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Controllability of the Lasso graph

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Abstract

In this article we study the Lasso graph with a boundary control at its leaf, and with scaling invariant or Robin-type vertex conditions at its loop vertex. We solve the wave equation for small times and establish the well-definedness of the control operator. For scaling invariant conditions, we show that the system attains controllability at a minimal time precisely when the scattering matrix fulfills $S_{21} \neq \pm S_{31}$. If this condition is not fulfilled then the system does not attain controllability, no matter the control time. A similar condition is derived for the Lasso graph with a magnetic potential and standard conditions at the loop vertex. Notably, none of the scaling invariant vertex conditions at the loop vertex fulfill the non-symmetry condition identified by P. Kurasov in [9] as sufficient for solvability of the inverse spectral problem.

Sammanfattning

I denna artikel studeras Lassografen med en randkontroll i hörnet av grad ett, och vars hörn av grad tre antingen har skalningainvarianta hörnvillkor eller Robinvillkor. För små tider löses vågekvationen och kontrolloperatorn visas vara väldefinierad. Givet skalningsinvarianta hörnvillkor visar vi att systemet är styrbart vid en minimal tid om och endast om spridningsmatrisen uppfyller $S_{21} \neq \pm S_{31}$. Om inte detta villkor uppfylls så är systemet inte styrbart, oavsett styrtiden. Ett liknande villkor härleds för Lassografen med en magnetisk potential och standardvillkor i hörnet med grad tre. Anmärkningsvärt är att inga skalningsinvarianta hörnvillkor för Lassografen uppfyller icke-symmetrikriteriet härlett av P. Kurasov i [9], som är ett tillräckligt villkor för det inversa spektrala problemets lösbarhet.

Contents

1 Introduction

1.1 Background

Differential operators on metric graphs (graphs whose edges are identified with intervals on the real line) is an active field of research. Quantum graphs are metric graphs associated with Shrödinger operators, whose differential expression $\tau_{a,q}$ includes an electric potential q and a magnetic potential a. If both q and a are identically zero then the Schrödinger operator coincides with the Laplacian.

$$\tau_{q,a} = (i\partial_x + a(x))^2 + q(x)$$

To assure the Schrödinger operator's self-adjointness one must impose certain conditions (called vertex conditions) on how elements from its domain behave at the vertices. At a given vertex, these vertex conditions can be expressed in terms of an irreducible, unitary matrix S, called the Scattering Matrix. These vertex conditions, the potential and the graph geometry determine the spectral properties of the quantum graph.

Inverse spectral problems are central questions in the study of quantum graphs and involve recovering the metric graph and the Schrödinger operator (the potentials and the vertex conditions) given only some kind of spectral data, such as the spectrum of the operator. Not all spectral problems are uniquely solvable. In 2001, Gutkin and Smilansky provided a pair of isospectral trees, showing that the spectrum alone does not necessarily determine the metric graph [7]. However, when the graph is simple and its edge-lengths are rationally independent, the spectrum does uniquely determine the metric graph [11].

Recovering the potentials typically requires more than a single spectrum, since the potential over an interval is determined by the spectrums corresponding to two distinct boundary conditions at an endpoint [12]. One alternative for the aforementioned spectral data is the M-function (based on the one-dimensional Titchmarsh-Weyl M-function), which is a matrix-valued function relating the eigenfunctions of the Dirichlet operator to their derivates over a subset of the vertices, called the contact set. A natural candidate for the contact set is the set of leaves (degree one vertices) in a graph.

There is a procedure for recovering the potentials given the M-function (or equivalent data) based on the Boundary Control method (hereafter referred to as the BC-method) pioneered by M. I. Belishev in the 1980s [4]. Originally the method was developed for solving inverse problems on manifolds; recovering the domain and operator of a dynamical system from boundary observations alone. In recent decades, the BC-method has been adapted for graphs. The graph governed by the wave equation[1] with boundary controls over the contact set comprises the dynamical system. Belishev describes the philosophy behind this approach for the BC-method on trees like this:

> *The approach follows a very general principle of the system theory: the richer the set of states of a dynamical system (in our case, the set of waves $u^f(,T)$) which an external observer can create by means of controls, the richer the information about this system which the observer can extract from external measurements (from inverse data). In particular, a possibility of creating the Dirac δ-functions in the interior points (that is the richest set of states because any state is a superposition of δ-functions) provides a way to solve inverse problems.* ([3])

The BC-method is explicit when solving inverse problems on trees. In [1] and [2], Avdonin, Kurasov and Nowaczyk develop an iterative process, called the leaf peeling procedure, for recovering the electric potential, alongside the metric tree and its vertex conditions.

[1] wherein the Schrödinger operator substitutes the Laplacian

Graphs with cycles require a modified approach, since, unlike in the case of trees, the magnetic potential gives rise to magnetic fluxes that must be accounted for. This relates to the Aharonov-Bohm effect in quantum physics. The modified approach is called the Magnetic BC-method (henceforth the MBC-method) and was developed by P. Kurasov in [8] and [9][2]. In both the BC- and MBC-methods the magnetic potential is eliminated through a unitary transform described in Section 5.1. Over cycles, this transform results in so called 'vertex phases', whereas for trees the elimination bears no consequence for the vertex conditions.

In [9], P. Kurasov studies the inverse problem of reconstructing the electric potential for the Lasso graph (depicted in Figure 2), by examining the M-function (given for two different magnetic fluxes) with the lone leaf comprising the contact set. The author inverstigates how the unique solvability of the inverse spectral problem depends on the vertex conditions at the loop-vertex, whose Scattering matrix S has order three. For several choices of vertex conditions (such as standard conditions), the M-function does not uniquely determine the electric potential over the Lasso graph, even when known for all magnetic fluxes ([8], Theorem 6.2) However, it turns out that when S satisfies the non-symmetry condition (1), the inverse spectral problem is uniquely solvable.

$$\textbf{Non-symmetry condition}: \qquad S_{12}S_{23}S_{31} \neq S_{13}S_{21}S_{32} \tag{1}$$

This non-symmetry condition can equivalently be expressed as $|S_{32}S_{13}| \neq |S_{31}S_{23}|$, which the author assigns the following interpretation in terms of boundary observations from the contact set.

> *Consider the wave evolution on the Lasso graph. Assume that an observer is sending waves along the outgrowth and tries to determine the potential on the graph. To determine the potential on the loop one needs to study waves coming back after passing along the loop in one or the other direction. There are precisely two such (shortest) trajectories. Crossing the internal vertex v_0 these waves are multiplied by the scattering coefficients $S_{32}S_{13}$ and $S_{31}S_{23}$. [The condition] implies that the corresponding amplitudes are different and one may distinguish between the waves coming after having passed the loop in different directions.* ([9] section VII)

Although the BC-method draws from control theory, the boundary controllability of the Lasso graph has not yet been addressed. Inverse spectral problems are not studied in this thesis, but serve to motivate inquiries into the controllability properties of the Lasso graph with scaling invariant vertex conditions[3] at its loop-vertex and a boundary control at its leaf. In this article, we establish that the Lasso graph, governed by the wave equation and with a zero initial state, attains controllability whenever $S_{21} \neq \pm S_{31}$ (see Section 4.3), which is remarkable because no Scaling invariant conditions at the loop vertex fulfill the non-symmetry condition (1) identified by P. Kurasov (see Section 4.4).

[2]A more general exposition can be found in chapters 22 and 23 of [10]

[3]This corresponds to S being Hermitian, in addition to unitary and irreducible

1.2 Overview and results

As a dynamical system obeying the wave operator, the Lasso graph propagates boundary inputs from its leaf as standing waves with a fixed speed. These waves are then scattered at the loop-vertex, eventually resulting in a state of superposition on the graph's edges. Scaling invariant vertex conditions yield a particularly transparent relation between the boundary control and the produced graph state, inviting the question of whether one can attain any desired state through an appropriate choice of boundary control. The main research question of this project is thus *for which scaling invariant vertex conditions can the Lasso graph attain controllability?*

Our inquiries are limited to the case of the Schrödinger operator's electric potential q being identically zero. Also, the magnetic potential a is assumed to be identically zero, apart from in Section 5. Throughout most of this text, the wave operator containing the Schrödinger operator will just be the regular wave equation containing the Laplacian.

The text is strutured such that Section 2 covers the necessary background on quantum graphs, introducing two relevant graphs as well as the Laplacian and vertex conditions. In Section 3 the wave equation is explicitly solved on different geometries with different boundary conditions and a proof of uniqueness is given. In Section 4 the inverse control problem on the Lasso graph is introduced and solved for certain scaling invariant vertex conditions. Lastly, in Section 5 boundary controllability is established for the Lasso graph with a magnetic potential and standard vertex conditions.

The principal results of this study are listed as follows.

- When scaling invariant vertex conditions prevail at the loop vertex and there is no magnetic potential present:
  - Controllability is attained at time $T = 3\ell$ if and only if the scattering matrix S fulfills $S_{31} \neq \pm S_{21}$ (Theorem 4.6, wherein we also invert the control operator for $T = 3\ell$)
  - If $S_{31} \neq \pm S_{21}$, then the minimal control time is $T_{\min} = 3\ell$ (Theorem 4.3), otherwise the system never attains controllability (Section 4.4)
  - The family of scaling invariant vertex conditions that yield controllability is parametrized (Theorem 4.9)
- When standard vertex conditions prevail at the loop-vertex and there is a magnetic potential $a \not\equiv 0$ present:
  - Controllability is attained at time $T = 3\ell$ if and only if the magnetic potential a fulfills a non-resonance condition (Theorem 5.2)

Suggestions for further research include investigating which Robin conditions yield controllability and identifying vertex conditions fulfilling the non-symmetry condition (1) that do (or do not) yield controllability of the Lasso graph.

2 Quantum graphs

The aim of this chapter is to gently introduce the reader to the field of quantum graphs, which is the study of differential operators acting on functions on metric graphs. One such family of second-order operators are *Schrödinger operators*, whose differential expression

$$\tau_{q,a} = \left(i\frac{d}{dx} + a(x)\right)^2 + q(x)$$

includes a 'magnetic' potential a and an 'electric' potential q. This section exclusively concerns Laplacians on metric graphs, i.e. Schrödinger operators with $q \equiv 0$ but $a \not\equiv 0$, but magnetic Schrödinger operators (having $q \equiv 0$ and $a \not\equiv 0$) are revisited in Section 5. As stated in the introduction, Schrödinger operators with electric potentials are beyond the scope of this text.

In Section 2.1 we define metric graphs and introduce the two recurring examples of metric graphs (the Y-graph and the Lasso graph). Then, Section 2.2 presents some notation for functions on metric graphs, allowing us to define Sobolev spaces over graphs in Section 2.3. Finally, in Section 2.4 we define the Laplacian on metric graphs and introduce vertex conditions, explaining their role and providing examples.

2.1 Metric graphs

A **metric graph** is a graph $\Gamma = (V, E)$ whose edges are intervals (from separate copies) of $\mathbb{R}$ and whose vertices constitute a partition of the set of edge endpoints. In general, half-infinite intervals are admissible as edges, but in this article we are only concerned with compact intervals.

$$\text{Edges:} \qquad E = \{E_n\}_{n=1}^{N} \qquad E_n = [x_{2n-1}, x_{2n}] \subset \mathbb{R}$$

$$\text{Vertices:} \qquad V = \{v_m\}_{m=0}^{M-1} \qquad \bigcup_{m=0}^{M-1} v_m = \{x_n\}_{n=1}^{2N} \qquad v_i,\ v_j \text{ disjoint unless } i = j$$

There are several ways to construct Γ as a space. The most obvious way to make a metric space out of Γ that reflects vertex connectivity is as the following quotient space

$$\Gamma = \bigsqcup_{n=1}^{N} E_n \big/ V$$

However, as a domain for functions it proves more convenient to merely use the disjoint union

$$\Omega_\Gamma = \bigsqcup_{n=1}^{N} E_n$$

this is for convenience of notation when working with functions that are not continuous at the vertices. Γ refers to the topological space induced by the graph (V, E), although functions "on Γ" are essentially defined on Ω_Γ.

