction* to the image $u(L)$ of u)

$$\begin{aligned} &\tilde{\gamma} \circ \iota = \gamma \circ (u^{\text{corest.}})^{-1} \\ \Leftrightarrow\ &\tilde{\gamma} \circ \iota \circ u^{\text{corest.}} = \gamma \\ \Leftrightarrow\ &\tilde{\gamma} \circ u = \gamma, \qquad \text{since } \iota \circ u^{\text{corest.}} = u, \end{aligned}$$

as depicted in the diagram below,



Hence $\tilde{\gamma} \circ \iota$ does the job of completing the triangle in 5.4.1.

Proceeding with the proof, we let

$$\mathsf{N} := \left\{ (N', \gamma') \,\middle|\, N \subseteq N' \subseteq M \text{ submodules and } \gamma' : N' \to I \text{ extends } \gamma \right\}.$$

We define a partial order $(\mathsf{N}, \leqslant)$ with $(N_1, \gamma_1) \leqslant (N_2, \gamma_2)$ if $N_1 \subseteq N_2$ and $\gamma_2|_{N_1} = \gamma_1$. Then a relatively straightforward check (using that a chain is exactly the restriction of a partial order to a subset where we get a total order) shows that every chain $\{(N_i, \gamma_i)\}_{i \in \mathcal{I}} \subset \mathsf{N}$ for some index set $\mathcal{I}$ has an upper bound, and so by Zorn's lemma there is a maximal element (N', γ'). We claim that $N' = M$. Assume not. Then there is an element $m \in M \setminus N'$. Let $N'' := N' + R \cdot m$, so that $N' \subsetneq N'' \subseteq M$ (the first inclusion since $0 + 1 \cdot m = m \in N''$ and the second since $N', R \cdot m \subset M$).

Let $\mathfrak{r} := \{r \in R \,:\, r \cdot m \in N'\}$. This is an ideal of R. Consider the sequence

$$0 \to \mathfrak{r} \xrightarrow{\alpha} N' \oplus R \xrightarrow{\beta} N'' \to 0,$$

with R-module homomorphisms $\alpha(r) = (r \cdot m, -r)$ and $\beta(n,r) = n + r \cdot m$. We see that

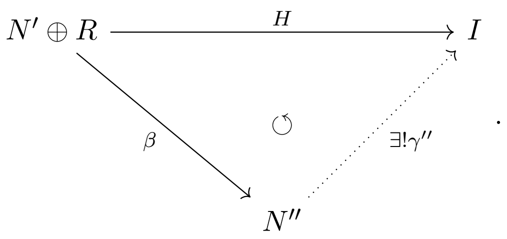
$$\begin{aligned}\beta(\alpha(r)) &= \beta(r \cdot m, -r) \\ &= r \cdot m - r \cdot m \\ &= 0,\end{aligned}$$

so that $\mathsf{im}(\alpha) \subset \mathsf{ker}\,(\beta)$, for $r \in \mathfrak{r}$. If $(n,r) \in \mathsf{ker}\,(\beta)$ then $n + r \cdot m = 0 \Leftrightarrow n = -r \cdot m \in N'$, which means that $r \in \mathfrak{r}$. But then $\alpha(-r) = (-r \cdot m, r) = (n,r)$ so that $(n,r) \in \mathsf{im}(\alpha)$. The surjectivity of β is clear from definition, and $\alpha(r) = (r \cdot m, -r) = (0,0)$ implies that $-r = 0 \Rightarrow r = 0$, so that α is injective. Hence the sequence is *exact*.

Define a homomorphism $\phi : \mathfrak{r} \to I$ by $\phi := \gamma' \circ \pi_{N'} \circ \alpha$ where $\pi_{N'} : N' \oplus R \twoheadrightarrow N'$ is the projection to the first factor. By assumption, there is an extension $\Phi : R \to I$ of ϕ. There is then an R-module homomorphism $H : N' \oplus R \to I$ defined by $H(n,r) = \gamma'(n) + \Phi(r)$. For $r \in \mathfrak{r}$ we have that

$$\begin{aligned}H(\alpha(r)) &= H(r \cdot m, -r) \\ &= \gamma'(r \cdot m) - \Phi(r) \\ &= \gamma'(r \cdot m) - \phi(r) \\ &= \gamma'(r \cdot m) - \gamma'(r \cdot m), \qquad \text{by definition of } \phi \\ &= 0,\end{aligned}$$

so that $\mathsf{im}(\alpha) \subset \mathsf{ker}(H)$. By exactness, it follows that $\mathsf{ker}(\beta) \subset \mathsf{ker}(H)$. By the universal property of the quotient there is then a unique map $\gamma'' : N'' \to I$ such that $H = \gamma' \circ \beta$, i.e. so that the following diagram commutes



It follows that for $(n,0) \in N' \oplus R$, we have that

$$\begin{aligned}\gamma'(n) &= \gamma'(n) + \Phi(0) \\ &= H(n,0) \\ &= \gamma''(\beta(n,0)) \\ &= \gamma''(n),\end{aligned}$$

so that γ'' extends γ' (and hence also γ). But since also $N' \subsetneq N''$ we have that $(N', \gamma') \leqslant (N'', \gamma'')$ with respect to $(\mathsf{N}, \leqslant)$, contradicting maximality of (N', γ'). $\square$

Definition 5.4.6 (Divisible module)**.** A module $M \in \mathsf{Mod}_R$ is **divisible** if for every non-zero element $r \in R$, the induced map (from the action $R \times M \to M$)

$$M \xrightarrow{r} M, \qquad m \mapsto rm,$$

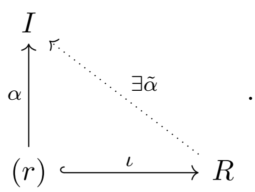
is surjective.

Theorem 5.4.7. *If R is an integral domain, then* *(a) All injective modules are divisible.*

(b) If R is also a prinicipal ideal domain, then the converse is true, i.e. all divisible modules are injective.

Proof. (a): Assume I is an injective module in Mod_R, and consider a non-zero element $r \in R$. For any $m \in I$ we must find $x \in I$ such that $rx = m$.

Consider the prinicipal ideal $(r) \subseteq R$, and define a map $\alpha : (r) \to I$ by $\alpha(ar) = am$. If $a'r = ar$ for $a \in R$, then $(a' - a)r = 0$. Since $r \neq 0$ and R is an integral domain, it follows that $a' = a$ and so $a'm = am$, so that α is well-defined. A routine check gives that α is an R-module homomorphism. The inclusion $\iota : (r) \to R$ is injective. Since I is injective, we get an induced R-module homomorphism $\tilde{\alpha}$ such that the following diagram commutes,



Let $x := \tilde{\alpha}(1) \in I$, and note that

$$\begin{aligned} rx &= r\tilde{\alpha}(1) \\ &= \tilde{\alpha}(r) \\ &= \alpha(r) \\ &= \alpha(1 \cdot r) \\ &= 1 \cdot m \\ &= m. \end{aligned}$$

Hence $I \xrightarrow{r} I$ is surjective.

(b): By Baer's criterion 5.4.5 it is enough to show that for any ideal $\mathfrak{r}$ and any R-module homomorphism $\psi : \mathfrak{r} \to I$ there is an extension $\Psi : R \to I$. To wit, let $\psi : \mathfrak{r} \to I$ be an R-module homomorphism. Since R is principal, $\mathfrak{r} = (r)$ for some element $r \in R$. If $r = 0$ then the zero-homomorphism $R \to I$ extends ψ. If $r \neq 0$ then since I is divisible, the map $I \xrightarrow{r} I$ is surjective. Therefore, there exists an $x \in I$ such that $rx = \psi(r)$. Define $\Psi : R \to I$ by $\Psi(r') = r'x$ for $r' \in R$, i.e. as the induced R-module homomorphism from the action $R \times I \to I$ defined by freezing x in the second variable. Then we find for arbitrary $ar \in (r)$, we have

$$\begin{aligned} \Psi(ar) &= a\Psi(r), && \text{since } \Psi \text{ is } R\text{-linear} \\ &= a(rx) \\ &= a\psi(r) \\ &= \psi(ar), && \text{since } \psi \text{ is } R\text{-linear,} \end{aligned}$$

which means that $\Psi|_{\mathfrak{r}} = \psi$. $\square$

Theorem 5.4.8. *Let $S \xrightarrow{f} R$ be a ring homomorphism. Then there are adjunctions as indicated in the diagram below*

$$\mathsf{Mod}_S \underset{\mathsf{Mod}_S(R,-)}{\overset{(\,\cdot\,)\otimes_S R}{\underset{\perp}{\overset{\perp}{\leftarrow\!\!-\!\!-\, F \,-\!\!-\!\!\!\!\!\rightrightarrows}}}} \mathsf{Mod}_R$$

Here F is the forgetful functor $\mathsf{Mod}_R \xrightarrow{F} \mathsf{Mod}_S$.

Remark 5.4.9. Observe that $S \xrightarrow{f} R$ is a witness to the fact that R is an S-algebra. Hence the forgetful functor F sends an R-module M to the same underlying abelian group but now equipped with the action $s \cdot m := f(s) \cdot m$. Hence one might as well call this the "*restriction of scalar*" functor (note; with respect to a *fixed* ring-homomorphism $S \xrightarrow{f} R$). On morphisms $L \xrightarrow{g} N$ of R-modules, we set $F(g) = g$ since g respects the R-action it will also induced the action induced by restriction of scalars to S, i.e. $g(s \cdot l) = g(f(s) \cdot l) = f(s) \cdot g(l) = s \cdot g(l)$ for all $l \in L$ and all $s \in S$.

Remark 5.4.10. On morphisms $g : M \to M' \in \mathsf{Mod}_S$, the functor $(\,\cdot\,) \otimes_S R : \mathsf{Mod}_S \to \mathsf{Mod}_R$ is defined as $(g) \otimes_S R := g \otimes \mathrm{id}_R : M \otimes_S R \to M' \otimes_S R$, while $\mathsf{Mod}_S(R, -)$ takes morphisms $g : N \to N'$ in Mod_S to $\mathsf{Mod}_S(R, g) : \mathsf{Mod}_S(R, N) \to \mathsf{Mod}_S(R, N')$ by postcomposition $\mathsf{Mod}_S(R, N) \ni u \mapsto g \circ u \in \mathsf{Mod}_S(R, N')$.

Remark 5.4.11. For any S-module N, we have that $\mathsf{Mod}_S(R, N)$ is an R-module by the action $(a \cdot g)(r) := g(ar)$ for $a \in R$ and $g \in \mathsf{Mod}_S(R, N)$, at least under the assumption that R is an S-algebra and that R is commutative (observe that if R is commutative and we have a ring-homomorphism from S to R, then this gives R an S-algebra structure), so certainly with S, R commutative and $S \xrightarrow{f} R$ witnessing ring-homomorphism for the S-algebra structure on R.

Remark 5.4.12. For an introduction to adjunctions, see e.g. [Rie16, Chapter 4.1].

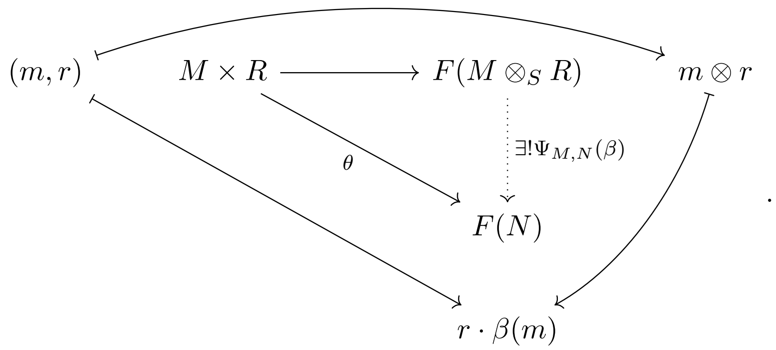
Proof. $\underline{(\,\cdot\,) \otimes_S R \dashv F}$: We need to show that for each pair of objects M, N with $M \in \mathsf{Mod}_S$ and $N \in \mathsf{Mod}_R$, there is a bijection

$$\mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N) \xrightarrow[\approx]{\Phi_{M,N}} \mathsf{hom}_{\mathsf{Mod}_S}(M, F(N)),$$

natural in both M and N (in the sense of [Rie16, 2nd and 3rd diagram on p. 131]). To this end, define $\Phi_{M,N}$ by sending an R-module homomorphism $M \otimes_S R \xrightarrow{\alpha} N$ to the map $\Phi_{M,N}(\alpha) := m \mapsto \alpha(m \otimes_S 1)$. A routine check gives that this is an S-module homomorphism. Correspondingly, for each such pair M, N, we define a map

$$\mathsf{hom}_{\mathsf{Mod}_S}(M, F(N)) \xrightarrow{\Psi_{M,N}} \mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N)$$

by $\Psi_{M,N}(\beta)(m \otimes r) := r \cdot \beta(m)$ for $\beta \in \mathsf{hom}_{\mathsf{Mod}_S}(M, F(N))$. This is the map induced by the universal property of $F(M \otimes_S R) \in \mathsf{Mod}_S$, as below,



One checks that θ is S-bilinear (recall that we are assuming the rings are commutative; otherwise exchange S-bilinear with S-balanced and R an S-algebra, i.e. $f(S) \subset Z(R)$) hence factors through a unique S-module homomorphism $\Psi_{M,N}(\beta)$ and the canonical map $(m, r) \overset{\iota}{\mapsto} m \otimes r$. A routine calculation gives that the induced map $\Psi_{M,N}(\beta)$ is in fact R-linear.

We check that Ψ and Φ are mutual inverses. If $\alpha \in \mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N)$ then

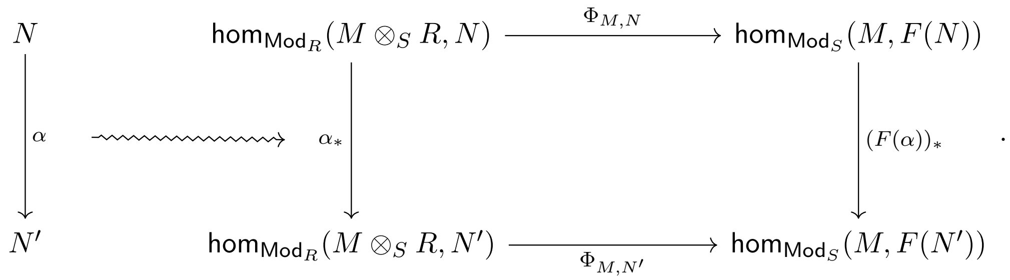
$$\begin{aligned} \Psi_{M,N}(\Phi_{M,N}(\alpha))(m \otimes r) &= r \cdot (\Phi_{M,N}(\alpha)(m)) \\ &= r \cdot \alpha(m \otimes 1) \\ &= \alpha(m \otimes r), \qquad \text{since } \alpha \text{ is } R\text{-linear,} \end{aligned}$$

and if $\beta \in \mathsf{hom}_{\mathsf{Mod}_S}(M, F(N))$ then

$$\begin{aligned} \Phi_{M,N}(\Psi_{M,N}(\beta))(m) &= \Psi_{M,N}(\beta)(m \otimes_S 1) \\ &= 1 \cdot \beta(m) \\ &= \beta(m). \end{aligned}$$

Hence $\Phi_{M,N}$ is an isomorphism. We check naturality (recall our earlier parenthesis in the proof about this).

We first check that the following square commutes, for all $\alpha : N \to N'$ in Mod_R,



For $\beta \in \mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N)$ we have

$$(F(\alpha))_* \circ \Phi_{M,N}(\beta))(m) = \alpha(\beta(m \otimes 1)), \qquad \text{since } F(\alpha) = \alpha,$$

while

$$\begin{aligned} \Phi_{M,N'} \circ \alpha_*(\beta))(m) &= \alpha_* \beta(m \otimes 1) \\ &= \alpha(\beta(m \otimes 1)), \end{aligned}$$

which agrees with our other computation, and so in natural language, one would perhaps say that we have just shown “naturality with respect to N”.

To check “naturality in M”, instead consider any morphism $h : M' \to M$ in Mod_S, and contemplate the diagram to the right below,

$$\begin{array}{ccccc} M' & & \mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N) & \xrightarrow{\Phi_{M,N}} & \mathsf{hom}_{\mathsf{Mod}_S}(M, F(N)) \\ \Big\downarrow h & \rightsquigarrow & \Big\downarrow (h \otimes_S R)^* & & \Big\downarrow h^* \\ M & & \mathsf{hom}_{\mathsf{Mod}_R}(M' \otimes_S R, N) & \xrightarrow[\Phi_{M',N}]{} & \mathsf{hom}_{\mathsf{Mod}_S}(M', F(N)) \end{array} .$$

We have, for $\gamma \in \mathsf{hom}_{\mathsf{Mod}_R}(M \otimes_S R, N)$ and arbitrary $m' \in M'$ that

$$\begin{aligned} (h^* \circ \Phi_{M,N}(\gamma))(m') &= \Phi_{M,N}(\gamma)(h(m')) \\ &= \gamma\big(h(m') \otimes 1\big) \end{aligned}$$

while

$$\begin{aligned} (\Phi_{M',N} \circ (h \otimes_S R)^*(\gamma))(m') &= (h \otimes_S R)^*(\gamma)(m' \otimes 1) \\ &= \gamma(h \otimes \mathrm{id}_R(m' \otimes 1)) \\ &= \gamma(h(m') \otimes 1). \end{aligned}$$

Both computations agree. Hence we conclude that $(\,\cdot\,) \otimes_S R \dashv F$.

<u>$F \dashv \mathsf{Mod}_S(R, -)$</u>: For all pairs N, M with $N \in \mathsf{Mod}_R$ and $M \in \mathsf{Mod}_S$, define

$$\Phi_{N,M} : \mathsf{Mod}_S(F(N), M) \to \mathsf{Mod}_R(N, \mathsf{Mod}_S(R, M)),$$

by taking an S-module homomorphism $F(N) \xrightarrow{\alpha} M$ to $\Phi_{N,M}(\alpha)(n)(r) := \alpha(r \cdot n)$. One checks that $\Phi_{N,M}(\alpha)(n)$ is S-linear for each $n \in N$, and that $\Phi_{N,M}(\alpha)$ is R-linear, by using that R is commutative.