A recurring set in this text is the 'vertex-less graph' $\Gamma \backslash V$, which is just the interior of Ω_Γ

$$\Gamma \backslash V = \Omega_\Gamma^\circ$$

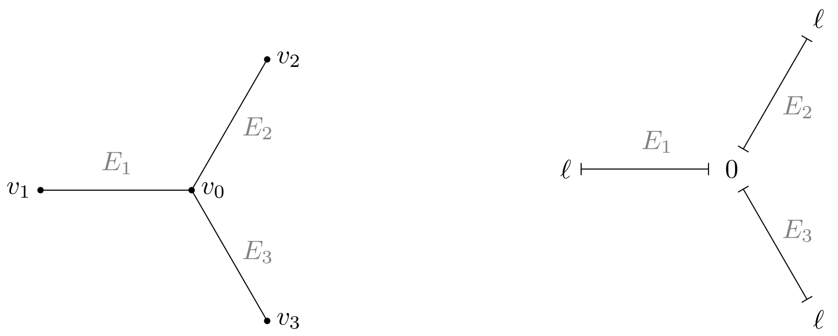
Now we introduce the two recurring graphs of this text: the Y-graph of Example 2.1, and the Lasso graph of Example 2.2.

Example 2.1 [The Y-graph]
The Y-graph is a star graph on three edges, as depicted in Figure 1a. For convenience we assume it is equilateral.

We shall use the following parametrisation, which is illustrated in Figure 1b

$$\begin{aligned} &\text{Edges:} && E_n = [0,\ell] \qquad \forall n = 1,2,3 \\ &\text{Vertices:} && v_n = \{\ell\} \subset E_n \qquad \text{for } n = 1,2,3 \quad \text{and} \quad v_0 = \{\underbrace{0}_{\in E_1}, \underbrace{0}_{\in E_2}, \underbrace{0}_{\in E_3}\} \end{aligned}$$

Hereafter we will only indicate which edge an endpoint beongs to when strictly necessary.



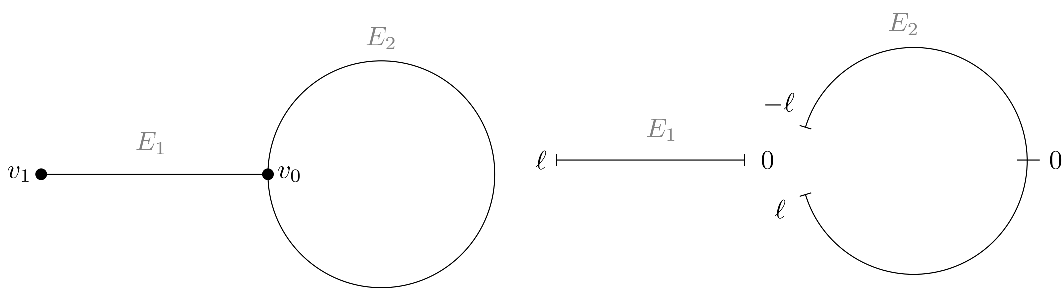
(a) Γ (b) Ω_Γ

Figure 1: The Y-graph illustrated as an equilateral star graph and as a disjoint union of $E_1 = E_2 = E_3 = [0, \ell]$

Example 2.2 [The Lasso Graph]
The principal object of study of this text is the Lasso graph, which - as depicted in Figure 2a - consists of a pendant edge connected to a loop at a central vertex. Again, for the sake of simplicity, assume that the loop is twice the length of the pendant. We shall use the following parametrisation

$$\begin{aligned} &\text{Edges:} && E_1 = [0,\ell] \qquad E_2 = [-\ell,\ell] \\ &\text{Vertices:} && v_1 = \{\ell\} \subset E_1 \qquad v_0 = \{0,-\ell,\ell\} \end{aligned}$$

as illustrated in Figure 2b



(a) Γ (b) Ω_Γ

Figure 2: The Lasso graph with a pendant edge E_1 and a loop E_2, here depicted both as a metric graph and a disjoint union

2.2 Functions on metric graphs

Functions u on a metric graph Γ will follow the notation

$$u : \Omega_\Gamma \longrightarrow \mathbb{C} \qquad\qquad u(x) \text{ for } x \text{ in some } E_n \in E$$

Sometimes, when convenient, we will use vector notation, indexing the components of u in accordance with edges.

$$\vec{u}(x) = (u_1(x_1), \dots, u_N(x_N)) \qquad \vec{x} \in \bigotimes_{n=1}^{N} E_n,\ x_n \in E_n$$

The value of u at a vertex v_0 is thus not well-defined as a scalar, and is instead represented with a vector containing u evaluated at all endpoints x_i contained in v_n:

$$\vec{u}(v_m) = (u(x_i))_{x_i \in v_m}$$

Likewise, u lacks a well defined derivate at the vertices, why a vector of **normal derivates** is taken instead. The normal derivate of u at an endpoint $x \in E_n$ is defined as

$$\partial u(x) = \lim_{y \to x,\ y \in E_n^\circ} \frac{u(x) - u(y)}{|x - y|} \qquad = \begin{cases} \lim_{y \to x^+} u'(y) & \text{if } x \text{ is a left endpoint} \\ -\lim_{y \to x^-} u'(y) & \text{if } x \text{ is a right endpoint} \end{cases} \tag{2}$$

and the derivate of u at a vertex v_m is analogously written as

$$\partial \vec{u}(v_m) = (\partial u(x_i))_{x_i \in v_m}$$

The pair $\vec{u}(v_m)$, $\partial \vec{u}(v_m)$ will be referred to as the **vertex state of u at v_m**.

Example 2.3 [Loop vertex state for the Y-graph and the Lasso graph]
Keeping the parametrisation of Examples 2.1 and 2.2, consider a function u on the Y-graph and on the Lasso graph. Both graphs have a central valence-3 vertex v_0, at which the state of u is written as follows

$$\text{Y-graph}: \qquad \vec{u}(v_0) = \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} \qquad\qquad \partial \vec{u}(v_0) = \begin{bmatrix} u_1'(0) \\ u_2'(0) \\ u_3'(0) \end{bmatrix}$$

$$\text{Lasso graph}: \qquad \vec{u}(v_0) = \begin{bmatrix} u_1(0) \\ u_2(-\ell) \\ u_2(\ell) \end{bmatrix} \qquad\qquad \partial \vec{u}(v_0) = \begin{bmatrix} u_1'(0) \\ u_2'(-\ell) \\ -u_2'(\ell) \end{bmatrix}$$

The integral of u over Γ is carried out edge-wise,

$$\int_\Gamma u(x)\, dx = \sum_{n=1}^{N} \int_{E_n} u(x)\, dx$$

Throughout this text, functions on metric graphs are assumed to be Lebesgue-integrable, in the sense of the Lebesgue measure on Γ induced by the Lebesgue measure on the real line.

2.3 Sobolev spaces over metric graphs

To properly define the Laplacian on a metric graph we will first introduce Sobolev spaces over metric graphs.

Let $\Omega = (a, b) \subset \mathbb{R}$ be a bounded open interval and recall that $v \in L^1(\Omega)$ is a **weak derivate** of $u \in L^1(\Omega)$ if

$$\int_\Omega u(x)\varphi'(x)\,dx = -\int_\Omega v(x)\varphi'(x)\,dx \qquad \forall \varphi \in C_0^\infty(\Omega)$$

The first order Sobolev space over this domain is defined as folllows.

Definition 2.4 [First order Sobolev space on a segment]
For $\Omega = (a, b)$ and $p \in [1, \infty)$, the first-order Sobolev space over Ω is defined as

$$W^{1,p}(\Omega) = \{u \in L^1(\Omega) \;:\; u \text{ has a weak derivate } u' \in L^1(\Omega)\}$$
$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_a^b |u|^p + |u'|^p\,dx\right)^{\frac{1}{p}}$$

Example 2.5 [Functions in and not in $W^{1,p}(\Omega)$]
Let $\Omega = (-1, 1)$. Then:
i) the Heaviside function $u = \chi_{\mathbb{R}_{\geq 0}}(x)$ in not weakly differentiable, although it is in $L^1(\Omega)$
ii) the absolute value $u(x) = |x|$ is weakly differentiable, with the weak derivate

$$u'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Since Ω is bounded, $L^p(\Omega) \subset L^1(\Omega)$ for $p \in (1, \infty)$ by Jensen's inequality;

$$\left(\frac{1}{b-a}\int_\Omega |u(x)|\,dx\right)^p \leq \frac{1}{b-a}\int_\Omega |u|^p\,dx$$
$$\|u\|_{L^1} = \int_\Omega |u|\,dx \leq \left((b-a)^{p-1}\int_\Omega |u|\,dx\right)^{1/p} = (b-a)^{\frac{p-1}{p}}\|u\|_{L^p}$$

and the same holds for the Sobolev space; $W^{1,p}(\Omega) \subset W^{1,1}(\Omega)$.

Theorem 2.6 [Characterization of $u \in W^{1,p}(\Omega)$]
Every $u \in W^{1,1}(\Omega)$ has a representative $g \in C(\overline{\Omega})$.
Further, u is classically differentiable a.e. on Ω.

Proof. As a member of $L^1(a, b)$, u is continuous a.e. on (a, b) ([6] Theorem 2.14.1), so there exists a point $x_0 \in (a, b)$ at which u is continuous. Define g as follows

$$g(x) = u(x_0) + \int_{x_0}^x u'(t)\,dt \qquad x \in [a, b]$$

By [6] thm 2.14.1, since $u' \in L^1(\Omega)$, g is classically differentiable a.e and $g' = u'$ a.e.
Let $(u_n) \subset C^1(a, b)$ approximate u in the Sobolev norm; then (u_n') converges to u' in mean and $u_n(x_0) \to u(x_0)$ as $n \to \infty$ by continuity at x_0 (otherwise $\int_{B_r(x_0)} |u_n - u|\,dx$ would not vanish as $n \to \infty$, contradicting convergence in mean).

By the fundamental theorem of analysis, u_n has the following expression and can be continuously extended to $[a, b]$

$$u_n(x) = u_n(x_0) + \int_{x_0}^x u_n'(t)\,dt \qquad x \in [a, b]$$

Note that $u_n \to g$ uniformly on $[a,b]$

$$\begin{aligned}|g(x) - u_n(x)| &\le |u(x_0) - u_n(x_0)| + \int_{x_0}^{x} |u'(t) - u_n'(t)|\,dt \\ &\le \underbrace{|u(x_0) - u_n(x_0)|}_{\to 0 \text{ as } n\to\infty} + \underbrace{\int_a^b |u'(t) - u_n'(t)|\,dt}_{\to 0 \text{ as } n\to\infty}\end{aligned}$$

By the uniqueness of limits, $u = g$ a.e. on $\overline{\Omega}$ and, as a uniform limit of continuous functions, $g \in C([a,b])$. □

Although the boundary limits of $u \in W^{1,1}(\Omega)$ may exist, the same need not hold for u'.

Example 2.7 [u' **may lack boundary limits**]
Consider $u = x^2 \sin(\frac{1}{x}) \in C^1(0,1)$, which can be continuously extended to $[0,1]$. Its derivate, however

$$u' = \cos(\frac{1}{x}) + 2x\sin(\frac{1}{x}) \in C((0,1)), \qquad \notin C([0,1])$$

is bounded and continuous on $(0,1)$, but does not converge when $x \to 0$.

To ascribe both u and u' boundary values, one can require u' to have a weak derivate of its own. For this endeavor we analogously define the second order Sobolev space.

Definition 2.8 [Second order Sobolev space on a segment]
For $\Omega = (a,b)$ and $p \in [1,\infty)$, the second-order Sobolev space over Ω is defined as

$$W^{2,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) \ : \ u' \text{ has a weak derivate } u'' \in L^1(\Omega)\right\}$$

$$\|u\|_{W^{2,p}(\Omega)} = \left(\int_a^b |u|^p + |u'|^p + |u''|^p\,dx\right)^{\frac{1}{p}}$$

Now, if $u \in W^{2,p}(\Omega)$, then $u' \in W^{1,p}(\Omega)$ has a representative in $C([a,b])$. Functions $u \in W^{2,p}(\Omega)$ can by Theorem 2.6 thus be characterized as $C^1(\overline{\Omega})$ with a classical second derivate a.e. We shall now translate this to the context of graphs.

Definition 2.9 [Sobolev space on a graph]
Given a graph Γ with N (compact) edges $E_n \subset \mathbb{R}$, we define the first order Sobolev space

$$W^{1,p}(\Gamma\backslash V) := \bigoplus_{n=1}^{N} W^{1,p}(E_n^\circ) \qquad \|u\|_{W^{1,p}(\Gamma\backslash V)} = \left(\sum_{n=1}^{N} \|u\|^p_{W^{1,p}(E_n^\circ)}\right)^{\frac{1}{p}}$$

Remark. The second-order Sobolev space on a graph is defined analogously, and since functions $u \in W^{2,2}(\Gamma\backslash V)$ are $C^1(\Omega_\Gamma)$ (by Theorem 2.6), we have that $\vec{u}(v)$ and $\partial\vec{u}(v)$ are well-defined at all vertices $v \in V$.

2.4 Laplacians on metric graphs

The Laplacian L on a graph Γ is defined over the Sobolev space of square-integrable, twice weakly differentiable functions on Γ:

$$L : W^{2,2}(\Gamma\backslash V) \longrightarrow L^2(\Gamma) \qquad u \longmapsto -u'' \text{ (weak derivate)}$$

where $W^{2,2}(\Gamma\backslash V)$ and $L^2(\Gamma)$ are defined as

$$W^{2,2}(\Gamma\backslash V) = \bigoplus_{n=1}^{N} W^{2,2}(E_n^\circ) \qquad L^2(\Gamma) = \bigoplus_{n=1}^{N} L^2(E_n^\circ)$$

So far the connectivity of Γ has had no bearing on the Laplacian or the Sobolev space, but this connectivity will be reflected in the conditions imposed on $\mathrm{Dom}(L)$ to make L symmetric.

For any $u, w \in \mathrm{Dom}(L) \subset W^{2,2}(\Gamma\backslash V)$

$$\begin{aligned}
\langle Lu, w\rangle_{L^2(\Gamma\backslash V)} &= -\int_{\Gamma\backslash V} u''\overline{w}\, dx = \int_{\Gamma\backslash V} u'\overline{w}'\, dx - [u'\overline{w}]_{x\in\partial(\Gamma\backslash V)} \\
\langle u, Lv\rangle_{L^2(\Gamma\backslash V)} &= -\int_{\Gamma\backslash V} u\overline{w}''\, dx = \int_{\Gamma\backslash V} u'\overline{w}'\, dx - [u\overline{w}']_{x\in\partial(\Gamma\backslash V)} \\
[u'\overline{w}]_{\partial(\Gamma\backslash V)} &= \sum_{E_n\in E} [u'\cdot\overline{w}]_{\partial E_n} = \sum_{n=1}^{N} (u'(x_{2n})\cdot\overline{w}(x_{2n}) - u'(x_{2n-1})\cdot\overline{w}(x_{2n-1})) \\
&= -\sum_{v\in V} \vec{w}(v)^{\dagger}\cdot\partial\vec{u}(v)
\end{aligned}$$

L is in general not symmetric on $W^{2,2}(\Gamma\backslash V)$ and its symmetry is contingent on the boundary form vanishing, which restricts the admissible vertex states of elements in $\mathrm{Dom}(L)$:

$$\begin{aligned}
\langle Lu, w\rangle_{L^2(\Gamma\backslash V)} - \langle Lv, u\rangle_{L^2(\Gamma\backslash V)} &= [u\overline{w}' - u'\overline{w}]_{x\in\partial(\Gamma\backslash V)} \\
&= \sum_{v\in V} \langle\partial\vec{u}(v), \vec{w}(v)\rangle_{\mathbb{C}^{|v|}} - \langle\vec{u}(v), \partial\vec{w}(v)\rangle_{\mathbb{C}^{|v|}} \qquad (3)
\end{aligned}$$

Such conditions on $(\vec{u}(v), \partial\vec{u}(v))_{v\in V}$ are called **vertex conditions**.

The boundary form (3) can be made to vanish at a vertex v by, for instance, imposing homogeneous Dirichlet-, von Neumann- or Robin conditions on the elements of $\mathrm{Dom}(L)$

$$\begin{aligned}
\textbf{Dirichlet:}\quad & \vec{u}(v) = 0 \\
\textbf{von Neumann:}\quad & \partial\vec{u}(v) = 0 \\
\textbf{Robin:}\quad & \partial\vec{u}(v) = H\vec{u}(v) \qquad H \in \mathbb{M}_{|v|}(\mathbb{C}),\ H = H^{\dagger}
\end{aligned}$$

In fact, all vertex conditions that make the boundary form vanish can be written on the form $A\vec{u}(v) = B\partial\vec{u}(v)$, provided $AB^{\dagger}$ is Hermitian and $\mathrm{rank}(A; B) = |v|$ ([10] theorem 3.2). However, this does not necessarily reflect the connectivity of the graph (in the sense that some vertex v splits into two or more non-adjacent vertices).

The set of vertex conditions that make the boundary form (3) vanish and reflect the connectivity at v is parametrized by the family of all unitary, irreducible[4] matrices $S \in \mathbb{M}_{|v|}(\mathbb{C})$ ([10] chapter 3.3.3 provides a thorough exposition). The matrix S associated with a vertex v is called the **vertex scattering matrix** and corresponds to the vertex condition

$$i(S - I)\vec{u}(v) = (S + I)\partial\vec{u}(v) \qquad S \in \mathbb{M}_{|v|}(\mathbb{C}),\ S^{-1} = S^{\dagger}$$

As shall be shown in Section 3, S determines the transmissive and reflective behaviour of a wave incident to v - hence the name 'scattering matrix'. Of particular interest to us is the case when this scattering behaviour is described by mere reflection- and transmission coefficients (i.e. the scattering merely re-scales the incoming wave along each edge). This occurs when S is Hermitian, which (because S is unitary and irreducible) is equivalent to S having the eigenvalues $\sigma(S) = \{-1, 1\}$. Such vertex conditions are called **scaling invariant**. Now we provide two examples of scaling invariant conditions; Standard conditions and Householder conditions.

[4] a matrix is said to be irreducible if it cannot be transformed into upper block-triangular form by permutation of coordinates ([10] chapter 3.3.3)

Example 2.10 [Standard conditions]
Out of all vertex conditions, one in particular conforms to physical intuition; **standard vertex conditions** require u to be continuous at each vertex (the continuity condition), and require the sum of normal derivates at v_0 to be zero (the Kirchoff condition).

$$\text{Continuity condition:} \qquad \vec{u}(v) = u(v) \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{C}^{|v|} \tag{4}$$

$$\text{Kirchoff condition:} \qquad \sum_{x_j \in v} \partial u_j(x_j) = 0 \tag{5}$$

The corresponding scattering matrix at a vertex of degree d is given by[5]

$$S^{\text{st}} = 2 \begin{bmatrix} \frac{1}{\sqrt{d}} \\ \vdots \\ \frac{1}{\sqrt{d}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{d}} & \cdots & \frac{1}{\sqrt{d}} \end{bmatrix} - I = \begin{bmatrix} \frac{2-d}{d} & \frac{2}{d} & \frac{2}{d} & \cdots \\ \frac{2}{d} & \frac{2-d}{d} & \frac{2}{d} & \cdots \\ \frac{2}{d} & \frac{2}{d} & \frac{2-d}{d} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

We shall return to standard conditions in Section 3.2.

Example 2.11 [Householder conditions]
Consider the following family of matrices, called Householder matrices[6], which is defined as

$$\mathcal{H} = \{A = I - 2v^*v \mid v \text{ is a unit vector}\}$$

To see that A is Hermitian:

$$(v^*v)^T = v^T\overline{v} = \overline{(v^*v)} \qquad \Rightarrow \qquad v^*v = (v^*v)^* \qquad \Rightarrow \qquad I - 2v^*v = (I - 2v^*v)^*$$

To see that A is unitary:

$$A \cdot A^* = A^2 = I^2 - 4v^*v + 4v^* \underbrace{vv^*}_{=1} v = I - 4v^*v + 4v^*v = I$$

i.e. $A^* = A^{-1}$. A must have both ± 1 as eigenvalues because if it only had one or the other then $A = \pm I$. This would require either $v^*v = I$ or $v^*v = 0$, which are both impossible when v is a unit vector.

All Householder matrices do not describe scaling invariant vertex conditions; if any component v_i of v is zero, then row i and column i of $A = I - 2v^*v$ are zero outside the diagonal, and A is hence reducible. On the other hand, if all components of v are non-zero then all entries of A outside the diagonal (and at least one diagonal element) are non-zero and A is hence irreducible.

Hence the Householder matrix $A = I - 2v^*v$ is an instance of scaling invariant vertex conditions precisely when all components of v are nonzero. We shall call the vertex conditions described by these matrices **Householder conditions**.

To a lesser extent, we shall also study Robin conditions, which occur when $\sigma(S) \cap \{-1, 1\} = \varnothing$, allowing the vertex conditions to be expressed with a Hermitian matrix H

$$\partial\vec{u}(v_0, t) = H\vec{u} \qquad\qquad H := i\frac{S - I}{S + I} \tag{7}$$

In Section 3 we return to and elaborate on both scaling invariant and Robin conditions.

[5] see [10] chapter 3.5.3, for derivation of this form
[6] named after the American mathematician Alan Scott Householder

3 Wave scattering

For the purpose of defining the control operator in Section 4, this section is dedicated to solving the wave equation on different geometries with different vertex conditions and establishing the uniqueness of the solution. This is because a well posed PDE-problem with an input (the boundary control) constitutes a deterministic system, making the input-to-output map (the control operator) well defined.

Instead of tackling the Lasso graph head on, Section 3.1 starts slow by solving the wave equation with a boundary control (from now on referred to as the PDE-problem) on the half line and establishing uniqueness using an energy method. In Section 3.2 we solve the corresponding PDE-problem on the Y-graph (from Example 2.1) with standard vertex conditions, demonstrating their influence on wave scattering. Uniqueness of the solution is shown by adapting the same energy method to the Y graph. Section 3.3 is a brief elaboration on energy methods for general graphs, providing a framework for showing uniqueness when the vertex conditions are homogeneous Dirichlet, scaling invariant or Robin-type. Sections 3.4 and 3.5 solve the PDE-problem on the Y-graph with scaling invariant and Robin conditions respectively. Uniqueness is also shown, but is for Robin conditions only established when the matrix H (in Equation (7)) is positive semidefinite. Lastly, in Section 3.6 the PDE-problem on the Lasso graph is solved for both scaling invariant- and Robin vertex conditions, with uniqueness proofs transferring directly from the Y-graph.

3.1 Wave Propagation on the half line

Consider a simple PDE problem; that of the wave equation on the half line with a boundary control.

$$\begin{aligned} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0 \qquad && (x,t) \in \mathbb{R}^2_{>0} \\ u(x,0) &\equiv 0 && \\ \partial_t u(x,0) &\equiv 0 && \\ u(0,t) &= f(t) && f \equiv 0 \text{ on } \mathbb{R}_{\leq 0} \end{aligned} \tag{8}$$

A priori, we only consider boundary controls $f \in C^2(\mathbb{R})$.

Theorem 3.1 [Uniqueness of solution: Half Line]
Over the space

$$X = \left\{ u(x,t) \in C^2(\mathbb{R}^2_{\geq 0}) : \ \forall t \, (u(\cdot\,,t) \text{ has compact support}) \right\}$$

The problem (8) admits at most one solution.