Let $\Psi_{N,M} : \mathsf{Mod}_R(N, \mathsf{Mod}_S(R, M)) \to \mathsf{Mod}_S(F(N), M)$ be defined by $\Psi_{N,M}(\beta)(n) = \beta(n)(1_R)$ (we suppress R in 1_R after this), for each $\beta \in \mathsf{Mod}_R(N, \mathsf{Mod}_S(R, M))$ and $n \in N$ so that $\beta(n) \in \mathsf{Mod}_S(R, M)$. Then one may check that $\Psi_{N,M}(\beta)$ is S-linear by using how the module-structures interact, in particular using remark 5.4.11, that $\beta(n)$ is S-linear for each $n \in N$ and that β is R-linear.

We check that $\Phi_{N,M}$ and $\Psi_{N,M}$ are mutual inverses. Let $\alpha \in \mathsf{hom}_{\mathsf{Mod}_S}(F(N), M)$. Then

$$\begin{aligned} \Psi_{N,M}(\Phi_{N,M}(\alpha))(n) &= \Phi_{N,M}(\alpha)(n)(1) \\ &= \alpha(1 \cdot n) \\ &= \alpha(n), \end{aligned}$$

and for $\beta \in \mathsf{Mod}_R(N, \mathsf{Mod}_S(R, M))$ we have

$$\begin{aligned} \Phi_{N,M}(\Psi_{N,M}(\beta))(n)(r) &= \Psi_{N,M}(\beta)(r \cdot n) \\ &= \beta(r \cdot n)(1) \\ &= (r \cdot \beta(n))(1), \qquad \text{since } \beta \text{ is } R\text{-linear} \\ &= \beta(n)(r), \qquad \text{by remark 5.4.11.} \end{aligned}$$

Hence $\Phi_{N,M}$ is an isomorphism. We check naturality.

Naturality in the first argument $F(-)$: Let $\alpha : N' \to N$ be an R-module homomorphism. Naturality amounts to checking that the following square commutes,

$$\begin{array}{ccccc} N' & & \mathsf{hom}_{\mathsf{Mod}_S}(F(N), M) & \xrightarrow{\Phi_{N,M}} & \mathsf{hom}_{\mathsf{Mod}_R}(N, \mathsf{Mod}_S(R, M)) \\ \downarrow{\scriptstyle \alpha} & \rightsquigarrow & \downarrow{\scriptstyle F(\alpha)^*} & & \downarrow{\scriptstyle \alpha^*} \\ N & & \mathsf{hom}_{\mathsf{Mod}_S}(F(N'), M) & \xrightarrow{\Phi_{N',M}} & \mathsf{hom}_{\mathsf{Mod}_R}(N', \mathsf{Mod}_S(R, M)) \end{array} .$$

Let $F(N) \xrightarrow{\beta} M$ be an element in the left-upper corner of the diagram above, let $n' \in N'$ and let $r \in R$. Then

$$\begin{aligned} \Phi_{N',M}(F(\alpha)^*(\beta))(n')(r) &= \Phi_{N',M}(\beta \circ F(\alpha))(n')(r) \\ &= (\beta \circ F(\alpha))(r \cdot n') \\ &= \beta(\alpha(r \cdot n')) \\ &= \beta(r \cdot \alpha(n')), \qquad \text{since } \alpha \text{ is } R\text{-linear.} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\alpha^* \circ \Phi_{N,M})(\beta)(n')(r) &= \alpha^*(\beta(r \cdot n)) \\ &= \beta(\alpha(r \cdot n) \\ &= \beta(r \cdot \alpha(n)), \qquad R\text{-linearity of } \alpha. \end{aligned}$$

We conclude that the diagram commutes.

Naturality in the second argument: Let $\alpha : M \to M'$ be an S-module homomorphism. Naturality then amounts to checking that the following diagram commutes,

$$\begin{array}{ccccc} M & & \mathsf{hom}_{\mathsf{Mod}_S}(F(N), M) & \xrightarrow{\Phi_{N,M}} & \mathsf{hom}_{\mathsf{Mod}_R}(N, \mathsf{Mod}_S(R, M)) \\ \downarrow{\scriptstyle \alpha} & \rightsquigarrow & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \mathsf{Mod}_S(R,\alpha)_*} \\ M' & & \mathsf{hom}_{\mathsf{Mod}_S}(F(N), M') & \xrightarrow[\Phi_{N,M'}]{} & \mathsf{hom}_{\mathsf{Mod}_R}(N, \mathsf{Mod}_S(R, M')) \end{array} ,$$

where

$$\mathsf{Mod}_S(R, \alpha)_* : \mathsf{Mod}_S(R, M) \to \mathsf{Mod}_S(R, M'), \quad b \mapsto \alpha \circ b$$

acts by postcomposition (as indicated by lower-case $*$).

We find that for $F(N) \xrightarrow{\beta} M$ R-linear homomorphism, $n \in N$ and $r \in R$ we have that

$$\begin{aligned} (\Phi_{N,M'} \circ \alpha_*)(\beta)(n)(r) &= \alpha_*(\beta(r \cdot n) \\ &= \alpha(\beta(r \cdot n)) \end{aligned}$$

while

$$\begin{aligned} (\mathsf{Mod}_S(R, \alpha)_* \circ \Phi_{N,M})(\beta)(n)(r) &= \mathsf{Mod}_S(R, \alpha)_*(\beta(r \cdot n)) \\ &= \alpha(\beta(r \cdot n)). \end{aligned}$$

Since the computations agree, the diagram commutes. □

Corollary 5.4.13. *If $F(M) \to N \in \mathsf{Mod}_S$ is an injection, the corresponding morphism (under the isomorphism given by theorem 5.4.8) $M \to \mathsf{Mod}_S(R, N)$ in Mod_R is an injection.*

Proof. By investigating the proof of theorem 5.4.8 we see that Φ sends an injection $F(M) \xrightarrow{\beta} N$ to $\tilde{\beta} := \Phi(\beta)$ defined by

$$\tilde{\beta}(n)(r) = \beta(r \cdot n).$$

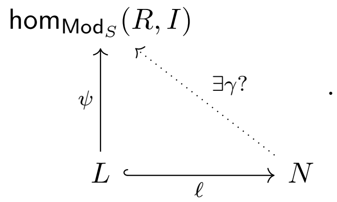
Assume that $\tilde{\beta}(n_1) = \tilde{\beta}(n_2)$ for some $n_1, n_2 \in N$. This means that

$$\begin{aligned} \tilde{\beta}(n_1)(r) &= \tilde{\beta}(n_2)(r), \qquad \forall r \in R \\ \Rightarrow \tilde{\beta}(n_1)(1) &= \tilde{\beta}(n_2)(1) \\ \Leftrightarrow \beta(n_1) &= \beta(n_2) \\ \Rightarrow n_1 &= n_2, \qquad \text{by injectivity of } \beta. \end{aligned}$$

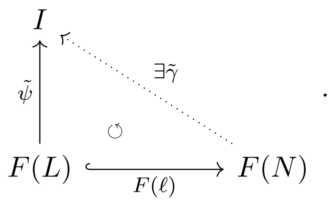
□

Corollary 5.4.14. *The functor $\mathsf{hom}_{\mathsf{Mod}_S}(R, -)$ preserves injective objects.*

Proof. Assume $I \in \mathsf{Mod}_S$ is an injective object. Then we want to check whether we can complete the triangle as below in Mod_R with an R-module homomorphism γ where ψ and ℓ are R-module homomorphism with ℓ an injection,



Observe that since $F(\ell) = \ell$ we then get an injection $F(\ell) : F(L) \to F(N)$ in Mod_S. The isomorphism Ψ in the proof of 5.4.8 gives us a map $\tilde{\psi} := \Psi(\psi) : F(L) \to I$, defined by $\tilde{\psi}(l) = \psi(l)(1)$. Since I is injective we get a map $\tilde{\gamma} : F(N) \to I$ so that the diagram below commutes,



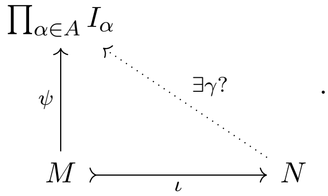
Then the natural isomorphism Φ gives us a map $\gamma := \Phi(\tilde{\gamma}) : N \to \mathsf{Mod}_S(R, I)$, defined by $\gamma(n)(r) = \tilde{\gamma}(r \cdot n)$. We then see that

$$\begin{aligned} (\gamma \circ \ell)(l)(r) &= \gamma(\ell(l))(r) \\ &= \tilde{\gamma}(r \cdot \ell(l)) \\ &= \tilde{\gamma}(\ell(r \cdot l)) \\ &= \tilde{\psi}(r \cdot l) \\ &= \psi(r \cdot l)(1) \\ &= (r \cdot \psi(l))(1), \qquad \text{since } \psi \text{ is } R\text{-linear} \\ &= \psi(l)(r), \end{aligned}$$

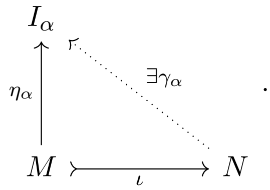
where we in the last step used the R-module structure on $\mathsf{Mod}_S(R, I)$ (we remind the reader of remark 5.4.11). ☐

Lemma 5.4.15. *Injective objects are preserved under products (as long as the product exists).*

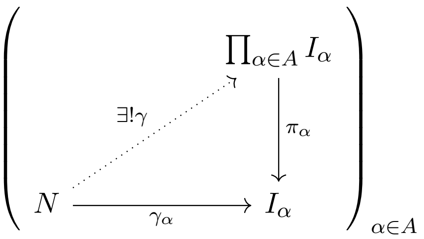
Proof. Let $(I_\alpha)_{\alpha \in A}$ be a family of injective objects in an abelian category $\mathcal{A}$. Consider a diagram on the following form in $\mathcal{A}$, with $M \xrightarrow{\iota} N$ a monomorphism,



We want to show the existence of γ completing the triangle. Since $\prod_{\alpha \in A} I_\alpha$ is a product there are canonical maps $\pi_\alpha : \prod_{\alpha \in A} I_\alpha \to I_\alpha$. This gives us maps $\eta_\alpha := \pi_\alpha \circ \psi : M \to I_\alpha$ and since each I_α is injective, we may complete the following diagram with a map γ_α as below,



By the universal property of the product applied to this A-indexed family of maps



there exists a unique morphism $\gamma : N \to \prod_{\alpha \in A} I_\alpha$ so that the diagram above commutes for all $\alpha \in A$, i.e. $\pi_\alpha \circ \gamma = \gamma_\alpha$.

We then find that

$$\begin{aligned} \pi_\alpha \circ \gamma \circ \iota &= \gamma_\alpha \circ \iota \\ &= \eta_\alpha \\ &= \pi_\alpha \circ \psi, \end{aligned}$$

for all $\alpha \in A$. By another application of the universal property of the product we must have that $\gamma \circ \iota = \psi$, and the conclusion follows. ☐

Lemma 5.4.16. $\mathbb{Q}/\mathbb{Z}$ *is divisible as a* $\mathbb{Z}$*-module over the principle ideal domain* $\mathbb{Z}$.

Proof. We must show that the map $\mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z}$ is surjective, for any $n \in \mathbb{Z}$. Let $q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ be arbitrary, with $q \in \mathbb{Q}$ non-zero. Then $\frac{q}{n} + \mathbb{Z}$ is sent to $q + \mathbb{Z}$. Clearly $0 + \mathbb{Z}$ is sent to $0 + \mathbb{Z}$ so we are done. ☐

Theorem 5.4.17. *$\mathbb{Q}/\mathbb{Z}$ is injective as a $\mathbb{Z}$-module.*

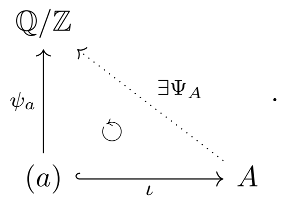
Proof. By lemma 5.4.16 $\mathbb{Q}/\mathbb{Z}$ is divisible, so by 5.4.7.(b) it is injective as a $\mathbb{Z}$-module. □

Theorem 5.4.18. *Let $A \in \mathsf{Ab}$. Then for all non-zero $a \in A$ there exists a homomorphism $A \xrightarrow{\Psi_a} \mathbb{Q}/\mathbb{Z}$ in Ab sending a to a non-zero element $\Psi_a(a) \neq 0$ of $\mathbb{Q}/\mathbb{Z}$.*

Proof. If a has *finite* order then since $a \neq 0$ its order is greater than one. Define a homomorphism $\psi_a : (a) \to \mathbb{Q}/\mathbb{Z}$ by $\psi_a(ka) := \frac{k}{n} + \mathbb{Z}$. Since $n > 1$ we see that $\psi_a(a) = \frac{1}{n} + \mathbb{Z} \neq \mathbb{Z}$ so that $\psi_a(a)$ is non-zero. This is well-defined since if $ka = \ell a \Leftrightarrow a(k-\ell) = 0$ which means that $n \mid k - \ell$ so that $\frac{k}{n} - \frac{\ell}{n} \in \mathbb{Z} \Leftrightarrow \frac{k}{n} + \mathbb{Z} = \frac{\ell}{n} + \mathbb{Z}$ and a routine calculation gives that this is a homomorphism.

If a has *infinite* order, then define e.g. $\psi_a(ka) = \frac{k}{2} + \mathbb{Z}$. Straightforward checks gives that this is well-defined (using that a has infinite order so every ka is uniquely defined), is an (additive) homomorphism and $\psi_a(a) = \frac{1}{2} + \mathbb{Z} \neq 0$.

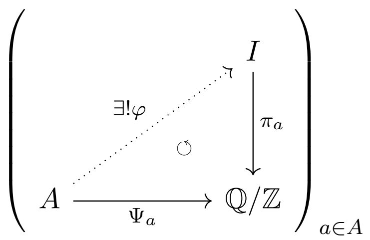
Since $\mathbb{Q}/\mathbb{Z}$ is injective, we may extend $\psi_a : (a) \to \mathbb{Q}/\mathbb{Z}$ as below for each non-zero a,



Since Ψ_a *extends* ψ_a, $\Psi_a(a) \neq 0$. □

Corollary 5.4.19. *For any object $A \in \mathsf{Ab}$, there is an injection $A \hookrightarrow \prod_{a\in A} \mathbb{Q}/\mathbb{Z}$.*

Proof. In accordance with theorem 5.4.18 for each non-zero $a \in A$ we may choose a homeomorphism $\Psi_a : A \to \mathbb{Q}/\mathbb{Z}$ such that $\Psi_a(a) \neq 0$ and for 0 we may choose the zero-homomorphism. Let $I := \prod_{a\in A} \mathbb{Q}/\mathbb{Z}$. Since I is a product, by the universal property we get a unique map φ making the family of A-indexed triangles commute, as below,



We claim that φ is an injection: Assume that $\varphi(a) = 0$. Then $\pi_a \circ \varphi(a) = \Psi_a(a) = 0$, which implies that $a = 0$. □

Theorem 5.4.20. Mod_R *has enough injectives.*

Proof. Let $M \in \mathsf{Mod}_R$ be given. Since every ring is a $\mathbb{Z}$-algebra, we have adjunctions as in theorem 5.4.8. By 5.4.19 there is then an injection $F(M) \hookrightarrow \prod_{m\in M} \mathbb{Q}/\mathbb{Z} =: I$ in Ab, so the same injection lives in $\mathsf{Mod}_\mathbb{Z}$. By 5.4.15 this gives an injection $M \overset{\zeta}{\hookrightarrow} \mathsf{Mod}_\mathbb{Z}(R, I)$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective object in $\mathsf{Mod}_\mathbb{Z}$ by 5.4.17 it follows by 5.4.14 that $\mathsf{Mod}_\mathbb{Z}(R, I)$ is an injective object. By 5.4.3 we have that ζ is a monomorphism. Therefore, by definition 5.4.4, Mod_R has enough injectives. $\square$

Summarizing § 5.3 and § 5.4: We call $\mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$ an **injective cogenerator**, and we call $R \in \mathsf{Mod}_R$ a **projective generator**.

5.5 Cell complexes and lifting

In the context of wanting to construct projective resolutions, let's define some notions. First we define

$$\begin{array}{c} S^n \\ \Big\downarrow{\scriptstyle i_n} \\ D^{n+1} \end{array} \quad := \quad \begin{array}{ccc} 0 & \longrightarrow & R \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{id}_R} \\ R & \xrightarrow[\mathsf{id}_R]{} & R \end{array} \tag{5.5.1}$$

by which we mean that i_n here is the chain map, S^n is the upper complex, D^{n+1} is the lower complex with zeroes everywhere but where indicated above, and where the *vertical* id_R lives in degree n. It is immediate that $\Sigma^{-n}S^n = S^0$ and that $\Sigma^{-n}D^n = D^0$ if one forgets the differentials. However, with respect to said differentials, when n is odd this introduces a sign in front of the lower horizontal id_R [5]

Theorem 5.5.1. *For any complex $M \in \mathsf{Ch}_R$, there are natural isomorphisms*

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \approx \mathsf{Z}_n(M) \qquad \textit{and} \qquad \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \approx M_{n+1}.$$

Proof. $\underline{\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \approx \mathsf{Z}_n(M)}$: Observe that by definition of S^n, a chain map $S^n \xrightarrow{f} M$ is determined solely by $f_n(1) \in M_n$. Since $d_n^M f_n(1) = f_{n-1} \underbrace{d_n^{S^n}(1)}_{=0} = 0$ it follows that the assignment $f \mapsto f_n(1)$ gives an element $f_n(1) \in \mathsf{Z}_n(M)$.