Proof. Suppose there are two solutions u_1, u_2 to the wave equation. Then their difference $u = u_1 - u_2$ solves problem (8) but with $f \equiv 0$. At any point in time t, the difference u can be assigned an **energy** $E(t)$

$$E_u(t) = \frac{1}{2} \int_0^\infty |u_t(x,t)|^2 + |u_x(x,t)|^2 \, dx = \frac{1}{2} \int_0^\infty |\nabla u|^2 \, dx$$

Note that the energy of u does not change over time

$$\begin{aligned}
2\cdot\frac{dE}{dt} &= \frac{d}{dt}\int_0^\infty |u_t|^2+|u_x|^2\,dx = \int_0^\infty \frac{\partial}{\partial t}(|u_t|^2+|u_x|^2)\,dx \\
&= \int_0^\infty u_t\overline{u_{tt}} + u_{tt}\overline{u_t} + u_{xt}\overline{u_x} + u_x\overline{u_{xt}}\,dx \\
&= \int_0^\infty u_t\overline{u_{tt}} + u_{tt}\overline{u_t} - u_t\overline{u_{xx}} - u_{xx}\overline{u_t}\,dx + [u_t\overline{u_x} + u_x\overline{u_t}]_{x=0}^\infty \\
&= \int_0^\infty \underbrace{u_t(\overline{u_{tt}} - \overline{u_{xx}})}_{=0\text{ by PDE}} + \underbrace{\overline{u_t}(u_{tt} - u_{xx})}_{=0\text{ by PDE}}\,dx + [u_t\overline{u_x} + u_x\overline{u_t}]_{x=0}^\infty \\
&= \underbrace{\overline{u_t(0,t)}u_x(0,t) + \overline{u_x(0,t)}u_t(0,t)}_{=0\text{ by initial cond.}} - \underbrace{\overline{u_t(\infty,t)}u_x(\infty,t) - \overline{u_x(\infty,t)}u_t(\infty,t)}_{=0\text{ by compact supp.}} = 0
\end{aligned}$$

Together with the fact that $E_u(0) = 0$, this implies that $E_u(t) \equiv 0$, forcing $\nabla u \equiv 0$ which together with the initial conditions means that $u \equiv 0$. Hence $u_1 = u_2$. □

The solution of problem (8) is acquired with a partial Laplace-transform with respect to t:

$$\hat{u}(x,s) := \int_0^\infty e^{-st}u(x,t)\,dt \qquad \mathrm{Re}(s) > 0$$

The PDE becomes a second order ODE

$$s^2\hat{u}(x,s) - s\underbrace{u(x,0)}_{=0} - \underbrace{u_t'(x,0)}_{=0} - \hat{u}_x''(x,t) = 0 \qquad\qquad \hat{u}(0,s) = \hat{f}(s)$$

having the solution

$$\hat{u}(x,s) = a(s)e^{isx} + b(s)e^{-isx}$$

Note that isx is not pure imaginary when $\mathrm{Re}(s) > 0$ and that e^{isx} hence diverges when $x \to \infty$, which is not the Laplace transform of a bounded or integrable function. Hence $a(s) \equiv 0$. The boundary condition thus yields $\hat{u}(0,s) = b(s) = \hat{f}(s)$ and consequently the solution is

$$u(x,t) = f(t-x)\theta(t-x) \qquad\qquad \theta(t-x) = \begin{cases} 1 & \text{if } t \geq x \\ 0 & \text{if } t < x \end{cases}$$

This is a right-traveling wave with the left-right inverted shape of f. Most importantly, the state $u(x,T)$ over $x \in [0,T)$ can be selected through our choice of signal $f(t)$.

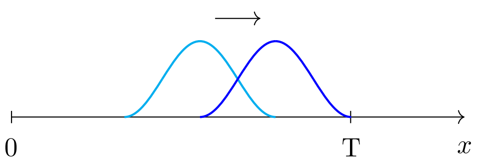


Figure 3: A boundary control at $x = 0$ producing a right-traveling wave

This behaviour of disturbances radiating from a source at a fixed speed is characteristic of the wave equation. Compare the boundary control in this problem on the half-line to someone vertically shaking the end of an infinite rope; the hand's exact motion determines the shape traveling down the rope.

3.2 Wave scattering on the Y-graph with standard conditoins

Before studying the wave equation on the Lasso graph, we shall illustrate the role of vertex conditions in wave scattering using a simpler example: the Y graph. Using the same approach with regards to vertex conditions, we will start with standard vertex conditions before transitioning to scaling invariant conditions generally and finally consider Robin conditions.

In plain language the question is: if the wave in problem (8), instead of traveling down a perfect line, somewhere met a junction, how would it scatter? Suppose the Y-graph of Section 2.1 had a boundary control at its leaf, as depicted in Figure 4.

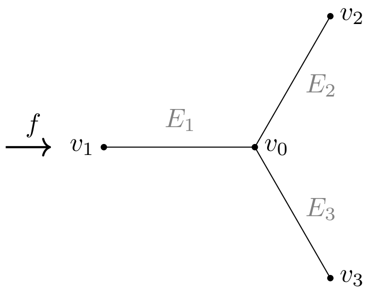


Figure 4: The Y-graph of Section 2.1 with a time-dependent boundary control $f(t)$ at v_1.

Momentarily setting aside the vertex conditions at v_0, consider the following PDE-problem:

$$\begin{aligned} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0 & & t \in (0,\infty), x \in \Gamma \backslash V \\ u(x,0) = \partial_t u(x,0) &\equiv 0 & & \\ u(v_1,t) &= f(t) & & f \equiv 0 \text{ on } \mathbb{R}_{\leq 0} \\ u(v_2,t) = u(v_3,t) &= 0 & & \end{aligned} \tag{9}$$

Drawing on our physical intuition, we expect the Y-graph to initially behave like the half line, with a lone right-traveling wave incident to v_0.

$$t \in (0,\ell) \qquad u(x,t) = \begin{cases} f(t-(\ell-x)) & \text{if } x \in E_1 \\ 0 & \text{if } x \in E_2 \\ 0 & \text{if } x \in E_3 \end{cases} \tag{10}$$

Once the incoming wave reaches the loop vertex, an outgoing wave a arises: a reflected wave on E_1 and a transmitted wave on each of E_2 and E_3.

$$t \in (\ell,2\ell) \qquad u(x,t) = \begin{cases} f(t-(\ell-x)) + a_1(t-x) & \text{if } x \in E_1 \\ a_2(t-x) & \text{if } x \in E_2 \\ a_3(t-x) & \text{if } x \in E_3 \end{cases} \tag{11}$$

$$u_x(x,t) = \vec{f}'(t-(\ell-x)) - \vec{a}'(t-x) = 2\vec{f}'(t-(\ell-x)) - \vec{u}_t'(t-x) \tag{12}$$

The vertex conditions at v_0 determine a. Let them be standard, meaning that u is continuous at v_0 and that the sum of normal derivates at v_0 is zero.

$$u(0_{E_1},t) = u(0_{E_2},t) = u(0_{E_3},t) \qquad \sum_{n=1}^{3} u_x(0_{E_n},t) = 0 \tag{13}$$

This yields

$$\sum_{n=1}^{3} \partial_x u_i(0,t) = 2f'(t-\ell) - 3u_t(0,t) = 0$$

i.e. $u_t(0,t) = \frac{2}{3}f'(t-\ell)$. Hence

$$a_3'(t) = a_2'(t) = a_1'(t) + f'(t-\ell) = \underbrace{\frac{2}{3}}_{=T} f'(t-\ell) \ \Rightarrow\ a_1'(t) = \underbrace{-\frac{1}{3}}_{=R} f'(t-\ell)$$

In other words, the outgoing wave can be expressed in terms of the reflection- and transmission coefficients R and T

$$\vec{a}(t-x) = \begin{bmatrix} R \\ T \\ T \end{bmatrix} f(t-(\ell+x))$$

For $t \in (\ell, 2\ell)$ the solution becomes

$$t \in (\ell, 2\ell) \qquad\qquad u(x,t) = \begin{cases} f(t-(\ell-x)) - \frac{1}{3}f(t-(\ell+x)) & \text{if } x \in E_1 \\ \frac{2}{3}f(t-(\ell+x)) & \text{if } x \in E_2 \\ \frac{2}{3}f(t-(\ell+x)) & \text{if } x \in E_3 \end{cases} \tag{14}$$

Using the same principles, one could compute u for even greater t, but this example has served its purpose of illustrating the role of vertex condititons. Before continuing to more general vertex conditions, we prove that the solution provided is the only solution.

Theorem 3.2 [Uniqueness of solution: standard conditions]
The problem (9) admits at most one solution $u(x,t) \in C^2((\Gamma \backslash V) \times \mathbb{R}_{\geq 0})$ that fulfills the standard vertex conditions (13) at v_0.

Proof. As in the proof of Theorem 3.1 we show that the difference u of two solutions u_1, u_2 has energy $E_u(t) \equiv 0$. Note that u satisfies standard conditions at v_0 and homogeneous Dirichlet conditions at v_1, v_2, v_3, which implies that $u_t \equiv 0$ at the degree one vertices. Thus, all endpoint terms vanish in the computation of $E_u'(t)$:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\frac{1}{2}\int_\Gamma |u_t|^2 + |u_x|^2\, dx = \frac{1}{2}\sum_{n=1}^{3} \underbrace{\int_{E_n} \frac{\partial}{\partial t}\left(|u_t|^2 + |u_x|^2\right) dx}_{=[\overline{u_t}u_x + u_t\overline{u_x}]_{\partial E_n}} \\ &= \sum_{n=1}^{3} \Re\Big(\underbrace{u_t(\ell_{E_n}, t)}_{\equiv 0 \text{ as } u(\ell_{E_n},t)\equiv 0} \ \overline{u_x(\ell_{E_n}, t)} \ - \ \underbrace{u_t(0_{E_n}, t)}_{=u_t(0,t) \text{ by continuity}} \ \overline{u_x(0_{E_n}, t)} \Big) \\ &= -\Re\big((u_t(0,t) \cdot \underbrace{\sum_{i=1}^{3} \overline{u_x(0_{E_n}, t)}}_{\equiv 0 \text{ by Kirchoff}}\big) = 0 \end{aligned}$$

where $\Re(\cdot)$ denotes taking the real part of a complex number. Since the initial condition are identically zero, we have that $E_u(0) = 0$, which in turn implies that $E_u(t) \equiv 0$, making the solution of (9) unique. □

3.3 Energy methods for graphs

While still on the note of uniqueness, we shall examine how energy methods can be used to prove uniqueness for arbitrary graphs with homogeneous Dirichlet-, scaling invariant, or certain Robin conditions. Our approach is a modified version of the one used by Evans in [5].

In this section, Γ denotes an arbitrary compact graph and $u(x,t)$ is assumed to satisfy the homogeneous wave equation on Γ with trivial initial conditions.

$$(\partial_t^2 - \partial_x^2)u(x,t) = 0 \qquad x \in \Gamma\backslash V,\ t \in [0,T] \tag{15}$$
$$u(x,0) = u_t(x,0) \equiv 0 \qquad x \in \Omega_\Gamma \tag{16}$$

For the sake of simplicity, we only consider classical solutions $u \in C^2(\Omega_\Gamma \times [0,T])$.

Define the **energy functional** $E_u(t)$ of u at a given time t as

$$E_u(t) = \frac{1}{2}\int_{\Gamma\backslash V} |\nabla u(x,t)|^2\, dx = \frac{1}{2}\int_{\Gamma\backslash V} \underbrace{|u_x|^2 + |u_t|^2}_{u_x\overline{u_x}+u_t\overline{u_t}}\, dx \tag{17}$$

The change in energy over time is given by

$$\begin{aligned}\frac{d}{dt}E_u(t) =& \frac{1}{2}\int_{\Gamma\backslash V} u_{xt}\overline{u_x} + u_{tt}\overline{u_t} + u_x\overline{u_{xt}} + u_t\overline{u_{tt}}\, dx \\ =& \frac{1}{2}\left(\int_{\Gamma\backslash V} u_t\overline{u_{tt}} + \overline{u_t}u_{tt} - (\overline{u_t}u_{xx} + u_t\overline{u_{xx}})\, dx + [\overline{u_t}u_x + u_t\overline{u_x}]_{x\in\partial(\Gamma\backslash V)}\right) \\ =& \frac{1}{2}\left(\int_{\Gamma\backslash V} u_t(\overline{u_{tt}} - \overline{u_{xx}}) + \overline{u_t}(u_{tt} - u_{xx})\, dx + [\overline{u_t}u_x + u_t\overline{u_x}]_{x\in\partial(\Gamma\backslash V)}\right)\end{aligned}$$

For any $u(x,t)$ satisfying the wave equation (15), the integral vanishes, leaving us with

$$\begin{aligned}\frac{d}{dt}E_u(t) &= \frac{1}{2}[\overline{u_t}u_x + u_t\overline{u_x}]_{x\in\partial(\Gamma\backslash V)} \\ &= -\frac{1}{2}\sum_{v\in V}\langle\partial\vec{u}(v,t), \vec{u}_t(v,t)\rangle_{\mathbb{C}^{|v|}} + \langle\vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}\end{aligned} \tag{18}$$

Where $\vec{u}(v,t) \in \mathbb{C}^{|v|}$.

Theorem 3.3 [Total energy over a graph]
Suppose $u \in C^2(\Omega_T, \mathbb{C})$ satisfies the wave equation. Then its energy at any time $T \geq 0$ can be expressed as

$$E_u(T) = E_u(0) - \frac{1}{2}\sum_{v\in V}\int_0^T \langle\partial\vec{u}(v,t), \vec{u}_t(v,t)\rangle_{\mathbb{C}^{|v|}} + \langle\vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}\, dt \tag{19}$$

Notably, the first term vanishes if u satisfies the homogeneous initial conditions (16).

A physical interpretation of this theorem is that the energy of the system can only be created or removed at the vertices, which either act as sources, sinks or preserve the energy. For convenience, let $e_v(t)$ denote the summand of (18) corresponding to a vertex v (which can be interpreted as the instantaneous source intensity of v)

$$e_v(t) := \frac{1}{2}\left(\langle\partial\vec{u}(v,t), \vec{u}_t(v,t)\rangle_{\mathbb{C}^{|v|}} + \langle\vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}\right) = \Re\big(\langle\vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}\big) \tag{20}$$

Example 3.4 [Energy conservation: Dirichlet- and standard conditions]
Clearly, if Homogeneous Dirichlet conditions prevail at v then the time derivates vanish

in equation (20), making $e_v(t) \equiv 0$.

Likewise, if standard conditions prevail at a vertex v, then (by continuity of u at v) the vector of time derivates $\partial_t \vec{u}(v,t)$ is parallell to the vector of ones. Meanwhile, by the Kirchoff condition, the sum of normal derivates at v is zero. Hence $e_v(t)$ vanishes;

$$\vec{u}_t(v,t) = u_t(v,t) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \Rightarrow \quad \langle \partial\vec{u}(v,t), \vec{u}_t(v,t)\rangle_{\mathbb{C}^{|v|}} = \overline{u_t(v,t)} \underbrace{\sum_{x_j \in v} \partial u(x_j,t)}_{=0 \text{ by Kirchoff}} = 0 \tag{21}$$

$$\Rightarrow \quad e_v(t) = \Re\big(\langle \vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}\big) = 0 \tag{22}$$

In other words, no energy is 'produced' or 'consumed' at vertices with homogeneous Dirichlet or standard conditions, and we shall show that this is also true for all scaling invariant vertex conditions. The underlying reason for this is that if v has scaling invariant conditions, then $\vec{u}(v)$ and $\partial\vec{u}(v)$ are orthogonal.

Theorem 3.5 [Unitary operators have orthogonal eigenspaces]
For any unitary operator A, if x, y are eigenvectors with distinct eigenvalues λ, μ, then x and y are orthogonal.

Proof. We use the fact that $y = \frac{1}{\mu}\mu y = \frac{1}{\mu}Ay$ and that $\frac{1}{\mu} = \overline{\mu}$ since $|\mu| = 1$;

$$\lambda\langle x, y\rangle = \langle Ax, y\rangle = \langle x, A^{-1}y\rangle = \langle x, \frac{1}{\mu}y\rangle = \mu\langle x, y\rangle$$

Hence $\langle x, y\rangle = 0$ if $\lambda \neq \mu$. □

Recall that scaling invariant vertex conditions are described by a unitary matrix S with the spectrum $\{-1, 1\}$.

$$i(S - I)\vec{u}(v) = (S + I)\partial\vec{u}(v) \qquad \text{where } \sigma(S) = \{-1, 1\}$$

Example 3.6 [Energy conservation: scaling invariant conditions]
Suppose scaling invariant vertex conditions prevail at v.

Let E_1 and E_{-1} denote the (orthogonal, by Theorem 3.5) eigenspaces of S and let P_1, P_{-1} denote their respective eigenprojectors, noting that S has the spectral decomposition $S = P_1 - P_{-1}$. For any vertex state $(\vec{u}, \partial\vec{u})$ fulfilling the vertex conditions (26), $\vec{u}$ must belong to E_1 while $\partial\vec{u}$ must belong to E_{-1}: since S commutes with its eigenprojectors we have that

$$P_{\pm 1}(S \mp I) = (\underbrace{P_{\pm 1}S}_{=SP_{\pm 1}} \mp P_{\pm 1}) = (S \mp I)P_{\pm 1} \equiv 0$$

$$P_{\pm 1}(S \pm I) = (\underbrace{P_{\pm 1}S}_{=SP_{\pm 1}} \pm P_{\pm 1}) = (S \pm I)P_{\pm 1} = 2P_{\pm 1}$$

Upon composition with the vertex conditions the eigenprojections yield

$$\left.\begin{aligned} P_{-1}i(S - I)\vec{u} = 2iP_{-1}\vec{u}(v) = 0 = P_{-1}(S + I)\partial\vec{u}(v) \\ P_1 i(S - I)\vec{u} = 0 = 2P_{-1}\partial\vec{u}(v) = P_1(S + I)\partial\vec{u}(v) \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} P_{-1}\vec{u}(v) = 0 \\ P_1\partial\vec{u}(v) = 0 \end{cases} \tag{23}$$

I.e. $\vec{u}(v)$ and $\partial\vec{u}(v)$ are orthogonal in $\mathbb{C}^{|v|}$, which in the time-dependent context extends to $\vec{u}_t(v,t)$ and $\partial\vec{u}(v,t)$. Consequently $e_v(t) \equiv 0$

$$e_v(t) = \Re\big(\underbrace{\langle \vec{u}_t(v,t), \partial\vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}}_{\equiv 0}\big) \equiv 0$$

Hence, energy is conserved at vertices with scaling invariant conditions. This yields a theorem that will be useful for proving uniqueness.

Theorem 3.7 [Uniqueness of trivial solution]
If each vertex in the graph Γ has homogeneous Dirichlet- or Scaling Invariant conditions then $u(x,t) \equiv 0$ is the unique solution to the wave equation described in (15)-(16).

Proof. Suppose that u is a solution to the PDE-problem (15)-(16). By Theorem 3.3 and equation (20), the total energy of u at time t can be written as

$$E(t) = \underbrace{E(0)}_{=0} - \sum_{v \in V} \int_0^t \underbrace{e_v(\tau)}_{\equiv 0} \, d\tau \equiv 0$$

where $e_v(t) \equiv 0$, according to Examples 3.4 and 3.6, because homogeneous Dirichlet- or scaling invariant conditions prevail at v. Since $E_u(t) = \|\nabla u(\cdot, 0)\|_{L^2(\Gamma \setminus V)} \equiv 0$ and $u(x,0) = u_t(x,0) \equiv 0$, we have that $u \equiv 0$.

□

What happens with other vertex conditions?

Example 3.8 [Energy conservation: some Robin conditions]
If Robin conditions prevail at a vertex v then

$$H\vec{u}(v) = \partial\vec{u}(v) \qquad \text{where } H = H^\dagger \tag{24}$$

The source density $e_v(t)$ can be written in terms of the time derivate of $\langle H\vec{u}(v,t), \vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}}$

$$\begin{aligned} e_v(t) &= \Re\big(\underbrace{\langle \partial\vec{u}(v,t), \vec{u}(v,t)\rangle}_{=\langle H\vec{u}_t(v,t), \vec{u}(v,t)\rangle}\big) \\ &= \frac{1}{2}\big(\langle H\vec{u}_t(v,t), \vec{u}(v,t)\rangle + \underbrace{\langle \vec{u}(v,t), H\vec{u}_t(v,t)\rangle}_{=\langle H\vec{u}(v,t), \vec{u}_t(v,t)\rangle}\big) = \frac{1}{2}\frac{\partial}{\partial t}\langle H\vec{u}(v,t), \vec{u}(v,t)\rangle \end{aligned} \tag{25}$$

Hence the total energy from v depends only on this inner product at time t

$$\begin{aligned} \int_0^t e_v(\tau)\, dt &= \frac{1}{2}\int_0^t \frac{\partial}{\partial \tau}\langle H\vec{u}(v,\tau), \vec{u}(v,\tau)\rangle_{\mathbb{C}^{|v|}} \, d\tau \\ &= \frac{1}{2}\big(\langle H\vec{u}(v,t), \vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}} \langle H\vec{u}(v,0), \underbrace{\vec{u}(v,0)}_{=0}\rangle_{\mathbb{C}^{|v|}} = \frac{1}{2}\langle H\vec{u}(v,t), \vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}} \end{aligned}$$

If H is positive semidefinite then the energy contribution to $E(t)$ from v is negative, by Theorem 3.3. Hence, in a graph where all Robin vertices have positive semi-definite matrices H and all other vertices are of homogeneous Dirichlet-, or scaling invariant type, the energy at time $t \in [0,T]$ is identically zero

$$0 \le E(t) = -\frac{1}{2}\sum_{v \in V_{\text{Robin}}} \langle H\vec{u}(v,t), \vec{u}(v,t)\rangle_{\mathbb{C}^{|v|}} \le 0$$

and then the only solution to the PDE-problem (15) - (16) is $u(x,t) \equiv 0$.

This allows uniqueness to be established whenever $\sigma(H) \subset \mathbb{R}_{\ge 0}$.

3.4 The Y-graph with scaling invariant vertex conditions

Now we consider the Y-graph with arbitrary scaling invariant vertex conditions at v_0, i.e.

$$i(S-I)\vec{u}(v_0,t) = (S+I)\partial\vec{u}(v_0,t) \qquad \sigma(S) = \{-1, 1\} \tag{26}$$

Theorem 3.9 [Problem (9) uniqueness: scaling invariant conditions]
The problem (9) admits at most one solution $u(x,t) \in C^2((\Gamma \backslash V) \times \mathbb{R})$ that fulfills scaling invariant vertex conditions at v_0 and homogeneous Dirichlet conditions at v_2, v_3.

Proof. As in the proof of Theorem 3.2, let u denote the difference of two solutions u_1, u_2 of problem 9. Note that u fulfills the same boundary conditions as u_1 and u_2 at v_0, i.e. u also fulfills the scaling invariant conditions (26). However, instead of having a boundary control at v_1, u fulfills Dirichlet conditions, just like at v_2 and v_3. Thus, by Theorem 3.7, $u \equiv 0$ and $u_1 = u_2$. The solution to the PDE-problem is thus unique. □

The solution is analogous to the one in Section 3.2, only we do not assume conditions are standard

$$u(x,t) = \begin{cases} f(t-(\ell - x)) & \text{if } x \in E_1 \\ 0 & \text{if } x \in E_2 \text{ or } E_3 \end{cases} \qquad t \in (0, \ell) \tag{27}$$

$$\vec{u}(x,t) = \vec{f}(t-\ell+x) + \vec{a}(t-x) \qquad t \in (\ell, 2\ell) \tag{28}$$

$$\vec{u}_x(x,t) = \vec{f}'(t-\ell+x) - \vec{a}'(t-x) \tag{29}$$

differentiating (28) with respect to t, equation (23) now yields that

$$\begin{aligned} P_{-1}\vec{a}'(t) &= -P_{-1}\vec{f}'(t-\ell) \\ P_1\vec{a}'(t) &= P_1\vec{f}'(t-\ell) \end{aligned} \quad \Rightarrow \quad \vec{a}'(t) = \underbrace{(P_1 - P_{-1})}_{=S}\vec{f}'(t-\ell)$$

Hence, $\vec{a}(t) = S\vec{f}(t-\ell)$, meaning waves are transmitted from E_1 to E_2, E_3 with transmission coefficients S_{21} and S_{31};

$$\vec{u}(x,t) = \begin{cases} f(t-\ell+x) + S_{11}f(t-(\ell+x)) & \text{if } x \in E_1 \\ S_{21}f(t-(\ell+x)) & \text{if } x \in E_2 \\ S_{31}f(t-(\ell+x)) & \text{if } x \in E_3 \end{cases} \qquad t \in (\ell, 2\ell) \tag{30}$$

Note that, for every $t \in (\ell, 2\ell)$, the transmitted and reflected waves have the same shape as $f(x)$ for $x \in [0, t-\ell]$, and that the scattering amplitudes do not depend f.