On the other hand, each n-cycle $x \in \mathsf{Z}_n(M)$ determines a chain map $1 \overset{g}{\longmapsto} x$ (where the assignment $1 \mapsto x$ happens in degree n) as indicated below,

$$\begin{array}{ccc} r & \overset{d^{S^n}}{\longmapsto} & 0 \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle g} \\ rx & \underset{d^M}{\longmapsto} & 0 \end{array} .$$

These are inverses hence gives a bijection of R-modules (since they are mutual inverses also the inverse assignment from an n-cycle to the corresponding chain map is R-linear). We claim the assignment $f \overset{\Phi_M}{\longmapsto} f_n(1)$ is *natural* (easy check gives we claim that it is also an R-module homomorphism), with $S^n \xrightarrow{f} M$ a chain map. To see this, consider the

[5] This is due to the fact that $d_j^{\Sigma^{-n}D^n} = (-1)^{-n} d_{j+n}^{D^n}$. There is no such issue for S^n since all differentials $d_j^{S^n} = 0$ are zero, hence so also the corresponding differentials after shifting by Σ^{-n}

following diagram for a chain map $\alpha : M \to M'$,

$$
\begin{array}{ccccc}
M & & \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) & \xrightarrow{\Phi_M} & \mathsf{Z}_n(M) \\
\Big\downarrow \alpha & \rightsquigarrow & \Big\downarrow \alpha_* & & \Big\downarrow \mathsf{Z}_n(\alpha) \\
M' & & \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M') & \xrightarrow[\Phi_{M'}]{} & \mathsf{Z}_n(M')
\end{array},
$$

where we recall the definition of $\mathsf{Z}_n(\alpha)$ (cf. theorem 2.3.3) is just the restriction (and corestriction) of α_n to $\mathsf{Z}_n(M)$ respectively $\mathsf{Z}_n(M')$. We then have, for any chain map $S^n \xrightarrow{f} M$, that

$$(\mathsf{Z}_n(\alpha) \circ \Phi_M)(f) = \alpha_n(f_n(1)),$$

while

$$
\begin{aligned}
\Phi_{M'}(\alpha_*(f)) &= \Phi_{M'}(\alpha \circ f) \\
&= (\alpha \circ f)_n(1) \\
&= \alpha_n(f_n(1)), \qquad \text{by definition.}
\end{aligned}
$$

Hence Φ is a natural isomorphism of the functors $\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -), \mathsf{Z}_n(\,\cdot\,) : \mathsf{Ch}_R \rightrightarrows \mathsf{Mod}_R$.

$\underline{\mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \approx M_{n+1}}$: Let $\mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \xrightarrow{\Phi_M} M_{n+1}$ be defined by the assignment $g \mapsto g_{n+1}(1)$. Note that since g is a chain map $D^{n+1} \xrightarrow{g} M$ and since $d_{n+1}^{D^{n+1}} = \mathsf{id}_R$, it follows that

$$
\begin{aligned}
g_n &= g_n \circ \mathsf{id}_R \\
&= g_n \circ d_{n+1}^{D^{n+1}} \\
&= d_{n+1}^{M} \circ g_{n+1}.
\end{aligned}
$$

Since all maps are R-module homomorphisms, g_n and hence g, is determined by the element $g_{n+1}(1) \in M_{n+1}$. On the other hand, given an element $y \in M_{n+1}$ we may define $\Psi_M : M_{n+1} \to \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M)$ by sending y to $g_y : D^{n+1} \to M$, where

$$
\begin{cases}
(g_y)_{n+1}(r) := ry, \quad \forall r \in R, \\
(g_y)_n(r) := r d_{n+1}^{M}(y), \qquad \forall r \in R, \\
(g_y)_j = 0, \quad \text{for } j \neq n, n+1.
\end{cases}
\tag{5.5.2}
$$

By using the module-structure on R we see that the maps above are R-linear. We then find that

$$
\begin{aligned}
\left(d_{n+1}^{M} \circ (g_y)_{n+1}\right)(r) &= d_{n+1}^{M}(ry) \\
&= r d_{n+1}^{M}(y) \\
&= (g_n)_y\left(\mathsf{id}_R(r)\right) \\
&= \left((g_n)_y \circ d_{n+1}^{D^{n+1}}\right)(r).
\end{aligned}
$$

One checks that g_y then in the other degrees defines a chain map $D^{n+1} \xrightarrow{g_y} M$ (observe that if $j \neq n, n+1$ then dg_y and $g_y d$ are maps *out of* $\left(D^{n+1}\right)_j = 0$). For $y \in M_{n+1}$ we have that

$$\begin{aligned}\Phi_M(\Psi_M(y)) &= \Phi_M(g_y)\\ &= (g_y)_{n+1}(1)\\ &= y, \qquad \text{by construction,}\end{aligned}$$

and for $D^{n+1} \xrightarrow{g} M$ chain map we have

$$\begin{aligned}\Psi_M(\Phi_M(g)) &= \Psi_M(g_{n+1}(1))\\ &= g_{g_{n+1}(1)},\end{aligned}$$

where

$$\begin{aligned}(g_{g_{n+1}(1)})_{n+1}(r) &= r g_{n+1}(1)\\ &= g_{n+1}(r), \qquad \text{by } R\text{-linearity,}\end{aligned}$$

and

$$\begin{aligned}(g_{g_{n+1}(1)})_{n}(r) &= r d^M_{n+1}(g_{n+1}(1))\\ &= r g_n\left(d^{D^{n+1}}_{n+1}(1)\right)\\ &= r g_n(\mathsf{id}_R(1))\\ &= g_n(r),\end{aligned}$$

and zero and all other degrees. Hence $\Psi_M(\Phi_M(g)) = g$. It is straightforward to check that Φ_M is R-linear and so it follows that (since Ψ_M is a mutual inverse) also Ψ_M is R-linear. Hence Φ_M is an isomorphism $\mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) \cong M_{n+1}$ of R-modules.

With respect to naturality, we check that the following square commutes, for every chain map $M \xrightarrow{\alpha} M'$ (one may check that $M \xmapsto{(\,\cdot\,)_{n+1}} M_{n+1}$ and $\alpha \xmapsto{(\,\cdot\,)_{n+1}} \alpha_{n+1}$ is a functor $\mathsf{Ch}_R \to \mathsf{Mod}_R$),

$$\begin{array}{ccccc} M & & \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M) & \xrightarrow{\Phi_M} & M_{n+1} \\ \big\downarrow{\scriptstyle\alpha} & \rightsquigarrow & \big\downarrow{\scriptstyle\alpha_*} & & \big\downarrow{\scriptstyle(\,\alpha\,)_{n+1}} \\ M' & & \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, M') & \xrightarrow[\Phi_{M'}]{} & M'_{n+1} \end{array}.$$

For a chain map $D^{n+1} \xrightarrow{g} M$ we have

$$(\alpha_{n+1} \circ \Phi_M)(g) = \alpha_{n+1}(g_{n+1}(1))$$

and

$$\begin{aligned}(\Phi_{M'} \circ \alpha_*)(g) &= (\alpha_*(g))_{n+1}(1)\\ &= (\alpha \circ g)_{n+1}(1)\\ &= \alpha_{n+1}(g_{n+1}(1)).\end{aligned}$$

Naturality follows. $\square$

We observe that the morphism $S^n \xhookrightarrow{i_n} D^{n+1}$ induces a morphism of functors (i.e. a natural transformation)

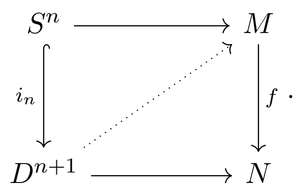
$$\mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, -) \xrightarrow{(i_n)^*} \mathsf{hom}_{\mathsf{Ch}_R}(S^n, -)$$

which by theorem 5.5.1 can be promoted to a natural transformation

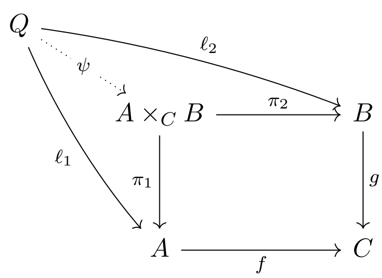
$$(\,\cdot\,)_{n+1} \xrightarrow{\theta} \mathsf{Z}_n(\,\cdot\,).$$

By analysing the proof of theorem 5.5.1 we see that for each object $M \in \mathsf{Ch}_R$, we get the map $M_{n+1} \xrightarrow{\theta_M} \mathsf{Z}_n(M)$ which is really $M_{n+1} \ni y \mapsto g_y \mapsto g_y \circ \iota_n \mapsto (g_y \circ \iota_n)_n(1) = d^M_{n+1}(y)$, i.e. $\theta_M = d^M_{n+1}$, or $\theta = d^{(\,\cdot\,)}_{n+1}$.

Definition 5.5.2 (Right lifting property)**.** Let $M \xrightarrow{f} N$ be a morphism in Ch_R. Then we say that f has the **right lifting property** *with respect to* i_n (or that i_n has the **left lifting property** with respect to f) if for every solid commutative diagram as below, there is a dashed lift such that the diagram below commutes.

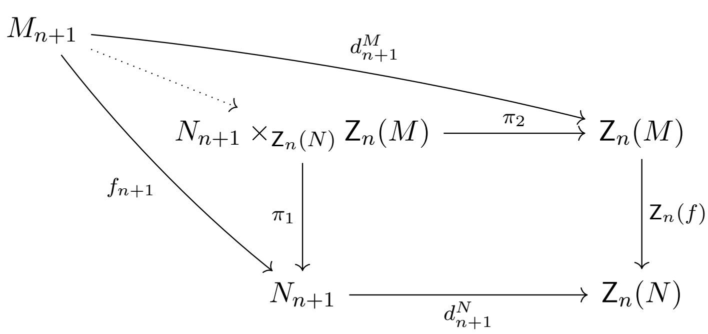


Recall that in Mod_R, the **pullback** of a diagram on the form $A \xrightarrow{f} C \xleftarrow{g} B$ (i.e. a *cospan*) is an object $A \times_C B := \{(a, b) \in A \oplus B \ : \ f(a) = g(b)\}$ together with pullback morphisms π_1, π_2 that satisfies the universal property that for any R-module Q and R-linear maps $\ell_1 :_Q\to A, \ell_2 : Q \to B$ such that $f \circ \ell_1 = g \circ \ell_2$, there exists a unique dashed R-linear map as indicated below, so that the diagram



commutes, i.e. so that $\pi_i \circ \psi = \ell_i$ for $i = 1, 2$.

Theorem 5.5.3. *A morphism $M \xrightarrow{f} N$ in Ch_R has the right-lifting property iff the dashed arrow indicated in the pullback-diagram below, is surjective.*



Proof. We first observe that the reason the "outer square" commutes in the pullback diagram above is due to f being a chain map. We further observe that the pullback is the kernel of the map $(y, z) \mapsto d^N_{n+1}(y) - \mathsf{Z}_n(f)(z)$ where recall (cf. 2.3.3) $\mathsf{Z}_n(f)$ is just the restriction to $\mathsf{Z}_n(M)$ of f_n. We let

$$\begin{aligned}\mathsf{P} &:= N_{n+1} \times_{\mathsf{Z}_n(N)} \mathsf{Z}_n(M) \\ &= \{(y, z) \in N_{n+1} \oplus \mathsf{Z}_n(M) \ : \ d^N_{n+1}(y) = f_n(z)\}.\end{aligned}$$

for brevity, and let $\theta : M_{n+1} \dashrightarrow \mathsf{P}$ be the dashed arrow in the pullback-diagram. It is straightforward to check (by using the projection maps π_1, π_2 defined by projection to the first or second factor of $\mathsf{P} \subset N_{n+1} \oplus \mathsf{Z}_n(M)$) that $\theta(m) = \big(f_{n+1}(m), d^M_{n+1}(m)\big)$.

$\underline{\Rightarrow}$: Let $(y, z) \in \mathsf{P}$ be arbitrary. Then

$$d^N_{n+1}(y) = f_n(z). \tag{5.5.3}$$

Consider the square

$$\begin{array}{ccc} S^n & \xrightarrow{\quad z \quad} & M \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow[\quad y \quad]{} & N \end{array}$$

where $S^n \to M$ is the chain map which sends 1 to $z \in \mathsf{Z}_n(M) \subset M_n$ in degree n and is zero otherwise. Furthermore recall from the proof of 5.5.1 that a chain map $D^{n+1} \to N$ is determined by where it sends 1 in degree $n + 1$, and we saw a way of turning an element $y \in N_{n+1}$ to a chain map $D^{n+1} \xrightarrow{g_y} N$. Hence the lower horizontal map is g_y, as defined in 5.5.2. Since S^n is zero in all other degrees than n, it is enough to check that the diagram commutes in degree n. We then have $(g_y)_n(r) = r d^M_{n+1}(y)$ by going down-right in the diagram, while going right-down we get $f_n(rz) = r f_n(z)$. These clearly agree by 5.5.3. Since f has the right lifiting property with respect to i_n, there exists a chain map $\psi : D^{n+1} \to M$ making the diagram above commute. Since this is chain a map *out of* D^{n+1}, it is determined by the element $\psi_{n+1}(1) = m \in M_{n+1}$. Furthermore, by the proof of theorem 5.5.1 we know that $\psi_n = d^M_{n+1} \circ \psi_{n+1}$. Hence it follows that

$$\psi_n(1) = d^M_{n+1}(m). \tag{5.5.4}$$

Since the diagram commutes, we have

$$\begin{aligned}(f \circ \psi)_{n+1}(1) &= (g_y)_{n+1}(1) \\ \Leftrightarrow f_{n+1}(m) &= y\end{aligned}$$

and

$$\begin{aligned}(i_n \circ \psi)_n(1) &= z(1) \\ \Leftrightarrow \psi_n(1) &= z \\ \Leftrightarrow d^M_{n+1}(m) &= z, \qquad \text{by 5.5.4.}\end{aligned}$$

But then we see that $\theta(m) = (y, z)$ (by definition). Since $(y, z) \in \mathsf{P}$ was arbitrary, θ is surjective.

$\Leftarrow$: Assume θ is surjective. Let

$$\begin{array}{ccc} S^n & \xrightarrow{h} & M \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow[q]{} & N \end{array}$$

be an arbitrary commuting square of chain maps. The map h is determined by $h_n(1) = z$ so in degree n we have that following the square right-down gives $1 \mapsto z \mapsto f_n(z)$ and following the square down-right gives $1 \mapsto 1 \mapsto q_n(1) = d^N_{n+1}(y)$ where $y = q_{n+1}(1)$. Therefore, we have $f_n(z) = d^N_{n+1}(y)$. Since h is a chain map and $d^{S^n}_n = 0$ we find that $z \in \mathsf{Z}_n(M)$ and so it follows that $(y, z) \in \mathsf{P}$. Since θ is surjective, there exists $m \in M_{n+1}$ such that $\theta(m) = (y, z)$. By definition, this means that $f_{n+1}(m) = y$ and $d^M_{n+1}(m) = z$.

We define a lift $\ell : D^{n+1} \to M$ to be the the chain map g_m as constructed in 5.5.2. We then observe that

$$\begin{aligned} (\ell \circ i_n)_n(1) &= \ell_n(1) \\ &= d^M_{n+1}(m) \\ &= z \\ &= h_n(1). \end{aligned}$$

Since maps out of S^n are completely determined by where they send 1 in degree n, it follows that $\ell \circ i_n = h$. Furthermore, we have

$$\begin{aligned} (f \circ \ell)_{n+1}(1) &= f_{n+1}(m) \\ &= y \\ &= q_{n+1}(1). \end{aligned}$$

Therefore $f \circ \ell = q$. Hence ℓ is a lift. $\square$

We may abbreviate that f has the right-lifting property with respect to i_n as $f \in \mathsf{rlp}(i_n)$ or $i_n \in \mathsf{llp}(f)$. We introduce the notation $I := \{i_n\}_{n\in\mathbb{Z}}$ and $\mathsf{rlp}(I) := \bigcap_{n\in\mathbb{Z}} \mathsf{rlp}(i_n)$.

Theorem 5.5.4. *If $M \xrightarrow{f} N \in \mathsf{rlp}(i_n)$ then $\mathsf{H}_n(f)$ is injective and $\mathsf{Z}_{n+1}(f)$ is surjective.*

Proof. Since $f \in \mathsf{rlp}(i_n)$, by theorem 5.5.3 the dashed arrow

$$\theta : M_{n+1} \to N_{n+1} \times_{\mathsf{Z}_n(N)} \mathsf{Z}_n(M) =: \mathsf{P}$$

is surjective, and $\theta(m) = \big(f_{n+1}(m), d^M_{n+1}(m)\big)$.

<u>Injectivity of $\mathsf{H}_n(f)$</u>: Let $[m] \in \mathsf{H}_n(M)$ for some $m \in \mathsf{Z}_n(M)$ such that $\mathsf{H}_n(f)([m]) = [f_n(m)] = 0$. This means that there is some $y \in N_{n+1}$ such that $d^N_{n+1}(y) = f_n(m)$. It follows that $(y, m) \in \mathsf{P}$, so by surjectivity of θ there is some $x \in M_{n+1}$ such that $\theta(x) = (y, m)$. Since the pullback-diagram with θ commutes, it follows that $d^M_{n+1}(x) = m$ so that $m \in \mathsf{B}_n(M)$, hence $[m] = 0$.

<u>Surjectivity of $\mathsf{Z}_{n+1}(f)$</u>: Let $y \in \mathsf{Z}_{n+1}(N) \subseteq N_{n+1}$ be arbitrary, such that $d^N_{n+1}(y) = 0$. Then $(y, 0) \in \mathsf{P}$. By surjectivity of θ there is an $m \in M_{n+1}$ such that $\theta(m) = (y, 0)$ and such that $f_{n+1}(m) = y$ and $d^M_{n+1}(m) = 0$. The latter identity tells us that $m \in \mathsf{Z}_{n+1}(M)$ (so is in the domain of $\mathsf{Z}_{n+1}(f)$), and the former identity tells us that $\mathsf{Z}_{n+1}(f)(m) = y$. $\square$

Corollary 5.5.5. *If $M \xrightarrow{f} N$ is in $\mathsf{rlp}(I)$, then f is a quasi-isomorphism.*

Proof. Fix $n \in \mathbb{Z}$. Since $f \in \mathsf{rlp}(i_n)$ theorem 5.5.4 tells us that $\mathsf{H}_n(f)$ is injective. Since $f \in \mathsf{rlp}(i_{n-1})$, theorem 5.5.4 gives that $\mathsf{Z}_n(f)$ is surjective. It follows (to convince yourself, recall the proof of theorem 2.3.3) that $\mathsf{H}_n(f)$ is surjective. Taken together this implies that $\mathsf{H}_n(f)$ is an isomorphism. Since $n \in \mathbb{Z}$ was arbitrary, f is a quasi-isomorphism. $\square$

Corollary 5.5.6. *If $M \xrightarrow{f} N$ is in $\mathsf{rlp}(I)$, then $M_n \xrightarrow{f_n} N_n$ is surjective for each $n \in \mathbb{Z}$.*

Proof. Fix $n \in \mathbb{Z}$. Then $f \in \mathsf{rlp}(i_n)$. By theorem 5.5.3 it follows that the dashed pullback map $\theta : M_{n+1} \to N_{n+1} \times_{\mathsf{Z}_n(N)} \mathsf{Z}_n(M) =: \mathsf{P}$ is surjective. Let $y \in N_{n+1}$ be arbitrary. Since $f \in \mathsf{rlp}(i_{n-1})$ it follows that $\mathsf{Z}_n(f)$ is surjective by theorem 5.5.4. Hence we may choose $z \in \mathsf{Z}_n(M)$ such that $f_n(z) = d^N_{n+1}(y) \in \mathsf{Z}_n(N)$. Then $(y, z) \in \mathsf{P}$ so there is some $m \in M_{n+1}$ such that $\theta(m) = \big(f_{n+1}(m), d^M_{n+1}(m)\big) = (y, z)$, i.e. $f_{n+1}(m) = y$. Therefore, $M_{n+1} \xrightarrow{f_{n+1}} N_{n+1}$ is surjective. Since n was arbitrary, f is level-wise surjective. $\square$

The theorem below shows the converse to Corollaries 5.5.5 and 5.5.6, i.e. so that $\mathsf{rlp}(I)$ consists precisely of quasi-isomorphism $M \xrightarrow{f} N$ that are *level*-wise surjective.