Remark. Note that, in the setting where we have boundary controls f_1, f_2, f_3 on the respective leaf vertices, then the fomula $\vec{a}(t) = S \cdot \vec{f}(t-\ell)$ would still hold. With this in mind, each component $S_{i,j}$ should be viewed as the scattering amplitude from $0 \in E_j$ to $0 \in E_i$; $S_{i,j}$ is a transmission coefficient when $i \neq j$ and a reflection coefficient when $i = j$.

3.5 The Y-graph with Robin conditions

Suppose instead that the spectrum of S does not contain ± 1. Then both $S \pm I$ are Robin and the vertex conditions can be written with a Hermitian matrix H

$$\partial\vec{u}(v_0,t) = H\vec{u} \qquad H := i\frac{S-I}{S+I} \tag{31}$$

To see that H is Hermitian, first note that $(S \pm I)$ commute, and that

$$H^\dagger = (-i)\frac{S^\dagger - I}{S^\dagger + I} = (-i)\frac{S^\dagger(I-S)}{S^\dagger(I+S)} = (-i)\frac{-(S-I)}{(S+I)} = H$$

With the same outgoing-wave-ansatz as before, equations (28) and (29) yield that

$$\vec{a}'(t) = H(\vec{a}(t) + \vec{f}(t-\ell)) + \vec{f}'(t-\ell)$$

which, as a linear time-invariant system, has the solution

$$
\begin{aligned}
\vec{a}(t) &= e^{H(t-t_0)} \underbrace{\vec{a}(\ell)}_{=0} + \int_\ell^t e^{H(t-\tau)} (H\vec{f}(\tau-\ell) + \vec{f}'(\tau-\ell))\, d\tau \\
&= \int_\ell^t 2e^{H(t-\tau)} H\vec{f}(\tau-\ell) + \frac{d}{d\tau}\left(e^{H(t-\tau)}\vec{f}(\tau-\ell)\right) d\tau \\
&= 2\int_\ell^t e^{H(t-\tau)} H\vec{f}(\tau-\ell)\, d\tau + e^{H(t-\ell)} \underbrace{\vec{f}(-\ell)}_{=0} - e^{H(t-t)}\vec{f}(t-\ell) \\
&= \vec{f}(t-\ell) + 2\int_\ell^t e^{H(t-\tau)} H\vec{f}(\tau-\ell)\, d\tau \qquad t \in (\ell, 2\ell) \qquad (32)
\end{aligned}
$$

Reinserting into equation (28) yields

$$
\vec{u}(x,t) = \vec{f}(t-\ell+x) + \vec{f}(t-\ell-x) + 2\int_\ell^{t-x} e^{H(t-x-\tau)} H\vec{f}(\tau-\ell)\, d\tau
$$

However, the integrand can be simplified further, using the fact that, as a Hermitian matrix, H is unitarily diagonizeable;

$$
H = P\Lambda P^{-1} \qquad \Lambda = \text{diag}(\lambda_i)_{i=1}^3 \qquad P = P^\dagger
$$

In its diagonalization, the matrix exponential shares the same basis; $e^H = Pe^\Lambda P^{-1}$. The integrand of (32) thus becomes

$$
\begin{aligned}
e^{H(t-\tau)} H\vec{f}(\tau-\ell) &= P\Lambda e^{\Lambda(t-\tau)} P^{-1} \cdot \hat{x}_1 f(\tau-\ell) \\
&= P\text{diag}(\lambda_i e^{\lambda_i(t-\tau)})_{i=1}^3 P^\dagger \hat{x}_1 f(\tau-\ell) \\
&= P\text{diag}(\lambda_i e^{\lambda_i(t-\tau)})_{i=1}^3 \begin{bmatrix} \overline{p_{11}} \\ \overline{p_{21}} \\ \overline{p_{31}} \end{bmatrix} f(\tau-\ell)
\end{aligned}
$$

where we have used that $[P^\dagger]_{ij} = \overline{p_{ji}}$. 3The nth component of which is

$$
\begin{aligned}
\hat{x}_n e^{H(t-\tau)} H\vec{f}(\tau-\ell) &= \begin{bmatrix} p_{n,1} & p_{n,2} & p_{n,3} \end{bmatrix} \text{diag}(\lambda_i e^{\lambda_i(t-\tau)})_{i=1}^3 \begin{bmatrix} \overline{p_{11}} \\ \overline{p_{21}} \\ \overline{p_{31}} \end{bmatrix} f(\tau-\ell) \\
&= \left(\sum_{i=1}^3 p_{n,i} \cdot \overline{p_{i,1}} \lambda_i e^{\lambda_i(t-\tau)} \right) \cdot f(\tau-\ell)
\end{aligned}
$$

Hence the state over E_2, E_3 for $(x,t) \in [0,\ell] \times (\ell, 2\ell)$ is thus

$$
u_n(x,t) = a_n(t-x) = 2\int_\ell^{t-x} \left(\sum_{i=1}^3 p_{n,i} \cdot \overline{p_{i,1}} \lambda_i e^{\lambda_i(t-x-\tau)} \right) \cdot f(\tau-\ell)\, d\tau \qquad (33)
$$

Theorem 3.10 [uniqueness when H is positive semi-definite]
The problem (9) admits at most one solution $u(x,t) \in C^2((\Gamma\backslash V) \times \mathbb{R})$ that fulfills the Robin vertex conditions in Equation (31) at v_0, with $\sigma(H) \subset \mathbb{R}_{\geq 0}$, and fulfills homogeneous Dirichlet conditions at v_2, v_3.

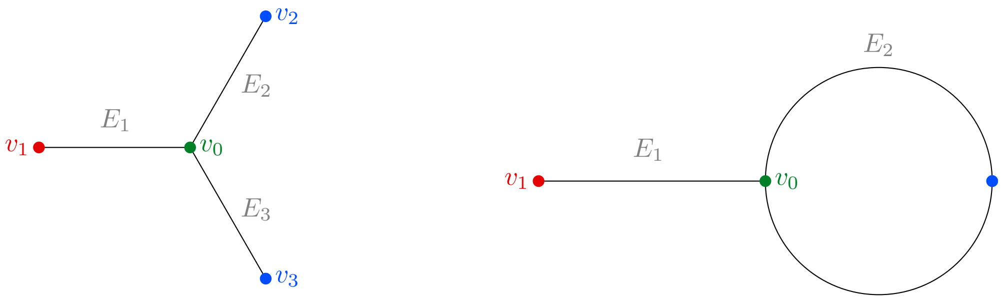
Proof. Let u be the difference of two solutions u_1, u_2 to the PDE-problem (9) on the Y graph with Robin conditions at v_0. By the exact same argument as in the proof of 3.9, u satisfies homogeneous Dirichlet conditions at v_1, v_2, v_3 and Robin conditions at v_0. Since the matrix H is assumed to be positive semi-definite, $E_u(t) = \int_0^t e_{v_0}(\tau)\, d\tau = 0$ for all $t \in [0,T]$, by Example 3.8. Thus $u \equiv 0$ and $u_1 = u_2$. The solution to the PDE-problem is thus unique. ☐

3.6 The wave equation on the Lasso graph

Consider the corresponding PDE-problem on the Lasso graph Γ, given a control f at v_1

$$\begin{aligned} (\partial_t^2 - \partial_x^2)u(x,t) &= 0 && x \in \Gamma \backslash V,\ t \in [0,T] \\ u(x,0) = \partial_t u(x,0) &\equiv 0 && x \in \Omega_\Gamma \\ u(v_1,t) &= f(t) && t \in [0,T],\ f = f' \equiv 0 \text{ on } \mathbb{R}_{\leq 0} \\ i(S-I)\vec{u}(v_0,t) &= (S+I)\vec{u}(v_0,t) && t \in [0,T] \end{aligned} \tag{34}$$

Note that, as a metric graph, the Lasso graph is just the quotient space of the Y-graph, where ℓ_{E_2} and ℓ_{E_3} have been identified, as illustrated in Figure 5. Both the proof of uniqueness and the solution of the PDE-problem (for $T < 2\ell$) are completely analogous to the Y-graph case.

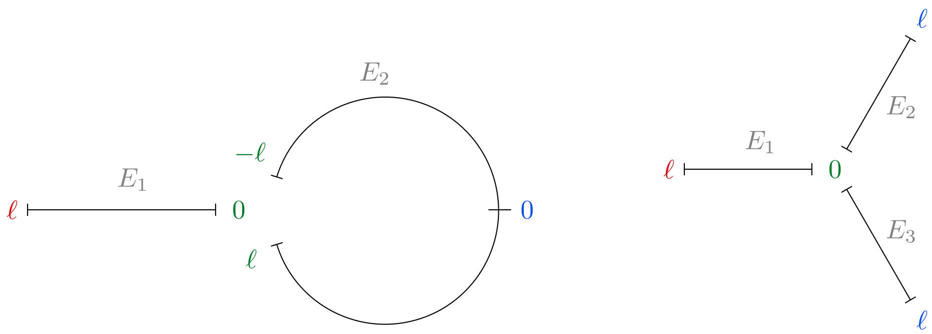


(a) Γ_Y (b) Γ_{Lasso}

Figure 5: Joining edges E_2, E_3 in the Y-graph along $\{v_2, v_3\}$ yields the Lasso Graph.

This means that the PDE solution for the Lasso graph is just the Y-graph solution composed with a transformation T;

$$T : \Omega_{\text{Lasso}} \longrightarrow \Omega_Y \qquad \begin{cases} x \longmapsto x & \text{if } x \in E_1 \\ x \longmapsto x + \ell & \text{if } x \in [-\ell, 0] \subset E_2 \\ x \longmapsto \ell - x & \text{if } x \in [0, \ell] \subset E_2 \end{cases} \tag{35}$$



(a) $\Omega_{\Gamma_{\text{Lasso}}}$ (b) Ω_{Γ_Y}

Figure 6: The transformation T in (35) splits the loop E_2 into two edges E_2 and E_3.

For $t \in (0, \ell)$, a standing wave travels from v_1, which at $t = \ell$ scattered at v_0, transmitting a wave into the upper and lower parts of E_2. At $t = 2\ell$ the transmitted waves meet at $x = 0$ on E_2, making the graph completely permeated by the signal. Instead of being reflected by Dirichlet conditions (as in the case of the Y-graph) at $x = 0$, the transmitted waves continue traveling towards opposite endpoints of E_2, which are reached at $t = 3\ell$.