Theorem 5.5.7. *If $M \xrightarrow{f} N$ is a level-wise surjective quasi-isomorphism, then $f \in \mathsf{rlp}(I)$.*

Proof. Let $K := \mathsf{ker}(f)$. We have a short exact sequence of complexes

$$0 \to K \overset{\iota}{\hookrightarrow} M \xrightarrow{f} N \to 0.$$

By theorem 3.4.5 we have an associated long exact sequence in homology

$$\cdots \to \mathsf{H}_{n+1}(M) \xrightarrow{\mathsf{H}_{n+1}(f)} \mathsf{H}_{n+1}(N) \xrightarrow{\partial_{n+1}} \mathsf{H}_n(K) \xrightarrow{\mathsf{H}_n(\iota)} \mathsf{H}_n(M) \xrightarrow{\mathsf{H}_n(f)} \mathsf{H}_n(N) \to \cdots,$$

with ∂ the connecting homomorphisms. Since f is a quasi-isomorphism, $\mathsf{H}_n(f)$ is an isomorphism for all n so by exactness it follows that $\mathsf{H}_n(\iota) : \mathsf{H}_n(K) \to \mathsf{H}_n(M)$ has both zero image and zero kernel, from which one may conclude that $\mathsf{H}_n(K) = 0$ for all n, i.e. K is *acyclic.*

Let $n \in \mathbb{Z}$ be arbitrary. To show that $f \in \mathsf{rlp}(i_n)$, it is by theorem 5.5.3 enough to show that the dashed arrow $\theta : M_{n+1} \to N_{n+1} \times_{\mathsf{Z}_n(N)} \mathsf{Z}_n(M) =: \mathsf{P}$ defined by $\theta(m) = \big(f_{n+1}(m), d^M_{n+1}(m)\big)$ is *surjective.* Let $(y, z) \in \mathsf{P}$ be arbitrary, so that $d^N_{n+1}(y) = f_n(z)$. Since f is levelwise surjective, choose $x \in M_{n+1}$ such that $f_{n+1}(x) = y$, and consider the element $d^M_{n+1}(x) - z \in M_n$. Since f is a chain map, and by how x, y, z were defined we see that $d^M_{n+1}(x) - z \in \mathsf{ker}(f_n) = K_n$. Furthermore,

$$d^M_n\left(d^M_{n+1}(x) - z\right) = d^M_n d^M_{n+1}(x) - d^M_n(z) = 0,$$

since z is a cycle and $d^2 = 0$. It follows that $d^M_{n+1}(x) - z \in \mathsf{Z}_n(K) = \mathsf{B}_n(K)$, which means that there is some $t \in K_{n+1}$ such that $d^M_{n+1}(t) = d^M_{n+1}(x) - z$. Let $m = x - t$. Then we see that

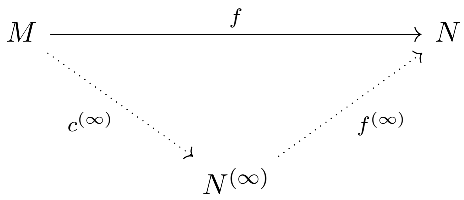
$$\begin{aligned} \theta(m) &= \left(f_{n+1}(x - t), d^M_{n+1}(x - t)\right) \\ &= (f_{n+1}(x) - f_{n+1}(t), z) \\ &= (y, z), \qquad \text{since } t \in K_{n+1} = \mathsf{ker}(f_{n+1}). \end{aligned}$$

Hence θ is surjective. We conclude that $f \in \mathsf{rlp}(i_n)$. Since $n \in \mathbb{Z}$ was arbitrary, we conclude that $f \in \mathsf{rlp}(I)$. $\square$

5.6 Projective resolutions as cellular approximations

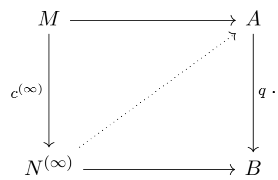
5.6.1 Introducing factorizations and arrow categories

In this section, we will construct projective resolutions $P \xrightarrow{\approx} M$ by a procedure called a **small object argument**. Given a morphism $M \xrightarrow{f} N$ in Ch_R, we will construct a **factorization** as indicated below,



such that $c^{(\infty)}$ and $f^{(\infty)}$ satisfies the following:

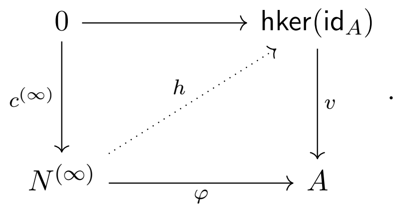
(a) The morphism $c^{(\infty)}$ has the left lifting property with respect to level-wise surjective quasi-isomorphisms, that is, $c^{(\infty)} \in \mathsf{llp}(\mathsf{rlp}(I))$. Said in another way, for every solid diagram on the form below, with $A \xrightarrow{q} B$ a levelwise surjective quasi-isomorphism, there exists a dashed lift as indicated.



(b) The morphism $f^{(\infty)}$ is a quasi-isomorphism.

Theorem 5.6.1. *Given a factorization of* $0 \to N$ *as above, it follows that* $N^{(\infty)}$ *is projective and that* $N^{(\infty)} \xrightarrow{f^{(\infty)}} N$ *is a projective resolution.*

Proof. Since $f^{(\infty)}$ is a quasi-isomorphism, that $f^{(\infty)}$ is a projective resolution follows directly if we can show that $N^{(\infty)}$ is projective. To that end, let $A \in \mathbf{A}_R$ be acyclic and let $N^{(\infty)} \xrightarrow{\varphi} A$ be an arbitrary chain map. By remark 5.2.4 we see that $\mathsf{hker}(\mathsf{id}_A) \to A$ is a levelwise surjective quasi-isomorphism. Therefore, since $c^{(\infty)} \in \mathsf{llp}(\mathsf{rlp}(I))$ and since the solid diagram below clearly commutes (both are chain maps $0 \to A$), there is a dashed lift h as indicated below,



By theorem 5.2.3, it follows that $N^{(\infty)}$ is projective. $\square$

Definition 5.6.2 (Arrow category)**.** Given a category $\mathcal{C}$, we define the associated **arrow category** $\mathsf{Arr}(\mathcal{C}) := \mathsf{Fun}([1], \mathcal{C})$ as the functor category, with objects functors $[1] \xrightarrow{F} \mathcal{C}$ where $[1]$ is the category with two objects $0, 1$ and one arrow $0 \to 1$, and where $\mathsf{Arr}(\mathcal{C})$ has as morphisms natural transformations between functors. Composition is defined as *vertical* composition of natural transformations (cf. [Rie16, Lemma 1.7.1]).

From the definition of $\mathsf{Arr}(\mathcal{C})$ we see that a morphism $F \overset{\alpha}{\Rightarrow} G$ in $\mathsf{Arr}(\mathcal{C})$ between functors $F, G : [1] \rightrightarrows \mathcal{C}$ is the data of a commutative square in $\mathcal{C}$ as below,

$$\begin{array}{ccccc} 0 & & F(0) & \xrightarrow{\alpha_0} & G(0) \\ \Big\downarrow{\scriptstyle *} & \rightsquigarrow & {\scriptstyle F(*)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(*)} \\ 1 & & F(1) & \xrightarrow[\alpha_1]{} & G(1) \end{array} . \tag{5.6.1}$$

If we consider $\mathsf{Arr}(\mathsf{Ch}_R)$, then we see that any chain map $M \xrightarrow{f} N$ determines a functor $F_f : [1] \to \mathsf{Ch}_R$ by letting $F_f(0) = M, F_f(1) = N, F_f(*) = f, F_f(\mathsf{id}_0) = \mathsf{id}_M$ and $F_f(\mathsf{id}_1) = N$. Hence we may in particular consider $\mathsf{hom}_{\mathsf{Arr}(\mathsf{Ch}_R)}(F_{i_n}, F_f)$ for any chain map $M \xrightarrow{f} N$ and $S^n \overset{i_n}{\hookrightarrow} D^{n+1}$. By diagram 5.6.1 morphisms α that belong to $\mathsf{hom}_{\mathsf{Arr}(\mathsf{Ch}_R)}(F_{i_n}, F_f)$ are precisely pairs of chain maps (α_0, α_1) such that the following diagram commutes,

$$\begin{array}{ccc} S^n & \xrightarrow{\alpha_0} & M \\ {\scriptstyle i_n}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow[\alpha_1]{} & N \end{array} .$$

We introduce the notation

$$\begin{aligned} I(f)_n &:= \mathsf{hom}_{\mathsf{Arr}(\mathsf{Ch}_R)}(F_{i_n}, F_f) \\ &\cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \times_{\mathsf{hom}_{\mathsf{Ch}_R}(S^n, N)} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N). \end{aligned}$$

The latter isomorphism is clear to see by taking the pullback (in Set or Mod_R) of the cospan

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, M) \xrightarrow{(f)_*} \mathsf{hom}_{\mathsf{Ch}_R}(S^n, N) \xleftarrow{(i_n)^*} \mathsf{hom}_{\mathsf{Ch}_R}(D^{n+1}, N).$$

Furthermore, we let $I(f) := \coprod_{n \in \mathbb{Z}} I(f)_n$. If $\beta \in I(f)$ then we write $n_\beta := n$ if $\beta \in I(f)_n$, i.e. we have the evident map $I(f) \xrightarrow{n_{(\,\cdot\,)}} \mathbb{Z}$ that takes an element $\beta = (n, \alpha) \in I(f)$ to n.

5.6.2 Construction of factorization $f^{(\infty)} \circ c^{(\infty)}$

We now aim to construct the factorization $f^{(\infty)} \circ c^{(\infty)}$ with respect to an arbitrary chain map $M \xrightarrow{f} N$ in Ch_R.

Theorem 5.6.3. *The category of chain complexes* Ch_R *has all small limits and colimits, i.e. is complete and cocomplete.*

Remark 5.6.4. We will take it as given that Mod_R is co/complete in the proof (sketch) below. There are several way to prove the above statement, but most proofs seem (naturally) to rely on knowing that Mod_R is co/complete.

Proof. Sketch: Cocomplete: If we take it as given that Ch_R is *abelian*, and that Mod_R is cocomplete, then in particular Mod_R has all set-indexed coproducts (i.e. direct sums) $\bigoplus_{a\in A} M_a$. By extension, Ch_R has all Set-indexed direct sums. By [Wei94, Prop. 2.6.8.(1) $\Rightarrow$ (2)] the conclusion follows.

Complete: Dual argument to the one for cocomplete in that (see [Wei94, Variation 2.6.9 (Limits)]) using that Mod_R has all Set-indexed products, by extension Ch_R has all Set-indexed products, and so the conclusion follows. □

We will furthermore take it as given that the direct sum plays the role of *coproduct* in Ch_R (analogous to $\oplus$ in Mod_R). Observe that each $I(f)_n$ as defined, lives in Set, and so hence so does $I(f)$ (it is a $\mathbb{Z}$-indexed disjoint union of sets).

It follows that $\bigoplus_{\alpha\in I(f)} S^{n_\alpha}, \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1}$ live in Ch_R. Now if $\alpha \in I(f)$ there is a *unique* integer n_α so that $\alpha \in I_{n_\alpha}$, which means that there are morphisms s_α and g_α as indicated in the diagram below, so that the diagram commutes,

$$\begin{array}{ccc} S^{n_\alpha} & \xrightarrow{s_\alpha} & M \\ {\scriptstyle i_{n_\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^{n_\alpha+1} & \xrightarrow[g_\alpha]{} & N \end{array} . \tag{5.6.2}$$

By the universal property of the coproduct applied to the diagrams

$$\left(\begin{array}{ccc} M & & \\ {\scriptstyle s}\uparrow & \nwarrow^{s_\alpha} & \\ \bigoplus_{\alpha\in I(f)} S^{n_\alpha} & \xleftarrow[k_\alpha]{} & S^{n_\alpha} \end{array}\right)_{\alpha\in I(f)}, \quad \left(\begin{array}{ccc} N & & \\ {\scriptstyle g}\uparrow & \nwarrow^{g_\alpha} & \\ \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} & \xleftarrow[\lambda_\alpha]{} & D^{n_\alpha+1} \end{array}\right)_{\alpha\in I(f)} \tag{5.6.3}$$

and

$$\left(\begin{array}{ccc} \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} & & \\ {\scriptstyle i}\uparrow & \nwarrow^{\lambda_\alpha\circ i_{n_\alpha}} & \\ \bigoplus_{\alpha\in I(f)} S^{n_\alpha} & \xleftarrow[k_\alpha]{} & S^{n_\alpha} \end{array}\right)_{\alpha\in I(f)}, \tag{5.6.4}$$

with $(k_\alpha)_{\alpha\in I(f)}$ and $(\lambda_\alpha)_{\alpha\in I(f)}$ the families of coproduct maps, we get uniquely determined maps

$$\begin{cases} \bigoplus_{\alpha\in I(f)} S^{n_\alpha} \xrightarrow{s} M \\ \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} \xrightarrow{g} N \\ \bigoplus_{\alpha\in I(f)} S^{n_\alpha} \xrightarrow{i} \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1}. \end{cases}$$

We claim that the associated square

$$\begin{array}{ccc} \bigoplus_{\alpha\in I(f)} S^{n_\alpha} & \xrightarrow{s} & M \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} & \xrightarrow[g]{} & N \end{array} \tag{5.6.5}$$

commutes: We find that

$$\begin{aligned}
f \circ \underbrace{s \circ k_\alpha}_{=s_\alpha} &= f \circ s_\alpha \\
&= \underbrace{g_\alpha}_{=g\circ\lambda_\alpha} \circ i_{n_\alpha}, && \text{since diagram 5.6.2 commutes} \\
&= g \circ \underbrace{\lambda_\alpha \circ i_{n_\alpha}}_{=i\circ k_\alpha} \\
&= g \circ i \circ k_\alpha.
\end{aligned}$$

Hence, for every $\alpha \in I(f)$, we see that $(f \circ s) \circ k_\alpha = (g \circ i) \circ k_\alpha$. By another application of the universal property of the coproduct, it follows that $f \circ s = g \circ i$, i.e. the square 5.6.5 commutes.

We let

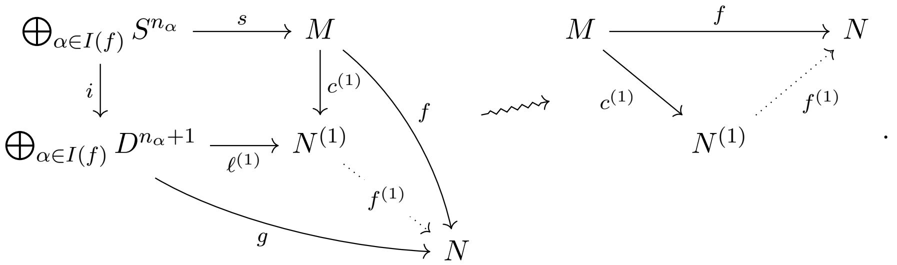
$$N^{(1)} := \mathsf{colim} \left(\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \xrightarrow{s} & M \\ {\scriptstyle i}\downarrow & & \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & & \end{array} \right). \tag{5.6.6}$$

Remark 5.6.5. Observe that for the statement that $N^{(1)}$ actually exists in Ch_R, one may use the weaker (as compared to theorem 5.6.3) theorem 3.1.12.

Then by definition 3.1.11, we get specified morphisms $c^{(1)}$ and $\ell^{(1)}$ as indicated below, so that the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha \in I(f)} S^{n_\alpha} & \xrightarrow{s} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle c^{(1)}} \\ \bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} & \xrightarrow[\ell^{(1)}]{} & N^{(1)} \end{array}, \tag{5.6.7}$$

commutes. Furthermore by the same definition (propert (b)) we get an induced $f^{(1)}$ as indicated below, so that the diagram to the left below commutes, leading to the factorization to the right.