For the Y-graph, the solution for $t \in (0, 2\ell)$ is:

$$u(x,t) = \begin{cases} f(t-\ell+x) & \text{if } x \in E_1 \\ 0 & \text{if } x \in E_2 \\ 0 & \text{if } x \in E_3 \end{cases} \qquad t \in (0,\ell)$$

$$u(x,t) = \begin{cases} f(t-\ell+x) + a_1(t-x) & \text{if } x \in E_1 \\ a_2(t-x) & \text{if } x \in E_2 \\ a_3(t-x) & \text{if } x \in E_3 \end{cases} \qquad t \in (\ell,2\ell)$$

where a depends on the vertex conditions at v_0. Transplanting this solution onto the Lasso graph with T yields

$$u(x,t) = \begin{cases} f(t-\ell+x) & \text{if } x \in E_1 \\ 0 & \text{if } x \in E_2 \end{cases} \qquad t \in (0,\ell)$$

$$u(x,t) = \begin{cases} f(t-\ell+x) + a_1(t-x) & \text{if } x \in E_1 \\ a_2(t-\ell-x) & \text{if } x \in [-\ell,0] \subset E_2 \\ a_3(t-\ell+x) & \text{if } x \in [0,\ell] \subset E_2 \end{cases} \qquad t \in (\ell,2\ell)$$

$$\Rightarrow \ u(x,t) = \begin{cases} f(t-\ell+x) + a_1(t-x) & \text{if } x \in E_1 \\ a_2(t-\ell-x) + a_3(t-\ell+x) & \text{if } x \in E_2 \end{cases} \qquad t \in (\ell,2\ell)$$

The traveling waves on E_2 do not change upon reaching $x = 0$, hence why u has the same expression on E_2 when $t \in (2\ell, 3\ell)$. On E_1, a reflected wave arises (without the control, homogeneous Dirichlet conditions would prevail over v_1) once the outgoing wave reaches v_1. For $t \in (2\ell, 3\ell)$, the state of the Lasso Graph is

$$u(x,t) = \begin{cases} f(t-\ell+x) + a_1(t-x) - a_1(t-2\ell+x) & \text{if } x \in E_1 \\ a_2(t-\ell-x) + a_3(t-\ell+x) & \text{if } x \in E_2 \end{cases} \qquad t \in (2\ell,3\ell) \tag{36}$$

Further, $\vec{a}$ relates to the vertex conditions at v_0 in the following way;

$$\textbf{Scaling invariant}: \quad \vec{a}(t) = S \cdot \vec{f}(t-\ell)$$

$$\textbf{Robin}: \quad \vec{a}(t) = \vec{f}(t-\ell) + 2\int_\ell^t e^{H(t-\tau)} H \vec{f}(\tau-\ell)\, d\tau$$

When **scaling invariant** conditions prevail at v_0 the state over the Lasso graph when $t \in (2\ell, 3\ell)$ is

$$u(x,t) = \begin{cases} f(t-\ell+x) + S_{11}\big(f(t-\ell-x) - f(t-3\ell+x)\big) & \text{if } x \in E_1 \\ S_{21} f(t-2\ell-x) + S_{31} f(t-2\ell+x) & \text{if } x \in E_2 \end{cases} \tag{37}$$

When **Robin** conditions prevail at v_0 the state over E_2 when $t \in (\ell, 3\ell)$ is

$$\begin{aligned} u(x,t) = 2\Bigg(&\int_\ell^{t-\ell-x} \sum_{i=1}^{3} p_{2,i}\overline{p_{i,1}}\lambda_i e^{\lambda_i(t-x-\tau)} f(\tau-\ell)\, dx \\ &+ \int_\ell^{t-\ell+x} \sum_{i=1}^{3} p_{3,i}\overline{p_{i,1}}\lambda_i e^{\lambda_i(t-x-\tau)} f(\tau-\ell)\, dx \Bigg) \end{aligned} \qquad x \in E_2 \tag{38}$$

4 Control problems on the Lasso graph

In this Section we establish the controllability properties of the Lasso graph with scaling invariant conditions at its loop vertex. These main results will be built upon in section 5 when a magnetic potential is introduced to the Lasso graph with standard condition at its loop vertex.

Section 4.1 formally defines the control operator, controllability and minimal control time and formulates the controllability questions studied in subsequent sections. In Section 4.2 we bound the minimal time required for the Lasso graph to attain controllability. The bound is shown to be optimal in 4.3, where we invert the control operator and derive an exact controllability condition for the scattering matrix. Section 4.4 provides a complete parametrization of all scaling invariant conditions at the loop-vertex and shows that no scattering matrices that yield controllability fullfill the non-symmetry condition (1) for the inverse spectral problem.

4.1 The Control Operator

Throughout this section Γ exclusively denotes the Lasso graph, which has scaling invariant vertex conditions at v_0. The PDE-problem for Γ, governed by the wave equation, having a boundary control at v_1 and a trivial initial state, is formulated as follows.

Problem 4.1 [PDE-problem on the Lasso graph]
For any given $T \in \mathbb{R}_{>0}$ and boundary control

$$f \in X := \left\{ f(t) \in C^2([0,T], \mathbb{C}) \;\middle|\; f(t) = f'(t) = 0 \text{ for } t \leq 0 \right\}$$

the PDE-problem on the Lasso graph Γ consists in finding a function

$$u \in Y := \left\{ u(x,t) \in C^2(\Omega_\Gamma \times [0,T], \mathbb{C}) \;\middle|\; i(S-I)\vec{u}(v_0,t) = (S+I)\partial\vec{u}(v_0,t) \quad \forall t \right\}$$

satisfying the following wave equation, initial- and boundary conditions

$$\begin{aligned} (\partial_t^2 - \partial_x^2)u(x,t) &= 0 & (x,t) &\in (\Gamma \backslash V) \times [0,T] & \text{(PDE)} \\ u(x,0) = \partial_t u(x,0) &\equiv 0 & x &\in \Omega_\Gamma & \text{(IC)} \\ u(v_1,t) &= f(t) & t &\in [0,T] & \text{(BC)} \end{aligned}$$

where $S \in \mathbb{M}_{3\times 3}(\mathbb{C})$ is a unitary, irreducible matrix having the spectrum $\sigma(S) = \{-1, 1\}$.

If the solution of Problem 4.1 is unique for a given boundary control f, then let u^f denote that solution. When a solution exists and is unique, the boundary condition f can be viewed as an **input signal** to a deterministic system returning an **output signal** u^f. The relationship between input and output at the end time T is formalized as the **control operator**.

Definition 4.2 [The control operator W^T]
If the PDE-problem is uniquely solvable for all $f \in X$ then it describes a deterministic system with a control operator W^T

$$W^T : X \longrightarrow Y \qquad\qquad f \longmapsto u^f(\,\cdot\,, T)$$

where $u^f(x,t)$ denotes the unique solution of the Problem 4.1 associated with the boundary-input f. Note that W^T is a linear operator on the linear space X.

A natural question that arises is whether the system, through an appropriate choice of input f, can reach any desired end state $u^f(x,T) = g(x)$, as well as the time T required for such an endeavor. This engineering of the end state is possible exactly when W^T is surjective, and then the system is said to be **controllable at time** T. Further, by delaying the input signal, any desired end state can be attained at a later time

$$W^T(f(t)) = g(x) \qquad \Rightarrow \qquad W^{T+\Delta t}(f(t-\Delta t)) = g(x)$$

so if the the system is controllable at time T then it is controllable for all times $T' > T$, which motivates the definition of the **minimal control time** $T_{\min}$.

$$T_{\min} := \inf\{T > 0 \ | \ \text{ the system is controllable at time T}\}$$

In the following subsections we will

(A) **Determine which scaling invariant vertex conditions make the Lasso graph controllable.**

(B) **Determine $T_{\min}$ for those scaling invariant vertex conditions**

(C) **Invert $W^{T_{\min}}$ whenever possible**

4.2 Bounding $T_{\min}$

Regardless of what kind of vertex conditions hold at v_0, the signal from v_1 must have permeated the entire Lasso graph before any kind of controllability is attained. Given that the signal propagates at a speed of 1 and that the diameter of Γ (the maximum distance between two points in Γ) is 2ℓ, $T_{\min}$ must at least be 2ℓ.

When scaling invariant vertex conditions prevail at v_0 a better bound is easily shown. The idea is to show that for T too small, the Lasso graph cannot reach a state where $u \equiv 0$ on some open set $\Omega \subset (-\ell, 0) \subset E_2$ while $u \neq 0$ somewhere on $-\Omega \subset (0, \ell) \subset E_2$.

Theorem 4.3 [$T_{\min} \geq 3\ell$]
The Lasso graph with scaling invariant vertex conditions at v_0 is not controllable at any $T < 3\ell$.

Proof. By equation (37), the state over E_2 for $t \in (2\ell, 3\ell)$ is

$$u(x,t) = S_{21}f(t - 2\ell - x) + S_{31}f(t - 2\ell + x) \qquad\qquad x \in E_2$$

For $|x| > 3\ell - t$ we note that $u(x,t)$ and $u(-x,t)$ are linearly dependent

$$\begin{aligned} u(x,t) &= S_{31}f(t - (2\ell - |x|)) \\ u(-x,t) &= S_{21}f(t - (2\ell - |x|)) \end{aligned} \qquad\qquad x \in (3\ell - t, \ell]$$

Then $u(x,t)$ and $u(-x,t)$ are linearly dependent on $E_2 \backslash (t - 3\ell, 3\ell - t)$

$$u(x,t) = \frac{S_{31}}{S_{21}} u(-x,t) \qquad\qquad x \in (3\ell - t, \ell]$$

When $T < 3\ell$ we cannot, for instance, attain a state which is identically zero on $[-\ell, T - 3\ell) \subset E_2$ and (somewhere) nonzero on $(3\ell - T, \ell] \subset E_2$. Hence the system is not controllable. $\square$

We will investigate whether the system is controllable at $T = 3\ell$.

4.3 Exact conditions for controllability at $T = 3\ell$

The control operator for $T = 3\ell$ is given by Equation (37)

$$W^{3\ell} f = \begin{cases} f(2\ell + x) + S_{11}\big(f(2\ell - x) - f(x)\big) & \text{if } x \in E_1 \\ S_{21} f(\ell - x) + S_{31} f(\ell + x) & \text{if } x \in E_2 \end{cases} \tag{39}$$

Recall that each term corresponds to a propagated, transmitted or reflected wave, as denoted by each coefficient. The state over E_2 depends only on $f|_{[0,2\ell]}$ because the 'signal' propagated from v_1 during $t \in [2\ell, 3\ell]$ has not yet reached E_2.

Suppose $g \in Y$ is the state we want to reach by appropriately selecting f. Then $g|_{E_2}$ imposes no restriction of f outside $t \in [0, 2\ell]$, and to attain $g|_{E_1}$ over E_1, we can select $f|_{[2\ell,3\ell]}$ as

$$f(2\ell + x) = g(x) - S_{11}(f(2\ell - x) - f(x)) \qquad x \in E_1 = [0, \ell] \tag{40}$$

$$\Rightarrow \qquad f(t) = g|_{E_1}(t - 2\ell) - S_{11}(f(4\ell - t) - f(t - 2\ell)) \qquad t \in [2\ell, 3\ell] \tag{41}$$

Showing controllability on the Lasso graph at $T = 3\ell$ boils down to showing controllability on E_2, because E_1 is controllable at time ℓ. Since the signal reaches the endpoints of E_2 simultaneously, there is a congruency between the transmitted waves. At $t = 3\ell$ they are superimposed over E_2, which results in a symmetry (or anti-symmetry) in $x = 0$ if the transmission coefficients are identical (or have opposing polarities).

Our approach to proving controllability therefore involves decomposing $u(x, 3\ell)$ and f into odd and even components.

Definition 4.4 [Odd and Even functions]
A function $f : [-\ell, \ell] \longrightarrow \mathbb{C}$ is said to be

- odd if $f(-x) = -f(x)$
- even if $f(-x) = f(x)$

Lemma 4.5 [Odd/Even-decomposition]
Every function f over $\mathbb{R}$ can be written as the sum of an even function f_{even} and an odd function f_{odd} where

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2} \qquad\qquad f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

Moreover, there is no other decomposition of f into even and odd functions. this decomposition of f into even and odd functions is unique.

Proof. It is clear that $f_{\text{even}} + f_{\text{odd}} = f$, and equally straightforward to verify that f_{even} and f_{odd} are indeed even and odd, respectively. As for uniqueness, suppose $f = f_1 + f_2$, where f_1 is odd and f_2 is even. Since the spaces of odd and even functions are both linearly closed, we have that $g_1 = f_1 - f_{\text{odd}}$ and $g_2 = f_2 - f_{\text{even}}$ constitutes an odd/even-decomposition of $g = f - f \equiv 0$. But then $g_1 = -g_2$ are simultaneously even and odd, which is only possible if $g_1 = g_2 = 0$. Hence $f_1 = f_{\text{odd}}$ and $f_2 = f_{\text{even}}$. □

Furthermore, just as $f = f_{\text{even}} + f_{\text{odd}}$, we have that

$$f(-x) = \frac{f(-x) + f(x)}{2} + \frac{f(-x) - f(x)}{2} = f_{\text{even}}(x) - f_{\text{odd}}(x)$$

Returning to the Lasso graph, its state over E_2 at 3ℓ as given by equation (37) is

$$u(x, 3\ell) = \underbrace{S_{21}}_{T_1 :=} \underbrace{f(\ell - x)}_{=h(-x)} + \underbrace{S_{31}}_{T_2 :=} \underbrace{f(\ell + x)}_{=:h(x)} \qquad\qquad x \in E_2$$

For the sake of convenience we introduce the function $h(x) = f(\ell + x)$, and the transmission coefficients S_{21} and S_{31} shall briefly be referred to as T_1 and T_2 respectively.