. (5.6.8)

Theorem 5.6.6. *The morphism $M \xrightarrow{c^{(1)}} N^{(1)}$ has the left lifting property with respect to any levelwise surjective quasi-isomorphism. That is, $c^{(1)} \in \mathsf{llp}(\mathsf{rlp}(I))$.*

Proof. Let $X \xrightarrow{q} Y$ belong to $\mathsf{rlp}(I)$, equivalently, q is a levelwise surjective quasi-isomorphism, and assume we have a commuting square on the form

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & X \\ {\scriptstyle c^{(1)}}\downarrow & & \downarrow{\scriptstyle q} \\ N^{(1)} & \xrightarrow[\psi]{} & Y \end{array} . \tag{5.6.9}$$

We want to show the existence of a lift $N^{(1)} \to X$. To this end, consider the following square

$$\begin{array}{ccc} S^{n_\alpha} & \xrightarrow{\varphi\circ s\circ k_\alpha} & X \\ {\scriptstyle i_{n_\alpha}}\downarrow & & \downarrow{\scriptstyle q} \\ D^{n_\alpha+1} & \xrightarrow[\psi\circ\ell^{(1)}\circ\lambda_\alpha]{} & Y \end{array} .$$

We have that

$$\begin{aligned} q\circ\varphi\circ s\circ k_\alpha &= \psi\circ c^{(1)}\circ s\circ k_\alpha, && \text{since the square 5.6.9 commutes} \\ &= \psi\circ\ell^{(1)}\circ i\circ k_\alpha, && \text{since the square 5.6.7 commutes} \\ &= \psi\circ\ell^{(1)}\circ\lambda_\alpha\circ i_{n_\alpha}, && \text{since the diagram in 5.6.4 commutes for all } \alpha\in I(f). \end{aligned}$$

Since $q \in \mathsf{rlp}(I)$ in particular $q \in \mathsf{rlp}(i_{n_\alpha})$, so there exists a dashed lift $D^{n_\alpha+1} \xrightarrow{h_\alpha} X$ such that the diagram below commutes

$$\begin{array}{ccc} S^{n_\alpha} & \xrightarrow{\varphi\circ s\circ k_\alpha} & X \\ {\scriptstyle i_{n_\alpha}}\downarrow & \overset{h_\alpha}{\nearrow} & \downarrow{\scriptstyle q} \\ D^{n_\alpha+1} & \xrightarrow[\psi\circ\ell^{(1)}\circ\lambda_\alpha]{} & Y \end{array} . \tag{5.6.10}$$

By the universal property of the coproduct applied to the $I(f)$-indexed diagrams below, we get a dashed arrow h as below such that the diagrams commutes

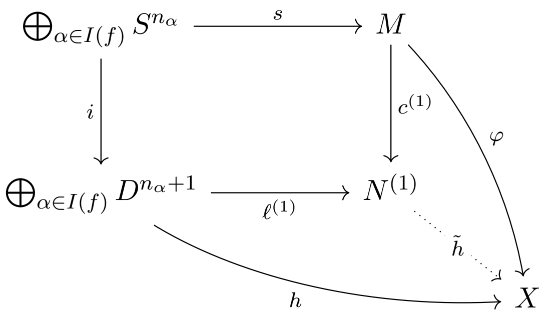
$$\left(\begin{array}{ccc} X & \xleftarrow{h_\alpha} & \\ {\scriptstyle h}\uparrow & & \\ \bigoplus_{\alpha\in I(f)} D^{n_\alpha+1} & \xleftarrow[\lambda_\alpha]{} & D^{n_\alpha+1} \end{array}\right)_{\alpha\in I(f)}, \tag{5.6.11}$$

i.e. so that $h\circ\lambda_\alpha = h_\alpha$ for all $\alpha \in I(f)$. We observe that

$$\begin{aligned} h\circ i\circ k_\alpha &= h\circ\lambda_\alpha\circ i_{n_\alpha}, && \text{by commutativity of diagram 5.6.4 for all } \alpha\in I(f) \\ &= h_\alpha\circ i_{n_\alpha} \\ &= \varphi\circ s\circ k_\alpha, && \text{since the associated diagram 5.6.10 commutes.} \end{aligned}$$

Since this hold for all α, by the universal property of the coproduct $\left(\bigoplus_{\alpha\in I(f)} S^{n_\alpha}, (k_\alpha)_{\alpha\in I(f)}\right)$

it follows that $h \circ i = \varphi \circ s$, i.e. the solid diagram commutes, giving us the dashed map $\tilde{h}$,



(5.6.12)

so that the full diagram commutes, i.e. $\tilde{h} \circ c^{(1)} = \varphi$ and $\tilde{h} \circ \ell^{(1)} = h$. We claim that $q \circ \tilde{h} = \psi$, i.e. we claim that $\tilde{h}$ is the sought after lift in 5.6.9. Observe that

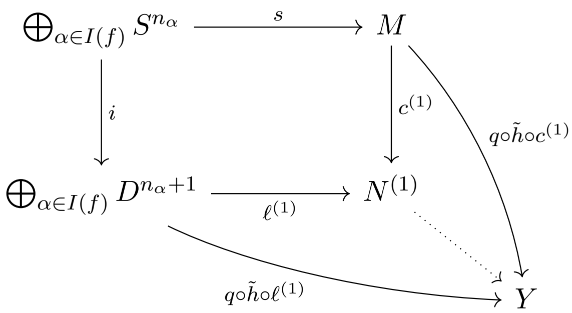
$$\begin{aligned} q \circ \tilde{h} \circ c^{(1)} &= q \circ \varphi \\ &= \psi \circ c^{(1)}, \qquad \text{by commutativity of 5.6.9,} \end{aligned}$$

and that $q \circ \tilde{h} \circ \ell^{(1)} = q \circ h$. We observe that

$$\begin{aligned} q \circ h \circ \lambda_\alpha &= q \circ h_\alpha, \qquad \text{by commutativity of 5.6.11 for } \alpha \in I(f) \\ &= \psi \circ \ell^{(1)} \circ \lambda_\alpha, \qquad \text{by commutativity of 5.6.10.} \end{aligned}$$

By the universal property of the coproduct it follows that $q \circ h = \psi \circ \ell^{(1)}$, i.e. $q \circ \tilde{h} \circ \ell^{(1)} = \psi \circ \ell^{(1)}$.

We check that the solid diagram



(5.6.13)

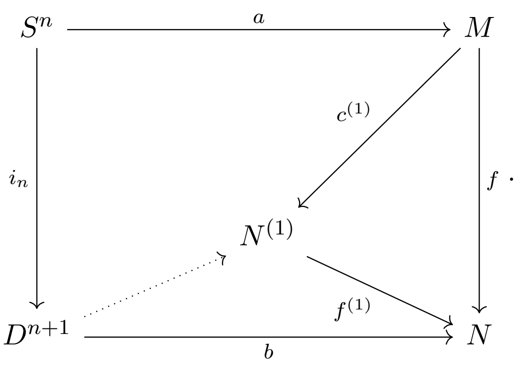
commutes. That the inner "inner square" commutes (i.e. that the pushout square commutes) is immediate from the commutativity of 5.6.12. Furthermore, we have that

$$q \circ \tilde{h} \circ \ell^{(1)} \circ i = q \circ \tilde{h} \circ c^{(1)} \circ s, \qquad \text{since } \ell^{(1)} \circ i = c^{(1)} \circ s.$$

Hence the outer square commutes so we get a unique dashed arrow as in 5.6.13 so that the diagram commutes. But by our earlier computations, both $q \circ \tilde{h}$ and ψ satisfies these conditions, hence $q \circ \tilde{h} = \psi$ by uniqueness of the dashed arrow $N^{(1)} \dashrightarrow Y$. It follows that $N^{(1)} \xrightarrow{\tilde{h}} X$ is the sought after lift in diagram 5.6.9. Hence $c^{(1)} \in \mathsf{llp}(\mathsf{rlp}(I))$. □

Theorem 5.6.7. *For any $n \in \mathbb{Z}$ and for any solid commutative diagram as below, there exists a dashed factorization $D^{n+1} \dashrightarrow N^{(1)}$ as indicated, making the full diagram com-*

mute,



(5.6.14)

Proof. We are given that $f \circ a = i_n \circ b$. Hence the outer commutative square is by definition an element $\alpha \in I(f)_n \subseteq I(f)$, so that $n = n_\alpha$ and consider that we have a coproduct inclusion $\lambda_\alpha : D^{n+1} \to \bigoplus_{\gamma \in I(f)} D^{n_\gamma+1}$. Letting $\ell^{(1)}$ be one of the structure morphisms (or pushout maps, say) as in the proof of 5.6.6, we then let ζ be the composition $\zeta := \left(D^{n+1} \xrightarrow{\lambda_\alpha} \bigoplus_{\alpha \in I(f)} D^{n_\alpha+1} \xrightarrow{\ell^{(1)}} N^{(1)} \right)$. We claim ζ is a factorization of diagram 5.6.14, i.e. is the requisite dashed arrow $D^{n+1} \dashrightarrow N^{(1)}$.

<u>The lower triangle in 5.6.14 commutes</u>: We need to show that $f^{(1)} \circ \zeta = b$. We have that

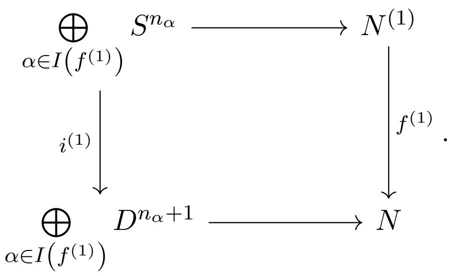
$$\begin{aligned} f^{(1)} \circ \zeta &= f^{(1)} \circ \ell^{(1)} \circ \lambda_\alpha \\ &= g \circ \lambda_\alpha, && \text{by commutativity of the left diagram in 5.6.8} \\ &= g_\alpha, && \text{by commutativity of the right diagram for this fixed } \alpha \text{ in 5.6.3} \\ &= b, && \text{by definition; c.f. diagram 5.6.2.} \end{aligned}$$

<u>The upper triangle in 5.6.14 commutes</u>: We need to show that $\zeta \circ i_n = c^{(1)} \circ a$. We compute that

$$\begin{aligned} \zeta \circ i_n &= \ell^{(1)} \circ \lambda_\alpha \circ i_n \\ &= \ell^{(1)} \circ i \circ k_\alpha, && \text{by commutativity of the } I(f)\text{-indexed diagrams in 5.6.4} \\ &= c^{(1)} \circ s \circ k_\alpha, && \text{since the square in 5.6.7 commutes} \\ &= c^{(1)} \circ s_\alpha, && \text{since the left } I(f)\text{-indexed squares in 5.6.3 commute} \\ &= c^{(1)} \circ a, && \text{since by definition } s_\alpha = a. \end{aligned}$$

The conclusion follows. □

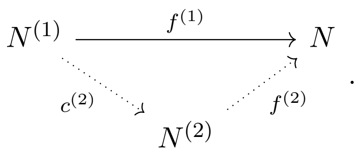
We reiterate this construction, but now applied to $f^{(1)}$ instead of f, giving us a commutative diagram



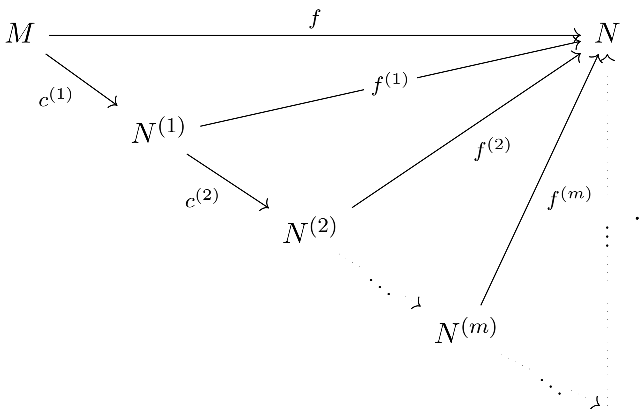
We may the take the colimit of the span below, setting

$$N^{(2)} := \mathsf{colim}\left(\begin{array}{ccc} \bigoplus_{\alpha\in I\left(f^{(1)}\right)} S^{n_\alpha} & \longrightarrow & N^{(1)} \\ {\scriptstyle i^{(1)}}\big\downarrow & & \\ \bigoplus_{\alpha\in I\left(f^{(1)}\right)} D^{n_\alpha+1} & & \end{array}\right).$$

With induced maps $N^{(1)} \xrightarrow{c^{(2)}} N^{(2)}$ and $N^{(2)} \xrightarrow{f^{(2)}} N$ which we put into a diagram as a factorization of $N^{(1)} \xrightarrow{f^{(1)}} N$ as



By repeated applicaton, observe that this gives a commutative diagram



(5.6.15)

We may then form the colimit

$$N^{(\infty)} := \mathsf{colim}\left(M \xrightarrow{c^{(1)}} N^{(1)} \xrightarrow{c^{(2)}} N^{(2)} \xrightarrow{c^{(3)}} \cdots\right).$$

We get a canonical structure morphism $M \xrightarrow{c^{(\infty)}} N^{(\infty)}$ into the colimit from M.

Furthermore, by property (b) of definition 3.1.11 applied to the object $A = N$ and maps $\left(f^{(i)}\right)_{i\in\mathbb{N}_{\geqslant 1}}$, which satisfy the required property since 5.6.15 commutes, it follows that we get a unique map $N^{(\infty)} \xrightarrow{f^{(\infty)}} N$ such that

$$f^{(\infty)} \circ \eta_m = f^{(m)} \tag{5.6.16}$$

where η_m are the canonical structure morphisms $N^{(m)} \xrightarrow{\eta_m} N^{(\infty)}$ with $N^{(0)} := M$ and $\eta_0 = c^{(\infty)}$.

We check that $c^{(\infty)}$ and $f^{(\infty)}$ satisfy property (a) and (b) as defined in §5.6.1.

(a): Assume $B \xrightarrow{g} C$ is levelwise surjective quasi-isomorphism (that is, $g \in \mathsf{rlp}(I)$), and that we have a commutative square

$$\begin{tikzcd} M \arrow[r, "a"] \arrow[d, "c^{(\infty)}"'] & B \arrow[d, "g"] \\ N^{(\infty)} \arrow[r, "b"'] & C \end{tikzcd} . \tag{5.6.17}$$

The diagram

$$\begin{tikzcd} M \arrow[r, "a"] \arrow[d, "c^{(1)}"'] & B \arrow[ddddddd, "g"] \\ N^{(1)} \arrow[d, "c^{(2)}"'] \arrow[ru, dotted, "a_1"] & \\ N^{(2)} \arrow[d, "c^{(3)}"'] \arrow[ruu, dotted, "a_2"] & \\ \vdots \arrow[ruuu, dotted] \arrow[d] & \\ N^{(m)} \arrow[d, "\eta_m"'] \arrow[ruuuu, dotted, "a_m"] & \\ N^{(\infty)} \arrow[r, "b"'] & C \end{tikzcd} \tag{5.6.18}$$

is meant to indicate that:

(1) By definition of the colimit (property (a) in definition 3.1.11), we have that $\eta_m \circ c^{(m)} \circ c^{(m-1)} \circ \cdots \circ c^{(1)} = \eta_0 = c^{(\infty)}$ for each $m \in \mathbb{N}_{\geqslant 1}$. Hence it follows that

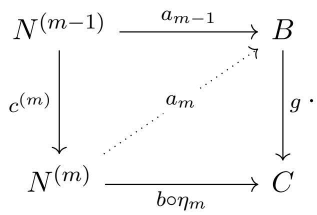
$$\begin{aligned} b \circ \eta_m \circ c^{(m)} \circ c^{(m-1)} \circ \cdots \circ c^{(1)} &= b \circ c^{(\infty)} \\ &= g \circ a, \qquad \text{since diagram 5.6.17 commutes.} \end{aligned}$$

(2) We construct the dashed lifts a_i in 5.6.18 inductively as follows: Start with the diagram

$$\begin{tikzcd} M \arrow[r, "a_0=a"] \arrow[d, "c^{(1)}"'] & B \arrow[d, "g"] \\ N^{(1)} \arrow[r, "b\circ\eta_1"'] \arrow[ru, dotted, "a_1"] & C \end{tikzcd}$$

where the solid diagram commutes by the computation in (1). By theorem 5.6.6 we get a lift a_1 as indicated, such that $g \circ a_1 = b \circ \eta_1$ and $a_1 \circ c^{(1)} = a$. Now suppose $N^{(m-1)} \xrightarrow{a_{m-1}} B$ have been constructed so that it satisfies $a_{m-1} \circ c^{(m-1)} = a_{m-2}$ and

$g \circ a_{m-1} = b \circ \eta_{m-1}$. Consider the solid square

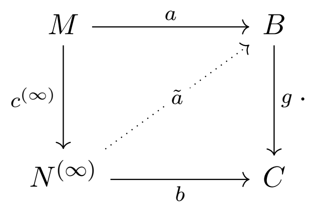


We have that

$$\begin{aligned} g \circ a_{m-1} &= b \circ \eta_{m-1} \\ &= b \circ \eta_m \circ c^{(m)}, \qquad \text{since } \eta_{m-1} = \eta_m \circ c^{(m)}. \end{aligned}$$

Hence the solid diagram commutes. Since $c^{(m)} \in \mathsf{llp}(\mathsf{rlp}(I))$ where $g \in \mathsf{rlp}(I)$ there exists a lift a_m such that $g \circ a_m = b \circ \eta_m$ and $a_m \circ c^{(m)} = a_{m-1}$.

By inductively [6] choosing such an a_m in each step, we get a family $(a_m)_{m \geqslant 0}$ of maps $N^{(m)} \xrightarrow{a_m} B$. By construction, for $r < m$, we have $a_r = a_m \circ c^{(m)} \circ \cdots \circ c^{(r+1)}$. By definition 3.1.11 (property (b)) it follows that there exists a unique map $N^{(\infty)} \xrightarrow{\tilde{a}} B$ such that $a_m = \tilde{a} \circ \eta_m$ for all $m \geqslant 0$. In particular, since recall, $\eta_0 = c^{(\infty)}$ and $a_0 = a$, we see that $a = \tilde{a} \circ c^{(\infty)}$. Also, for all $m \geqslant 0$, we see that $g \circ \tilde{a} \circ \eta_m = g \circ a_m = b \circ \eta_m$. Another application of 3.1.11 (property (b); with $f_i := b \circ \eta_i$) tells us that $g \circ \tilde{a} = b$. But taken together this means that $\tilde{a}$ is the sought after dashed lift



We conclude that $c^{(\infty)} \in \mathsf{llp}(\mathsf{rlp}(I))$.

(b): To show that $f^{(\infty)}$ is a quasi-isomorphism, we introduce a certain type of colimit, and prove a lemma.

Definition 5.6.8 (Filtered colimit; c.f. [KS26, Def. 2.6.2])**.** Let $\mathcal{I}$ be a category. Then we say that $\mathcal{I}$ is **directed** if the following holds:

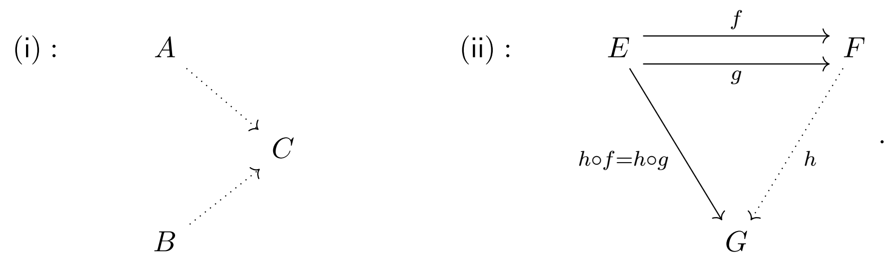
(i) $\mathcal{I}$ is non-empty, i.e. there is at least one object in $\mathcal{I}$.

(ii) For any pair of objects $A, B \in \mathsf{Ob}(\mathcal{I})$, there exists an object $C \in \mathsf{Ob}(\mathcal{I})$ and morphisms $A \to C$ and $B \to C$ in $\mathcal{I}$.

(iii) If $f, g : E \rightrightarrows F$ are a parallel pair of morphisms in $\mathcal{I}$, there is a morphism $h : F \to G$ in $\mathcal{I}$ such that $h \circ f = h \circ g$.

[6] If one wants to be more precise, one might say that we are using dependent choice with suitable set $X := \bigsqcup_{m \geqslant 0} \{u : N^{(m)} \to B \ : \ g \circ u = b \circ \eta_m\}$ and (entire) binary relation R on X such that $(m, u)R(m+1, v)$ iff $v \circ c^{(m+1)} = u$.

Condition (ii) and (iii) may be represented as below.



Given a $\mathcal{I}$-shaped diagram $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is *directed*, we say that the colimit $\mathsf{colim}_{\mathcal{I}}F$, if it exists, is a **filtered colimit**.

Definition 5.6.9 (Compact object)**.** An object S in a category $\mathcal{C}$ is **compact** if the associated functor $\mathsf{hom}_{\mathcal{C}}(S, -) : \mathcal{C} \to \mathsf{Set}$ *commutes with filtered* colimits, meaning that whenever we have $\mathcal{I}$-indexed diagram $F : \mathcal{I} \to \mathcal{C}$ valued in $\mathcal{C}$, where $\mathcal{I}$ is directed, then the filtered colimit (when it exists) $\mathsf{colim}_{\mathcal{I}}F$ is such that

$$\mathsf{hom}_{\mathcal{C}}\left(S, \mathsf{colim}_{\mathcal{I}}F\right) \cong \mathsf{colim}_{i\in\mathcal{I}}\,\mathsf{hom}_{\mathcal{C}}(S, F(i)).$$

To prove the actual lemma we want to use to show that $f^{(\infty)}$ is a quasi-isomorphism, we first want to give the following lemma.

Lemma 5.6.10. *Colimits are computed degreewise in Ch_R. That is, if $F : \mathcal{I} \to \mathsf{Ch}_R$ is a diagram whose associated colimit $\mathsf{colim}_{\mathcal{I}}F$ exists in Ch_R, then*

$$(\mathsf{colim}_{\alpha\in\mathcal{I}}F(\alpha))_k \cong \mathsf{colim}_{\alpha\in\mathcal{I}}F(\alpha)_k, \qquad \forall k \in \mathbb{Z}. \tag{5.6.19}$$

Proof. Let $C := \mathsf{colim}_{\alpha\in\mathcal{I}}F(\alpha)$, together with structure morphisms $q^\alpha : F(\alpha) \to C$. In degree k, we get a map $q^\alpha_k : F(\alpha)_k \to C_k$. For any morphism $u : \alpha \to \beta$ in $\mathcal{I}$, property (a) in definition 3.1.11 tells us that $q^\beta \circ F(u) = q_\alpha$. In degree k, this means that $q^\beta_k \circ F(u)_k = q^\alpha_k$ in Mod_R. Let $P_k := \mathsf{colim}_{\alpha\in\mathcal{I}}F(\alpha)_k$ (for future purposes do this for all $k \in \mathbb{Z}$) with structure maps $p^\alpha_k : F(\alpha)_k \to P_k$ such that for every morphism $\alpha \xrightarrow{u} \beta$ we have that $p^\beta_k \circ F(u)_k = p^\alpha_k$. Then we see that there is a unique map $\theta_k : P_k \to C_k$, from property (b) of definition 3.1.11, such that

$$q^\alpha_k = \theta_k \circ p^\alpha_k, \qquad \forall \alpha \in \mathcal{I}. \tag{5.6.20}$$

Now set $P := (P_k)_{k\in\mathbb{Z}}$. We want associate with P some system of differentials which make P a complex in Ch_R. To that end, consider, for fixed $j \in \mathbb{Z}$, that the system of maps $h_\alpha := p^\alpha_{j-1} \circ d^{F(\alpha)}_j : F(\alpha)_j \to P_{j-1}$ is a family of maps indexed by $\mathcal{I}$ in degree j, which satisfies property (b) of the colimit (use that $F(u)$ is a chain map for all $\alpha \xrightarrow{u} \beta$ in $\mathcal{I}$), so this induces a *unique* map $d^P_j : P_j \to P_{j-1}$, such that

$$\begin{aligned} d^P_j \circ p^\alpha_j &= h_\alpha \\ &= p^\alpha_{j-1} \circ d^{F(\alpha)}_j, \qquad \text{by definition.} \end{aligned} \tag{5.6.21}$$

For each $\alpha \in \mathcal{I}$, we then have

$$\begin{aligned} d^P_{j-1} \circ d^P_j \circ p^\alpha_j &= d^P_{j-1} \circ p^\alpha_{j-1} \circ d^{F(\alpha)}_j, \qquad \text{by 5.6.21} \\ &= p^\alpha_{j-2} \circ d^{F(\alpha)}_{j-1} \circ d^{F(\alpha)}, \qquad \text{by another application of 5.6.21} \\ &= 0, \qquad \text{since } p^\alpha_{j-2} \text{ is a homomorphism and } \left(d^{F(\alpha)}\right)^2 = 0. \end{aligned}$$

Hence $\left(P, \left(d_j^P\right)_{j\in\mathbb{Z}}\right)$ is a complex in Ch_R. Furthermore, the maps p_j^α assemble into a chain map $F(\alpha) \to P$ by 5.6.21. Since $p^\beta \circ F(u) = p^\alpha$ holds (it holds degreewise!) it follows from the universal property of C that there is a unique map $\rho : C \to P$ such that

$$\rho \circ q^\alpha = p^\alpha \tag{5.6.22}$$

for all $\alpha \in \mathcal{I}$. We claim that the θ_k assemble into a chain map $\theta : P \to C$: We want to show that $d_j^C \circ \theta_j = \theta_{j-1} \circ d_j^P$. By the universal property of the colimit P_j it is (for each j) enough to check this after precomposing with the structure morphisms p_j^α for each $\alpha \in \mathcal{I}$. We then see that

$$\begin{aligned} d_j^C \circ \theta_j \circ p_j^\alpha &= d_j^C \circ q_j^\alpha, && \text{by 5.6.20} \\ &= q_{j-1}^\alpha \circ d_j^{F(\alpha)}, && \text{since } F(\alpha) \xrightarrow{q^\alpha} C \text{ is a chain map,} \end{aligned}$$

while

$$\begin{aligned} \theta_{j-1} \circ d_j^P \circ p_j^\alpha &= \theta_{j-1} \circ p_{j-1}^\alpha \circ d_j^{F(\alpha)}, && \text{since } p^\alpha \text{ is a chain map by 5.6.21} \\ &= q_{j-1}^\alpha \circ d_j^{F(\alpha)}, && \text{by 5.6.20.} \end{aligned}$$

Comparison gives that the computations above agree, so that $\theta := (\theta_k)_{k\in\mathbb{Z}}$ is indeed a chain map. We show that θ and ρ are mutual inverses.

$\underline{\theta \circ \rho = \mathsf{id}_C}$: C is a colimit in Ch_R, and for every $\alpha \in \mathcal{I}$ we have that

$$\begin{aligned} \theta \circ \rho \circ q^\alpha &= \theta \circ p^\alpha, && \text{by 5.6.22} \\ &= q^\alpha, && \text{by 5.6.20 since it is enough to check degreewise.} \end{aligned}$$

By the universal property of C it follows that $\theta \circ \rho = \mathsf{id}_C$.

$\underline{\rho \circ \theta = \mathsf{id}_P}$: We check this degreewise. For fixed $k \in \mathbb{Z}$, we precompose by p_k^α and see that

$$\begin{aligned} (\rho \circ \theta)_k \circ p_k^\alpha &= \rho_k \circ \theta_k \circ p_k^\alpha \\ &= \rho_k \circ q_k^\alpha, && \text{by 5.6.20} \\ &= p_k^\alpha, && \text{by 5.6.22.} \end{aligned}$$

Since P_k is a colimit, with structure maps p_k^α it follows (for essentially the same reason as for $\theta \circ \rho = \mathsf{id}_C$ but in Mod_R) that $\rho_k \circ \theta_k = \mathsf{id}_{P_k}$. Since k was arbitrary it follows that $\rho \circ \theta = \mathsf{id}_P$.

We conclude that $C \cong P$. Degreewise, this is precisely the statement 5.6.19. □

Lemma 5.6.11. *The object S^n in Ch_R is compact, for all $n \in \mathbb{Z}$.*

Proof. From theorem 5.5.1 we see that

$$\mathsf{hom}_{\mathsf{Ch}_R}(S^n, \mathsf{colim}_{\mathcal{I}} F) \cong \mathsf{Z}_n(\mathsf{colim}_{\mathcal{I}} F). \tag{5.6.23}$$

By lemma 5.6.10 it follows that, using the commuting square

$$\begin{array}{ccc} P_n := \mathsf{colim}_{\alpha\in\mathcal{I}} F(\alpha)_n & \xrightarrow{d_n^P} & \mathsf{colim}_{\alpha\in\mathcal{I}} F(\alpha)_{n-1} =: P_{n-1} \\ \cong\downarrow & \circlearrowleft & \downarrow\cong \\ (\mathsf{colim}_{\mathcal{I}} F)_n & \xrightarrow[d_n]{} & (\mathsf{colim}_{\mathcal{I}} F)_{n-1} \end{array},$$

we have $\mathsf{Z}_n(\mathsf{colim}_{\mathcal{I}}(F)) \cong \ker\left(P_n \xrightarrow{d_n^P} P_{n-1}\right)$. By [Rie16, Theorem 3.8.9] (using that the kernel in Mod_R is the *equalizer* $\mathsf{Eq}\left(d_n^P, 0\right)$, hence a *finite limit*), we find that

$$\begin{aligned}
\mathsf{Z}_n(\mathsf{colim}_{\mathcal{I}} F) &\cong \ker\left(P_n \xrightarrow{d_n^P} P_{n-1}\right) \\
&\cong \mathsf{colim}_{\alpha\in I} \ker\left(F(\alpha)_n \xrightarrow{d_n^{F(\alpha)}} F(\alpha)_{n-1}\right) \\
&= \mathsf{colim}_{\alpha\in\mathcal{I}} \mathsf{Z}_n(F(\alpha)).
\end{aligned} \tag{5.6.24}$$

Since $\mathsf{hom}_{\mathsf{Ch}_R}(S^n, F(-)) \approx \mathsf{Z}_n(F(\,\cdot\,))$ is natural (since $\mathsf{hom}_{\mathsf{Ch}_R}(S^n, -) \approx \mathsf{Z}_n(\,\cdot\,)$ is natural by theorem 5.5.1), it follows from [Rie16, Corollary 3.3.3] that

$$\begin{aligned}
\mathsf{colim}_{\alpha\in\mathcal{I}} \mathsf{hom}_{\mathsf{Ch}_R}(S^n, F(\alpha)) &\cong \mathsf{colim}_{\alpha\in\mathcal{I}} \mathsf{Z}_n(F(\alpha)) \\
&\cong \mathsf{Z}_n(\mathsf{colim}_{\mathcal{I}} F), \qquad \text{by } 5.6.24 \\
&\cong \mathsf{hom}_{\mathsf{Ch}_R}(S^n, \mathsf{colim}_{\mathcal{I}} F), \qquad \text{by } 5.6.23
\end{aligned}$$

This is what we wanted to show. □

Remark 5.6.12. We will take it as given that one may realize the (abstract) isomorphism given in the statement of 5.6.11 by the map $\left(S^n \xrightarrow{g} F(\alpha)\right) \mapsto \left(S^n \xrightarrow{g} F(\alpha) \xrightarrow{\eta_\alpha} \mathsf{colim}_{\mathcal{I}} F\right)$ where η_α are the canonical structure morphisms.

Going back to $f^{(\infty)}$: Recall that showing that $f^{(\infty)}$ is a quasi-isomorphism it is enough to show that $f^{(\infty)} \in \mathsf{rlp}(I)$ by Corollary 5.5.5. To that end, consider a commutative square on the form

$$\begin{array}{ccc}
S^n & \xrightarrow{a} & N^{(\infty)} \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle f^{(\infty)}} \\
D^{n+1} & \xrightarrow[b]{} & N
\end{array} . \tag{5.6.25}$$

Since $N^{(\infty)} = \mathsf{colim}_{k\in\mathbb{N}} N^{(k)}$ is a filtered colimit, by lemma 5.6.11 and 5.6.12 we find that

$$\mathsf{colim}_{k\in\mathbb{N}} \mathsf{hom}_{\mathsf{Ch}_R}\left(S^n, N^{(k)}\right) \cong \mathsf{hom}_{\mathsf{Ch}_R}\left(S^n, N^{(\infty)}\right), \left(S^n \xrightarrow{g} N^{(k)}\right) \mapsto \left(S^n \xrightarrow{g} N^{(k)} \xrightarrow{\eta_k} N^{(\infty)}\right).$$

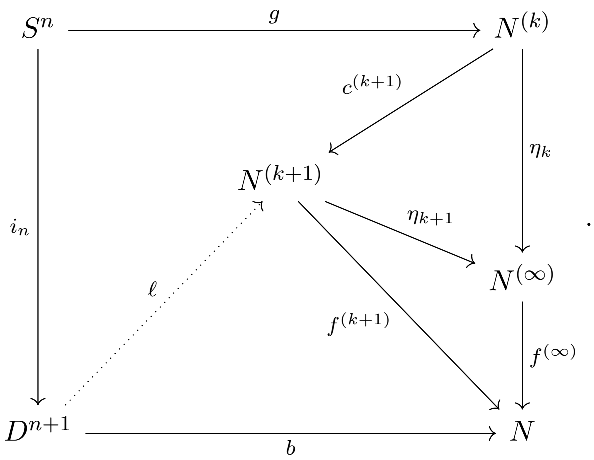
This means that the given map $S^n \xrightarrow{a} N^{(\infty)}$ in diagram 5.6.25 factors as

$$a = \eta_k \circ g \tag{5.6.26}$$

for some map $S^n \xrightarrow{g} N^{(k)}$. We then then have

$$\begin{aligned}
b \circ i_n &= f^{(\infty)} \circ a, \qquad \text{by commutativity of } 5.6.25 \\
&= f^{(\infty)} \circ \eta_k \circ g \\
&= f^{(k)} \circ g, \qquad \text{by } 5.6.16 \\
\Rightarrow b \circ i_n &= f^{(k)} \circ g.
\end{aligned}$$

Consider the *solid* part of the following diagram



The outer square commutes by the previous computation. The lower right triangle commutes by 5.6.16. The upper right triangle commutes since η_k are structure morphisms for the *colimit* $N^{(\infty)}$, and lastly we see that $f^{(k+1)} \circ c^{(k+1)} = f^{(\infty)} \circ \eta_k = f^{(k)}$ and $f^{(k+1)} \circ c^{(k+1)} = f^{(k)}$ follows from 5.6.15.

We claim one may then, completely analogous to the proof of 5.6.7, constuct a lift $D^{n+1} \xrightarrow{\ell} N^{(k+1)}$ such that $f^{(k+1)} \circ \ell = b$ and $\ell \circ i_n = c^{(k+1)} \circ g$. Set $\zeta := \eta_{k+1} \circ \ell$. Then

$$\begin{aligned} f^{(\infty)} \circ \zeta &= f^{(\infty)} \circ \eta_{k+1} \circ \ell \\ &= f^{(k+1)} \circ \ell \\ &= b, \end{aligned}$$

and

$$\begin{aligned} \zeta \circ i_n &= \eta_{k+1} \circ \ell \circ i_n \\ &= \eta_{k+1} \circ c^{(k+1)} \circ g \\ &= \eta_k \circ g \\ &= a, \qquad \text{by 5.6.26.} \end{aligned}$$

Hence ζ is the required lift of 5.6.25. Since n was arbitrary, $f^{(\infty)} \in \mathsf{rlp}(I)$.

Chapter 6

The derived ∞-category

6.1 Homotopy theory for $\mathbf{K}_R$, the dg-category of complexes

As we saw in §5, we have a diagram

$$\begin{array}{ccccc} \mathbf{P}_R & & & & \\ & \searrow & & & \\ & & \mathbf{K}_R & \hookleftarrow & \mathbf{A}_R \\ & \nearrow & & & \\ \mathbf{I}_R & & & & \end{array}$$

of *fully faithful* inclusions. Observe the order in which we defined them: We started with the full dg-subcategory of *acylic* complexes $\mathbf{A}_R$ and then defined what one might call (motivation forthcoming) its *homotopical left* and respectively, *right orthogonal subcategories* $\mathbf{P}_R$ and $\mathbf{I}_R$. The name "left" and "right" comes from the fact that, recall, $\mathbf{P}_R$ consists (objectwise) of all $M \in \mathbf{K}_R$ such that $\mathsf{hom}_{\mathbf{K}_R}(M, A)$ is acyclic for all acyclic complexes $A \in \mathbf{A}_R$, or said differently, $M \in \mathbf{K}_R$ so that for all acyclic complexes $A \in \mathbf{K}_R$ we have quasi-isomorphisms $\mathsf{hom}_{\mathbf{K}_R}(M, A) \xrightarrow{\approx} 0$. Dually, we stipulated that $\mathbf{I}_R$ consists of the objects $I \in \mathbf{K}_R$ such that for all acyclic complexes $A \in \mathbf{A}_R$ we have $\mathsf{hom}_{\mathbf{K}_R}(A, I) \xrightarrow{\approx} 0$. One may view this as a (*handed*) homotopically categorified notion of the concept of *orthogonality*, with respect to an inner product space $(V, \langle -, - \rangle)$ (although note that handedness introduces further distinctions as compared with say a *symmetric* inner product).

As we saw in §5 we have an associated projective resolution $M' \xrightarrow[f]{\approx} M$ for any object $M \in \mathbf{K}_R$. If we let $A := \mathsf{hker}(M' \to M)$, then we have a (by definition) homotopy kernel sequence $A \to M' \xrightarrow{f} M$.[1] For any projective object $P \in \mathbf{P}_R$, we then by theorem 4.3.4 get the associated homotopy kernel sequence

$$\mathsf{hom}_{\mathbf{K}_R}(P, A) \to \mathsf{hom}_{\mathbf{K}_R}(P, M') \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(P, f)} \mathsf{hom}_{\mathsf{Ch}_R}(P, M),$$

[1] N.B. Recall that this composition is not *strictly equal to zero but* it is nullhomotopic.

where we used that

$$\begin{aligned}\mathsf{hom}_{\mathbf{K}_R}(P,A) &= \mathsf{hom}_{\mathbf{K}_R}\left(P, \mathsf{hker}\left(M' \xrightarrow{f} M\right)\right)\\ &\cong \mathsf{hker}\left(\mathsf{hom}_{\mathbf{K}_R}(P,M') \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(P,f)} \mathsf{hom}_{\mathbf{K}_R}(P,M)\right).\end{aligned}$$

Since P was projective, and since $M' \xrightarrow{f} M$ was a quasi-isomorphism so that A is by Theorems 3.2.15, 3.3.2 and using that the shift-functor Σ^{-1} preserve acyclic complexes, $\mathsf{hom}_{\mathbf{K}_R}(P,A)$ is acyclic. [2] By the associated long exact sequence

$$\begin{aligned}\cdots \to \underbrace{\mathsf{H}_1\left(\mathsf{hom}_{\mathbf{K}_R}(P,A)\right)}_{=0} \to \mathsf{H}_1\left(\mathsf{hom}_{\mathbf{K}_R}(P,M')\right) \to \mathsf{H}_1\left(\mathsf{hom}_{\mathbf{K}_R}(P,M)\right)\\ \to \underbrace{\mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}(P,A)\right)}_{=0} \to \mathsf{H}_0\left(\mathsf{hom}_{\mathbf{K}_R}(P,M')\right) \to \cdots,\end{aligned}$$

in homology, we see that $\mathsf{hom}_{\mathbf{K}_R}(P,M') \xrightarrow{\approx} \mathsf{hom}_{\mathbf{K}_R}(P,M)$ is a quasi-isomorphism. So, up to quasi-isomorphism, whenever one takes a projective resolution $M' \xrightarrow{\approx} M$ then if one "tests" how close M' and M are as complexes to being equal by *mapping out of projective complexes P into them*, one finds that they can't be distinguished on the level of dg-hom-complexes.

To say more about this phenomena we just observed, we need to introduce a definition.

Definition 6.1.1 (Pointwise right adjoints)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Fixing an object $D \in \mathcal{D}$, a **pointwise right adjoint to F at D** is an object $C \in \mathcal{C}$ together with a morphism $F(C) \xrightarrow{\varepsilon_D} D$ such that the compositions

$$\mathsf{hom}_{\mathcal{C}}((\,\cdot\,),C) \xrightarrow{F_{(\,\cdot\,),C}} \mathsf{hom}_{\mathcal{D}}(F(\,\cdot\,),F(C)) \xrightarrow{(\varepsilon_D)_*} \mathsf{hom}_{\mathcal{D}}(F(\,\cdot\,),D) \tag{6.1.1}$$

assemble into a natural isomorphism $\mathsf{hom}_{\mathcal{C}}((\,\cdot\,),C) \Rightarrow \mathsf{hom}_{\mathcal{D}}(F(\,\cdot\,),D)$. In the case this holds, we may call the morphism ε_D the **pointwise counit**.

Remark 6.1.2. Observe that for fixed $C' \in \mathcal{C}$ and morphism $C \xrightarrow{h} C'$, the composition 6.1.1 is $h \mapsto \varepsilon_D \circ F(h)$.

Remark 6.1.3. If we denote the component morphisms at $X \in \mathcal{C}$ in 6.1.1 α_X, then *naturality* means that the following square commutes, for all $X \in \mathcal{C}$ and all morphisms $X' \xrightarrow{u} X$,

$$\begin{array}{ccc}\mathsf{hom}_{\mathcal{C}}(X,C) & \xrightarrow{\alpha_X} & \mathsf{hom}_{\mathcal{D}}(F(X),D)\\ \Big\downarrow u^* & & \Big\downarrow F(u)^*\\ \mathsf{hom}_{\mathcal{C}}(X',C) & \xrightarrow[\alpha_{X'}]{} & \mathsf{hom}_{\mathcal{D}}(F(X'),D)\end{array}.$$

Theorem 6.1.4. *Pointwise right adjoints are unique (up to isomorphism) whenever they exist.*

[2] Observe that this carries over different models of the homotopy kernel; since homology does not distinguish between homotopy equivalences and so in particular does not distinguish isomorphic complexes.

Remark 6.1.5. We will assume that the categories $\mathcal{C}, \mathcal{D}$ in the proof below are *locally small* (i.e. $\mathsf{hom}(X, Y)$ are actual sets for all objects X, Y).

Proof. Let $(C, F(C) \xrightarrow{\varepsilon_D} D)$ and $(C', F(C') \xrightarrow{\varepsilon'_D} D)$ be two pointwise right adjoints to $D \in \mathcal{D}$. Then by definition we have natural isomorphisms

$$\begin{aligned} \mathsf{hom}_{\mathcal{C}}\left((\,\cdot\,), C\right) &\approx \mathsf{hom}_{\mathcal{D}}\left(F(\,\cdot\,), D\right) \\ &\approx \mathsf{hom}_{\mathcal{C}}((\,\cdot\,), C') \\ \Rightarrow \mathsf{hom}_{\mathcal{C}}\left((\,\cdot\,), C\right) &\approx \mathsf{hom}_{\mathcal{C}}\left((\,\cdot\,), C'\right). \end{aligned}$$

By [Rie16, Prop. 2.3.1.(ii) $\Rightarrow$ (i)] the conclusion follows. □

Theorem 6.1.6. *The datum of a right adjoint G to F, denoted $F \dashv G$, is equivalent to a choice of a pointwise right adjoint to F at every object $D \in \mathcal{D}$.*

Remark 6.1.7. Below, we will use notation consistent with [Rie16, Chapter 4], in that whenever we have an adjunction $F \dashv G$, then we denote by $\left(f^\sharp, f^\flat\right)$ the pair of morphisms $f^\sharp : F(C) \to D$ and $f^\flat : C \to G(D)$ that correspond to each other under the (natural) bijection $\mathsf{hom}_{\mathcal{D}}(F(C), D) \approx \mathsf{hom}_{\mathcal{C}}(C, G(D))$. That is, $\mathsf{hom}_{\mathcal{D}}(F(C), D) \xrightarrow[\approx]{(\,\cdot\,)^\flat} \mathsf{hom}_{\mathcal{C}}(C, G(D))$, so that $\left(f^\sharp\right)^\flat = f^\flat$ and $\left(f^\flat\right)^\sharp = f^\sharp$.

Proof. $\underline{\Rightarrow}$: By [Rie16, Lemma 4.2.3] we have a **counit** natural transformation $\varepsilon : FG \Rightarrow \mathsf{id}_{\mathcal{D}}$. In particular, for any $D \in \mathcal{D}$, we have that $\varepsilon_D : F(G(D)) \to D$ is the (adjunct) **transpose** of $\mathsf{id}_{G(D)}$, which means that under the correspondence isomorphism

$$\mathsf{hom}_{\mathcal{D}}(F(G(D)), D) \approx \mathsf{hom}_{\mathcal{C}}(G(D), G(D)),$$

ε_D is sent to $\mathsf{id}_{G(D)}$. We claim that $(G(D), \varepsilon_D : F(G(D)) \to D)$ satisfies the property of being a pointwise right adjoint to F at D. Observe we have the pair $f^\sharp := \varepsilon_D$ and $f^\flat = \mathsf{id}_{G(D)}$ and so by [Rie16, p. 129, third diagram from the top] we find that for any morphism $h : C \to G(D)$ we have that $(\varepsilon_D \circ F(h))^\flat = \mathsf{id}_{G(D)} \circ h = h$. Since $(\,\cdot\,)^\flat : \mathsf{hom}_{\mathcal{D}}(F(C), D) \to \mathsf{hom}_{\mathcal{C}}(C, G(D))$ is a bijection and right inverses are unique it follows that the map $h \mapsto \varepsilon_D \circ F(h)$ must be a bijection $\mathsf{hom}_{\mathcal{C}}(C, G(D)) \to \mathsf{hom}_{\mathcal{D}}(F(C), D)$. Last, we check naturality: By remark 6.1.3 this amounts to checking that for any morphism $X' \xrightarrow{u} X$ in $\mathcal{C}$ and any morphism $X \xrightarrow{h} G(D)$, we have

$$\varepsilon_D \circ F(h \circ u) = (\varepsilon_D \circ F(h)) \circ F(u).$$

But this is immediate by functoriality of F.

$\underline{\Leftarrow}$: Assume we are given a pointwise right adjoint to F at D $\left(C_D, F(C_D) \xrightarrow{\varepsilon_D} D\right)$ for each $D \in \mathcal{D}$. We then may *construct* a right adjoint functor G to F explicitly as follows: For objects $D \in \mathcal{D}$, let $G(D) := C_D$. For a morphism $D \xrightarrow{f} D'$ in $\mathcal{D}$, the property of being pointwise right adjoint to F at D' tells us that for every $X \in \mathcal{C}$ we have a bijection

$$\mathsf{hom}_{\mathcal{C}}(X, C_{D'}) \approx \mathsf{hom}_{\mathcal{D}}(F(X), D'), \qquad h \mapsto \varepsilon_{D'} \circ F(h).$$

Letting $X = C_D$ we see that we get a (natural) isomorphism

$$\mathsf{hom}_{\mathcal{C}}(C_D, C_{D'}) \to \mathsf{hom}_{\mathcal{D}}(F(C_D), D').$$

In particular, we note that $F(C_D) \xrightarrow{f\circ\varepsilon_D} D'$ is in the codomain of this isomorphism. Then there is a unique map γ_f in $\mathsf{hom}_{\mathcal{C}}(C_D, C_{D'})$ such that $\varepsilon_{D'} \circ F(\gamma_f) = f \circ \varepsilon_D$. We then *define* $G(f) := \gamma_f$ (observe that this assignment is *unique* by what we just said).

<u>G is a functor</u>: Let $f := \mathsf{id}_D$. Then γ_f is the unique map such that $\varepsilon_D \circ F(\gamma_f) = \mathsf{id}_D \circ \varepsilon_D$. Since F is a functor, we have that id_{C_D} satisfies this, so by uniqueness, we find that $G(\mathsf{id}_D) = \mathsf{id}_{C_D}$.

Consider the composition $D \xrightarrow{f} D' \xrightarrow{g} D''$. We see that $G(g \circ f)$ is the unique morphism $\gamma_{g\circ f}$ such that

$$\varepsilon_{D''} \circ F\left(\gamma_{g\circ f}\right) = (g \circ f) \circ \varepsilon_D$$

If we can show that $G(g) \circ G(f)$ satisfies the same equation, then G respecting *composition* follows. We have

$$\begin{aligned}
\varepsilon_{D''} \circ F(G(g) \circ G(f)) &= \varepsilon_{D''} \circ F(G(g)) \circ F(G(f)), \qquad \text{by functoriality of } F \\
&= g \circ \varepsilon_{D'} \circ F(G(f)), \qquad \text{by definition of } G(g) \\
&= g \circ f \circ \varepsilon_D, \qquad \text{by definition of } G(f).
\end{aligned}$$

The conclusion follows.

<u>$F \dashv G$</u>: We want to construct natural isomorphisms $\mathsf{hom}_{\mathcal{C}}(X, G(D)) \approx \mathsf{hom}_{\mathcal{D}}(F(X), D)$. By how G acts on objects, this is the same as constructing natural isomorphisms $\mathsf{hom}_{\mathcal{C}}(X, C_D) \approx \mathsf{hom}_{\mathcal{D}}(F(X), D)$. But by definition, we are already given such an isomorphism by $h \mapsto \varepsilon_D \circ F(h)$ for $X \xrightarrow{h} C_D$. Furthermore, by remark 6.1.3 this is already natural in the first variable. It remains to show that this assignment is natural in the second variable, which amounts to checking that the following square commutes,

$$\begin{array}{ccc}
\mathsf{hom}_{\mathcal{C}}(X, C_D) & \xrightarrow{\quad\Phi\quad} & \mathsf{hom}_{\mathcal{D}}(F(X), D) \\
{\scriptstyle G(k)_*}\big\downarrow & & \big\downarrow{\scriptstyle k_*} \\
\mathsf{hom}_{\mathcal{C}}(X, C_{D'}) & \xrightarrow[\quad\Phi\quad]{} & \mathsf{hom}_{\mathcal{D}}(F(X), D')
\end{array},$$

for any morphism $D \xrightarrow{k} D'$, and where the *horizontal* arrows is $h \xmapsto{\Phi} \varepsilon_D \circ F(h)$ (upper horizontal) and $h \xmapsto{\Phi} \varepsilon_{D'} \circ F(h)$ (lower horizontal). We find that for $X \xrightarrow{h} C_D$ we have

$$\begin{aligned}
\Phi(G(k)_*(h)) &= \Phi(G_k \circ h) \\
&= \varepsilon_{D'} \circ F(G(k) \circ h) \\
&= \varepsilon_{D'} \circ F(G(k)) \circ F(h), \qquad \text{by functoriality of } F \\
&= k \circ \varepsilon_D \circ F(h), \qquad \text{by definition of } G(k) \\
&= k \circ \Phi(h), \qquad \text{by definition of } \Phi,
\end{aligned}$$

so that the diagram commutes. The (main) conclusion follows. □

Dually to pointwise right adjoint, we have the following definition.

Definition 6.1.8 (Pointwise left adjoint)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then a **pointwise left adjoint to a F at $D \in \mathcal{D}$** is an object $C \in \mathcal{C}$ together with a morphism $\eta_D : D \to F(C)$ such that

$$\mathsf{hom}_{\mathcal{C}}(C, X) \to \mathsf{hom}_{\mathcal{D}}(D, F(X)), \qquad h \mapsto F(h) \circ \eta_D$$

is a natural isomorphism for all objects $X \in \mathcal{C}$.

We see that the *dual* statement of theorem 6.1.6 tells us that the datum of a left adjoint $G \dashv F$ to F is equivalent to specifiying a pointwise left adjoint to F for each object $D \in \mathcal{D}$.

Relating this to our earlier observations: If we would like a right adjoint to the inclusion $\mathbf{P}_R \xrightarrow{\iota} \mathbf{K}_R$, a natural choice would be (using theorem 6.1.6) to let the pointwise right adjoint to ι at M be (M', ε_M), where M' is the projective object in $\mathbf{P}_R$ such that there is a projective resolution $M' \xrightarrow{\approx} M$ and where $\varepsilon_M := \left(\iota(M') = M' \xrightarrow{\approx} M\right)$. The reason for our choice of ε_M is that one finds that $\mathsf{hom}_{\mathbf{K}_R}(P, M') \xrightarrow{\mathsf{hom}_{\mathbf{K}_R}(P, \varepsilon_M)} \mathsf{hom}_{\mathbf{K}_R}(P, M)$ is a quasi-isomorphism for every $P \in \mathbf{P}_R$, realized by $h \mapsto \varepsilon_M \circ h = \varepsilon_M \circ \iota(h)$. Hence this *almost* defines a pointwise right adjoint to ι at M for each $M \in \mathbf{K}_R$, and in a sense *does* define a right adjoint if we treat quasi-isomorphisms as *actual* isomorphisms.

Dually, if we assume (we have not shown this) that there is an injective resolution $M \xrightarrow[\eta_M]{\approx} M'$ into an injective object M' for each $M \in \mathbf{K}_R$, then observe that we have an associated homotopy cokernel sequence $M \xrightarrow[\eta_M]{\approx} M' \to \mathsf{hcoker}\,(\eta_M)$. If we, for any injective object $I \in \mathbf{I}_R$, apply the hom-functor $\mathsf{hom}_{\mathbf{K}_R}(-, I)$ to this sequence, we by theorem 4.3.4 get a homotopy kernel sequence

$$\mathsf{hom}_{\mathbf{K}_R}(\mathsf{hcoker}(\eta_M), I) \to \mathsf{hom}_{\mathbf{K}_R}(M', I) \xrightarrow{\mathsf{hom}_{\mathbf{K_R}}(\eta_M, I)} \mathsf{hom}_{\mathbf{K}_R}(M, I).$$

Now the argument proceeds as in the projective case: Since η_M is a quasi-isomorphism theorem 3.2.15 gives that $\mathsf{hcoker}(\eta_M)$ is acyclic, so that since I is injective, $\mathsf{hom}_{\mathbf{K}_R}(\mathsf{hcoker}(\eta_M), I)$ is acyclic and so the long exact sequence in homology gives that $\mathsf{Hom}_{\mathbf{K}_R}(\eta_M, I)$ is a quasi-isomorphism realized by $h \mapsto h \circ \eta_M = \iota(h) \circ \eta_M$. Hence by what we said about the dual of statement 6.1.6 this *almost* defines a left adjoint to the inclusion $\mathbf{I}_R \hookrightarrow \mathbf{K}_R$.

6.2 Introducing (briefly) $\Bbbk$-linear ∞-categories

For a more in-depth treatment on dg-categories and dg-functors (which are only briefly talked about below), we may direct the reader to e.g. [Toë11] or [Yek19, Sections 3.4, 3.5 and 4.6]. To be safe, we will assume that every category is *small*. The treatment on ($\Bbbk$-linear) ∞-categories in this section will be very shallow, and there will not be many proofs provided (which is too say, we will have to take many things *on faith*). This is not an optimal situation, but it is the best we can do in our current circumstance.

Let us denote cat the category of categories, where objects are categories and morphisms are functors, and let $\mathsf{cat}^{\mathsf{dg}}_{\Bbbk}$ be the category of categories *enriched* over $\left(\mathsf{Ch}_{\Bbbk}, \otimes_{\Bbbk}, \underset{\sim}{\Bbbk}\right)$ in the sense of definition 4.1.5, with morphisms dg-functors, where we now define dg-functors.

Definition 6.2.1 (dg-functor; c.f. with [Toë11, § 2.3.2])**.** Let $(\mathcal{C}, \chi^{\mathcal{C}}, \iota^{\mathcal{C}}), (\mathcal{D}, \chi^{\mathcal{D}}, \iota^{\mathcal{D}})$ be enriched over $\left(\mathsf{Ch}_{\Bbbk}, \otimes_{\Bbbk}, \underset{\sim}{\Bbbk}\right)$ with $\chi^{\mathcal{C}}, \chi^{\mathcal{D}}$ denoting composition and $\iota^{\mathcal{C}}, \iota^{\mathcal{D}}$ units, in the *enriched* sense of definition 4.1.5. Then a **dg-functor** $\mathcal{C} \xrightarrow{F} \mathcal{D}$ consists of the following data:

- A map $\mathsf{Ob}(\mathcal{C}) \xrightarrow{F} \mathsf{Ob}(\mathcal{D})$ in Set.
- For any pair of objects $C, C' \in \mathcal{C}$, there is a morphism $\mathsf{hom}_{\mathcal{C}}(C, C') \xrightarrow{F_{C,C'}} \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathsf{Ch}_{\Bbbk}$.
- For any triple of objects C, C' and C'' in $\mathcal{C}$, we require the following diagram to commute,

$$\begin{array}{ccc}
\mathsf{hom}_{\mathcal{C}}(C', C) \otimes \mathsf{hom}_{\mathcal{C}}(C'', C') & \xrightarrow{\chi^{\mathcal{C}}_{C,C',C''}} & \mathsf{hom}_{\mathcal{C}}(C'', C) \\
\downarrow{\scriptstyle F_{C',C} \otimes F_{C'',C'}} & & \downarrow{\scriptstyle F_{C'',C}} \\
\mathsf{hom}_{\mathcal{D}}(F(C'), F(C)) \otimes \mathsf{hom}_{\mathcal{D}}(F(C''), F(C')) & \xrightarrow[\chi^{\mathcal{D}}_{F(C),F(C'),F(C'')}]{} & \mathsf{hom}_{\mathcal{D}}(F(C''), F(C))
\end{array} \tag{6.2.1}$$

- For any object C in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc}
\underset{\sim}{\Bbbk} & \xrightarrow{\iota^{\mathcal{C}}_{C}} & \mathsf{hom}_{\mathcal{C}}(C, C) \\
& {\scriptstyle \iota^{\mathcal{D}}_{F(C)}} \searrow & \downarrow{\scriptstyle F_{C,C}} \\
& & \mathsf{hom}_{\mathcal{D}}(F(C), F(C))
\end{array} .$$

Remark 6.2.2. Observe that [Toë11] has a different convention in that his μ is with our conventions the same as $\chi \circ \sigma$ where σ is the natural isomorphism coming from the symmetric monoidal structure on $\mathsf{Ch}_{\Bbbk}$ (recall theorem 2.5.4), i.e. so on pure tensors $f \otimes g$ this would be $\mu(f \otimes g) = g \circ f$. The *naturality* of σ together with the fact that all maps $F_{C,C'}$ lives in $\mathsf{Ch}_{\Bbbk}$ we claim gives that our requirement that diagrams on the form 6.2.1 commutes is *equivalent* to the corresponding diagram in [Toë11, § 2.3.2] commuting.

Remark 6.2.3. Observe the similarity of this definition with [Mal26, Def. 10.1.3] of a **simplicial functor** $\mathcal{C}_{\bullet} \xrightarrow{F} \mathcal{D}_{\bullet}$ (replace H_0 with π_0).

With this definition in place, we may introduce further notions related to dg-functors.

Definition 6.2.4 (Homotopically fully faithful; cf. [Toë11, Def. 2.3.2.(a)])**.** Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a dg-functor. Then we say that F is **homotopically fully faithful** if for all objects C, C' in $\mathcal{C}$, we have that the map $\mathsf{hom}_{\mathcal{C}}(C, C') \xrightarrow{F_{C,C'}} \mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$ in $\mathsf{Ch}_{\Bbbk}$ is a *quasi-isomorphism.*

We will take it as given that there is a category $\mathsf{H}_0(\mathcal{C})$ for any dg-category $\mathcal{C}$, with $\mathsf{Ob}(\mathsf{H}_0(\mathcal{C})) = \mathsf{Ob}(\mathcal{C})$ and $\mathsf{hom}_{\mathsf{H}_0(\mathcal{C})}(C, C') = \mathsf{H}_0(\mathsf{hom}_{\mathcal{C}}(C, C'))$ (c.f. with lower row in diagram 4.1.2), e.g. in the case that the dg-category is Ch_R for some $\Bbbk$-algebra R we get back chain maps $C \xrightarrow{f} C'$ *modulo nullhomotopy.* Composition in $\mathsf{H}_0(\mathcal{C})$ is defined using enriched composition χ and the canonical map $[f] \otimes [g] \mapsto [f \otimes g]$ (see [Toë11, p. 259]). This category, which we touched briefly upon in discussing how one produces $\mathsf{H}_0(\mathbf{D}_R)$ in

§2.6, is called the **homotopy category** of $\mathcal{C}$. Furthermore the claim is that this construction gives us a functor $\mathsf{cat}_{\Bbbk}^{\mathsf{dg}} \xrightarrow{\mathsf{H}_0} \mathsf{cat}$ ([Toë11, Def. 2.3.2]). This functor furnishes us with another notion.

Definition 6.2.5 (Homotopically essentially surjective; cf. [Toë11, Def. 2.3.2.(b)])**.** Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a dg-functor. Then we say that F is **homotopically essentially surjective** if the induced functor $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ is *essentially surjective.*

Any functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ which is both homotopically fully faithful and homotopically essentially surjective, we call a **weak equivalence**. The notation H_0 also gives us a wink at what happens when we let H_0 act on morphisms $F_{C,C'}$: Since this lives in $\mathsf{Ch}_{\Bbbk}$ and is a quasi-isomorphism whenever F is homotopically fully faithful, the claim is that H_0 converts this into an actual isomorphism of $\Bbbk$-modules $\mathsf{H}_0(\mathsf{hom}_{\mathcal{C}}(C, C')) \cong \mathsf{H}_0(\mathsf{hom}_{\mathcal{D}}(F(C), F(C'))$. Although we have not provided a completely rigorous justification for this, if we assume it for now, then we see that the term *weak equivalence* may be apt, since $\mathsf{H}_0(\ \cdot\ )$ then in fact turns weak equivalences into *actual* equivalences of categories $\mathsf{H}_0(\mathcal{C}) \xrightarrow{\mathsf{H}_0(F)} \mathsf{H}_0(\mathcal{D})$ (since, given our assumptions, $\mathsf{H}_0(F)$ becomes fully faithful and essentially surjective). If we localize $\mathsf{cat}_{\Bbbk}^{\mathsf{dg}}$ at the weak equivalences W, we get a category which we denote as $\mathsf{ho}\,(\mathsf{Cat}_{\Bbbk}) := \left(\mathsf{cat}_{\Bbbk}^{\mathsf{dg}}\right)[W^{-1}]$ (c.f. [Toë11, Def. 2.3.3]. We refer to this category as the **homotopy category of $\Bbbk$-linear ∞-categories**. The objects of this category we call **$\Bbbk$-linear ∞-categories**.

The claim is further that this localization furnishes us with a canonical localization-functor

$$\mathsf{cat}_{\Bbbk}^{\mathsf{dg}} \xrightarrow{Q^{\infty}} \mathsf{ho}\,(\mathsf{Cat}_{\Bbbk})$$

that takes all weak equivalences to isomorphisms (c.f. proposition 2.4.5). Observe however, that in contrast to the localization in definition 2.4.2 with respect to a class of morphisms as defined in 2.4.1, we don't necessarily need a hypothesis on the class of weak equivalences forming a multiplicative system of morphisms, since *Gabriel-Zisman* localization ([GZ67]) works for *any* class of morphisms.[3]

Since the canonical functor Q^{∞} acts as the identity map on objects (see [GZ67, p. 6]) it follows that Q^{∞} is essentially surjective. Furthermore, by construction, the class of weak equivalences turn into isomorphisms in $\mathsf{ho}\,(\mathsf{Cat}_{\Bbbk})$ ([GZ67, The claim of p. 6, 1.1.(i)]). Consequently, any $\Bbbk$-linear ∞-category $\mathcal{C}^{\infty}$ has a representative in the category $\mathsf{cat}_{\Bbbk}^{\mathsf{dg}}$, in the sense that there is an associated category $\mathcal{C}$ in $\mathsf{cat}_{\Bbbk}^{\mathsf{dg}}$ such that $Q^{\infty}(\mathcal{C}) = \mathcal{C}^{\infty}$. Since Q^{∞} (as noted) acts as the identity on objects, one may denote $\mathcal{C}^{\infty}$ as $\mathcal{C}$ or $\mathcal{C}^{\infty}$. We will not try define $\mathsf{Cat}_{\Bbbk}$ (see [Maz23, §9.5.3]).

By construction, the morphisms in $\mathsf{ho}\,(\mathsf{Cat}_{\Bbbk})$ are equivalence classes of morphisms in $\mathsf{cat}_{\Bbbk}^{\mathsf{dg}}$ on the form as in 2.4.7 but where the "backwards" $\approx$ are *weak equivalences*. One may refer to the data in $\mathcal{C}$ as **point-set** and the data in its image $Q^{\infty}(\mathcal{C}) = \mathcal{C}^{\infty}$ under Q^{∞} as **∞-categorical**.

Remark 6.2.6. We remark that in [Maz23, §6.3] he proceeds as above, and then defines $\mathbf{D}_R$ as the ∞-categorical localization $\mathbf{K}_R^{\infty} \xrightarrow{\pi} \mathbf{D}_R$ at the quasi-isomorphisms and that precomposition by π determines an inclusion $\mathsf{hom}_{\mathsf{ho}(\mathsf{Cat}_{\Bbbk})}(\mathbf{K}_R^{\infty}, \mathcal{E}) \xrightarrow{\pi^*} \mathsf{hom}_{\mathsf{hoCat}_{\Bbbk}}(\mathbf{D}_R, \mathcal{E})$ such that its image consists of those functors $\mathbf{K}_R^{\infty} \to \mathcal{E}$ which takes quasi-isomorphisms to *equivalences* (equivalences can be seen as the ∞-categorical notion of isomorphism).

[3] Regarding Gabriel-Zisman localization, we may refer the reader to [Toë11, Section 2.1, 2.2, and 2.4].

Furthermore, we are told that there are "canonical equivalences" $\mathbf{I}_R^\infty \simeq \mathbf{P}_R^\infty \simeq \mathbf{K}_R^\infty$, and how these categories participate in adjunctions between each other. As we understand it, this was the motivation for working with $\mathbf{I}_R$ and $\mathbf{P}_R$ in § 5. However, we found this all very confusing, perhaps because of lack of background knowledge. Therefore, we will proceed as in § 6.3, mainly sticking with the exposition given in [Mal26, Chapter 10].

6.3 The derived ∞-category of R-modules

The denouement of this thesis is the definition of the derived ∞-category of R-modules, $\mathbf{D}_R$.

Before saying something about how to define the derived ∞-category of R-modules, we want to say something about ∞-categories in general. There are *at least* two different ways to define ∞-categories [Mal26, §§ 10.1–10.2]. We will use the definition given in §10.2 of the cited source.

Definition 6.3.1 (Simplex category $\mathbf{\Delta}$)**.** We define the **simplex category $\mathbf{\Delta}$** as the category with objects non-empty finite ordinals $[n] := \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$ and morphisms $[m] \xrightarrow{f} [n]$ (non-strictly) *order-preserving* maps. It is straightforward to verify that this defines a category.

Remark 6.3.2. If one wants to be a bit more precise about the objects of $\mathbf{\Delta}$ one might say that the objects are the *finite Von Neumann ordinals* except 0, with the total order $<$ given by the membership relation $\in$, i.e. $\alpha < \beta \Leftrightarrow \alpha \in \beta$.

Definition 6.3.3 (Simplicial set)**.** A **simplicial set** X is a contravariant functor $\mathbf{\Delta} \xrightarrow{X} \mathsf{Set}$ into the category of sets, Set.

From $\mathbf{\Delta}$ we set $\Delta[n] := \mathsf{hom}_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta} \to \mathsf{Set}$, i.e. the contravariant hom-functor from $\mathbf{\Delta}$ to Set, and call this the **standard n-simplex**, we may write $\Delta[n]([m])$ as $(\Delta[n])_m$. we let $\mathbf{\Delta} \xrightarrow{\Lambda_i[n]} \mathsf{Set}$ be the contravariant functor defined on objects as

$$(\Lambda_i[n])_m := \Lambda_i[n]([m]) := \left\{[m] \xrightarrow{\alpha} [n] \mid \exists j \neq i \in [n] \text{ such that } j \notin \mathsf{im}(\alpha)\right\},$$

and which takes morphisms $[\ell] \xrightarrow{\beta} [k]$ in $\mathbf{\Delta}$ to

$$(\Lambda_i[n])_k \xrightarrow{\Lambda_i[n](\beta)} (\Lambda_i[n])_\ell, \qquad \alpha \mapsto \alpha \circ \beta.$$

Since $\mathsf{im}(\alpha \circ \beta) \subseteq \mathsf{im}(\alpha)$, this is well-defined, and we call $\Lambda_i[n]$ the i^{th} **horn** of $\Delta[n]$. If $0 < i < n$, then we call $\Lambda_i[n]$ an **inner horn** of $\Delta[n]$.

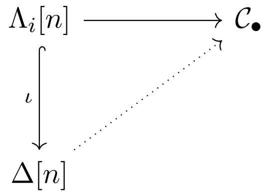
By construction, we see that $\Lambda_i[n]$ is a **subfunctor** of $\Delta[n]$, which we may write as $\Lambda_i[n] \subseteq \Delta[n]$. Furthermore, we see that both functors are *simplicial sets*. There is a natural transformation defined by inclusion $\Lambda_i[n] \overset{\iota}{\Rightarrow} \Delta[n]$,

$$\begin{array}{ccccc} [\ell] & & (\Lambda_i[n])_k & \xhookrightarrow{\iota_k} & (\Delta[n])_k \\ \downarrow{\scriptstyle\alpha} & \rightsquigarrow & \downarrow{\scriptstyle\alpha^*} & & \downarrow{\scriptstyle\iota(\alpha)^*} \\ [k] & & (\Lambda_i[n])_\ell & \xhookrightarrow[\iota_\ell]{} & (\Delta[n])_\ell \end{array} .$$

Definition 6.3.4 (Quasicategory $\mathcal{C}_\bullet$)**.** Let $\mathcal{C}_\bullet$ be a large simplicial set.[4] Then we say that

[4]Meaning: The levels $\mathcal{C}_\bullet([n])$ does not actually have to be *sets*. We will disregard any size-issues here.

$\mathcal{C}_\bullet$ is a **quasicategory** if any solid diagram on the form below can be completed by a dashed arrow to a commuting triangular diagram as indicated below, for any *inner* horn $\Lambda_i[n]$,



(6.3.1)

where solid and dashed arrows lives in the functor category $\mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{Set})$. If this property holds, we say that *every inner horn in $\mathcal{C}_\bullet$ can be filled.*

We may roughly think of ∞-categories and quasicategories as interchangeable, at least for present purposes. With this notation in place we let $\mathsf{sSet} := \mathsf{Fun}\left(\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{Set}\right)$. If we think of each finite set $[n] \in \boldsymbol{\Delta}$ as the poset-category $0 \to 1 \to \cdots \to n$, then we may define the **nerve** $N(\mathcal{C})$ of any (ordinary) category $\mathcal{C}$ as the simplical set $\boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{Fun}(-,\mathcal{C})} \mathsf{Set}$. Disregarding size issues one may think of N as a functor $\mathsf{Cat} \xrightarrow{N(-)} \mathsf{sSet}$, with Cat denoting the category of small categories.

From now on, we call ordinary categories 1-categories, and we may think of them (by [Mal26, Example 10.1.13, Lemma 10.1.14]) as precisely the simplicial sets in which every inner horn can be filled *uniquely.* Furthermore, by definition, we see that $(N(\mathcal{C}))_0$ precisely picks out an object of $\mathcal{C}$ and $(N(\mathcal{C}))_1$ picks out a morphism of $\mathcal{C}$, that is,

$$(N(\mathcal{C}))_0 \cong \mathsf{ob}(\mathcal{C}) \tag{6.3.2}$$

and

$$(N(\mathcal{C}))_1 \cong \mathsf{Mor}(\mathcal{C}). \tag{6.3.3}$$

Proposition 6.3.5. *For any ordinary category $\mathcal{C}$, the nerve $N(\mathcal{C})$ of $\mathcal{C}$ is a quasicategory.*

Proof. See [Mal26, Example 10.1.13]. □

We refer the reader to [Mal26, Def. 10.2.2] for what is called the **Dwyer-Kan** localization or DK-localization of a quasicategory. We may denote the Dwyer-Kan localization of a quasi-category $\mathcal{C}$ at a class of *weak equivalences* $W \subseteq \mathcal{C}_1 := \mathcal{C}([1])$ as

$$\mathcal{C}[\![W^{-1}]\!],$$

such that there is a canonical map $\mathcal{C} \xrightarrow{\delta} \mathcal{C}[\![W^{-1}]\!]$ where:

(a) Each morphism in W is sent to an isomorphism in $\mathcal{C}[\![W^{-1}]\!]$.

(b) For any other quasicategory $\mathcal{D}$, precomposition with δ determines what is called a **Joyal equivalence** ([Mal26, Def. 10.1.24])

$$\mathsf{Fun}\left(\mathcal{C}[\![W^{-1}]\!], \mathcal{D}\right) \xrightarrow{\sim} \mathsf{Fun}\left(\mathcal{C}, \mathcal{D}\right).$$

Proposition 6.3.6 ([Mal26, Prop. 10.2.3])**.** *For any quasicategory $\mathcal{C}$ and any subset $W \subseteq \mathcal{C}_1$, the Dwyer-Kan localization $\mathcal{C}[\![W^{-1}]\!]$ exists.*

Corollary 6.3.7. *Since we may take any* 1*-category* $\mathcal{C}$ *and produce its nerve* $N(\mathcal{C})$ *with* $N(\mathcal{C})_1 \cong \mathsf{Mor}(\mathcal{C})$*, for any morphisms* $W \subseteq \mathcal{C}$ *(but viewed from the perspective of the nerve* $N(\mathcal{C})$ *of* $\mathcal{C}$*), we may produce a quasicategory* $N(\mathcal{C})[\![W^{-1}]\!]$ *in which the morphisms in* W *are isomorphisms.*

Definition 6.3.8 (The underlying ∞-category of $(\mathcal{C}, W)$)**.** For any 1-category $\mathcal{C}$ and a class of morphisms W which we call "weak equivalences", we call the quasicategory produced as in Corollary 6.3.7 the **underlying ∞-category of** $(\mathcal{C}, W)$.

Finally, we may give a very expedient definition of the derived ∞-category of R-modules: It is the Dwyer-Kan localization at the quasi-isomorphisms qiso in Ch_R of the nerve $N(\mathsf{Ch}_R)$ of Ch_R, i.e.

$$\mathbf{D}_R := N(\mathsf{Ch}_R)[\![\mathsf{qiso}^{-1}]\!],$$

the underlying ∞-category of $(\mathsf{Ch}_R, \mathsf{qiso})$.

Remark 6.3.9. For a less expedient treatment on how to define $\mathbf{D}_R$, see [Lur17, Section 1.3].

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