By Lemma 4.5, every function can be decomposed into odd and even components, e.g. $h = h_{\text{odd}} + h_{\text{even}}$ and $h(-x) = h_{\text{even}}(x) - h_{\text{odd}}(x)$. This lets us express $u(x, 3\ell)|_{E_2}$ in terms of the odd and even components of h

$$\begin{aligned} u(x, 3\ell) &= T_1 \cdot h(-x) + T_2 \cdot h(x) \qquad\qquad x \in E_2 \\ &= T_1(h_{\text{even}}(x) - h_{\text{odd}}(x)) + T_2(h_{\text{even}}(x) + h_{\text{odd}}(x)) \\ &= (T_1 + T_2) \cdot h_{\text{even}} + (T_2 - T_1) \cdot h_{\text{odd}} \end{aligned}$$

Note that this expresses $u(x, 3\ell)|_{E_2}$ as the sum of an even function $(T_1 + T_2) \cdot h_{\text{even}}$ and an odd function $(T_2 - T_1) \cdot h_{\text{odd}}$. By Lemma 4.5, this is the unique odd/even-decomposition of $u(x, 3\ell)|_{E_2}$.

Hence, any desired state $u(x, 3\ell)|_{E_2} = g|_{E_2}(x)$ can be attained if

$$h_{\text{odd}} = \frac{g_{\text{odd}}}{T_2 - T_1} = \frac{g(x) - g(-x)}{2(T_2 - T_1)} \qquad h_{\text{even}} = \frac{g_{\text{even}}}{T_2 + T_1} = \frac{g(x) + g(-x)}{2(T_2 + T_1)} \qquad x \in E_2$$

which translates to

$$h(x) = \left(\frac{1}{(T_2 + T_1)} + \frac{1}{(T_2 - T_1)}\right)\frac{g(x)}{2} + \left(\frac{1}{(T_2 + T_1)} - \frac{1}{(T_2 - T_1)}\right)\frac{g(-x)}{2} \qquad x \in E_2$$

To translate this into terms of f, we use the fact that

$$h(x) = f(\ell + x), \text{ for } x \in [-\ell, \ell] \qquad \iff \qquad f(t) = h(t - \ell), \text{ for } t \in [0, 2\ell]$$

and compute the appropriate boundary control for $t \in (0, 2\ell)$

$$f(t) = \left(\frac{1}{(T_2 + T_1)} + \frac{1}{(T_2 - T_1)}\right)\frac{g|_{E_2}(t - \ell)}{2} + \left(\frac{1}{(T_2 + T_1)} - \frac{1}{(T_2 - T_1)}\right)\frac{g|_{E_2}(\ell - t)}{2} \tag{42}$$

Notably, to reach an arbitrary state $g(x)$ on E_2 at $t = 3\ell$ the transmission coefficients must fulfill $T_1 \neq \pm T_2$.

Recall that if $u(x, 3\ell) = g$ on E_1 is tantamount to $f(2\ell + x) = g(x) - S_{11}(f(2\ell - x) - f(x))$ for $x \in E_1$. If $u(x, 3\ell) = g$ on E_2, this lets us compute the reflected waves on E_1 as

$$\begin{aligned} f(2\ell - x) - f(x) &= h(\ell - x) - h(x - \ell) \\ &= 2h_{\text{odd}}(x - \ell) = \frac{g|_{E_2}(\ell - x) - g|_{E_2}(x - \ell)}{T_2 - T_1} \qquad x \in [0, \ell] \end{aligned}$$

Combining this with Equations (40) and (41), the appropriate boundary control for $t \in [2\ell, 3\ell]$ becomes

$$f(t) = g|_{E_1}(t - 2\ell) + S_{11}\frac{g|_{E_2}(t - 3\ell) - g|_{E_2}(3\ell - t)}{T_2 - T_1} \qquad t \in [2\ell, 3\ell] \tag{43}$$

These results are summarized in the following theorem.

Theorem 4.6 [Exact controllability conditions and inverse control operator]
The Lasso graph with scaling invariant vertex conditions at v_0 is controllable at $T = 3\ell$ precisely when the scattering matrix fulfills

$$S_{21} \neq \pm S_{31} \tag{44}$$

Under these conditions, the inverted control operator

$$(W^{3\ell})^{-1} : Y \longrightarrow X \qquad g \longmapsto f$$

maps g to

$$f(t) = 0 \qquad t \leq 0 \tag{45}$$

$$\begin{aligned} f(t) = &\left(\frac{1}{S_{31}+S_{21}} + \frac{1}{S_{31}-S_{21}}\right)\frac{g|_{E_2}(t-\ell)}{2} \\ &+ \left(\frac{1}{S_{31}+S_{21}} - \frac{1}{S_{31}-S_{21}}\right)\frac{g|_{E_2}(\ell-t)}{2} \qquad t \in [0, 2\ell] \end{aligned} \tag{46}$$

$$f(t) = g|_{E_1}(t-2\ell) + S_{11}\frac{g|_{E_2}(t-3\ell) - g|_{E_2}(3\ell - t)}{S_{31}-S_{21}} \qquad t \in [2\ell, 3\ell] \tag{47}$$

Example 4.7 [Householder matrices that yield controllability]
Recall that 3×3 Householder matrices are of the form

$$A = I - 2v^*v = \begin{bmatrix} 1-2|v_1|^2 & -2\overline{v_1}v_2 & -2\overline{v_1}v_3 \\ -2\overline{v_2}v_1 & 1-2|v_2|^2 & -2\overline{v_2}v_3 \\ -2\overline{v_3}v_1 & -2\overline{v_3}v_2 & 1-2|v_3|^2 \end{bmatrix} \qquad \|v\|_{\mathbb{C}^3} = 1 \tag{48}$$

And that A corresponds to the scattering matrix S of a controllable Lasso graph precisely when $v_1, v_2, v_3 \neq 0$ and $v_2 \neq \pm v_3$. A concrete example of Householder-conditions that yield controllability are given by $v = (\frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, which yields the matrix

$$v^*v = \frac{1}{3}\begin{bmatrix} 1 & i & 1 \\ -i & 1 & -i \\ 1 & i & 1 \end{bmatrix} \qquad \Rightarrow \qquad S = I - 2v^*v = \begin{bmatrix} \frac{1}{3} & \frac{2i}{3} & \frac{2}{3} \\ \frac{-2i}{3} & \frac{1}{3} & \frac{-2i}{3} \\ \frac{2}{3} & \frac{2i}{3} & \frac{1}{3} \end{bmatrix} \tag{49}$$

4.4 Characterizing Scaling Invariant Conditions

The following theorem provides a useful characterization of all Scattering matrices for the scaling invariant conditions at v_0.

Theorem 4.8 [Complete characterization]
Suppose $S \in \mathbb{M}_{3\times 3}(\mathbb{C})$ is unitary, Hermitian and irreduible. Then S assumes the form

$$S = \pm(I - 2v^*v) \qquad v \in \mathbb{C}^3,\ \|v\|_{\mathbb{C}^3} = 1,\ v_i \neq 0 \text{ for } i = 1,2,3 \tag{50}$$

Proof. S has the spectral decompoition $S = P_1 - P_{-1}$ where P_1, P_{-1} are the eigenprojectors of S, where both eigenprojectors have a rank of at least one. Because S is bijective, we have that $P_1 = I - P_{-1}$. Geometrically, this makes S a reflection in it's eigenspace E_{-1};

$$S = I - 2P_{-1} = -(I - 2P_1) \qquad \text{rank}(P_{-1}) \in \{1, 2\}$$

Either P_{-1} is a projection of rank 1, i.e. a projection onto the space spanned by a vector $v \neq 0$, and has the matrix expression v^*v, where $v \in \mathbb{C}^3$ is a unit vector, or P_1 has rank one with that same matrix expression. Hence S assumes the form

$$S = \pm(I - 2v^*v) \qquad v \in \mathbb{C}^3,\ \|v\|_{\mathbb{C}^3} = 1$$

Note that if any component $v_i = 0$ then row i and column i of S are zero outside the diagonal, making S reducible. On the other hand, if all components of v are nonzero then all entries of S are nonzero, making it an irreducible matrix. ☐

In other words, S has the explicit form

$$S = \pm(I - 2v^*v) = \pm \begin{bmatrix} 1-2|v_1|^2 & -2\overline{v_1}v_2 & -2\overline{v_1}v_3 \\ -2\overline{v_2}v_1 & 1-2|v_2|^2 & -2\overline{v_2}v_3 \\ -2\overline{v_3}v_1 & -2\overline{v_3}v_2 & 1-2|v_3|^2 \end{bmatrix} \qquad \|v\|_{\mathbb{C}^3} = 1 \tag{51}$$

where $v_1, v_2, v_3 \neq 0$. Hence the Householder conditions, described in Example 2.11, correspond to the case when P_{-1} has rank 1, and standard conditions, described in Example 2.10, are an example of the case where P_{-1} has rank 2.

For the characterization provided in Theorem 4.8, the controllability condition $S_{21} \neq \pm S_{31}$ in Theorem 4.6 translates to $v_2 \neq \pm v_3$.

Theorem 4.9 [Parametrization of all 3ℓ-controllable conditions]
Let S be the scattering matrix corresponding to a set of properly connecting, scaling invariant vertex conditions at v_0. The corresponding family of such scattering matrices that yield controllability at $T = 3\ell$ is:

$$\{S = \pm(I - 2v^*v) \in \mathbb{M}_3(\mathbb{C}) \ : \ \|v\|_{\mathbb{C}^3} = 1, v_1, v_2, v_3 \neq 0, \ v_2 \neq v_3\} \tag{52}$$

Remark. Notably, none of the scaling invariant conditions fulfill the Non-symmetry condition (1) that was sufficient for solvability of the inverse spectral problem. If S is the scattering matrix corresponding to scaling invariant conditions at v_0, then Theorem 4.8 implies that

$$\begin{aligned} S_{12}S_{23}S_{31} &= \pm(-2)^3\overline{v_1}v_2\overline{v_2}v_3\overline{v_3}v_1 = \mp 2^3|v_1|^2|v_2|^2|v_3|^2 \\ &= \pm(-2)^3\overline{v_1}v_3\overline{v_2}v_1\overline{v_3}v_2 = S_{13}S_{21}S_{32} \end{aligned}$$

Theorem 4.9 thus provides an entire family of scattering matrices that yield controllability of the lasso graph without fulfilling the sufficient criterium given by P. Kurasov.

Remark. Consider the case when the Lasso graph is not controllable at $T_{\min}$. If $v_2 = \pm v_3$ the scattering matrix assumes the form

$$S = \pm \begin{bmatrix} 1-2|v_1|^2 & -2\overline{v_1}v_2 & \mp 2\overline{v_1}v_2 \\ -2\overline{v_2}v_1 & 1-2|v_2|^2 & \mp 2|v_2|^2 \\ \mp 2\overline{v_2}v_1 & \mp 2|v_2|^2 & 1-2|v_2|^2 \end{bmatrix} = \begin{bmatrix} R_1 & \overline{T_2} & \pm\overline{T_2} \\ T_2 & R_2 & T_3 \\ \pm T_2 & T_3 & R_2 \end{bmatrix} \tag{53}$$

T_2 and $\pm T_2$ denote the transmission coefficients from $0 \in E_1$ to $-\ell$ and ℓ in E_2, respectively. Likewise, $\overline{T_2}$ and $\pm\overline{T_2}$ are the transmission coefficients from $-\ell$ and ℓ to 0. $\pm T_3$ is the transmission coefficient between $-\ell$ and ℓ on E_2. Furthermore, the reflection coefficients at $-\ell, \ell \in E_2$ are both R_2. Hence, all transmission and reflection occurs symmetrically or anti-symmetrically at $-\ell$ and ℓ, which correspondingly results in an even or an odd state over E_2. In other words, if the system is not controllable at $T_{\min}$ it is not controllable for any $T > T_{\min}$.

5 What if there was a magnetic potential

In prior sections we have shown that the Lasso graph with standard conditions at its loop vertex cannot attain controllability. However, this can be changed if a magnetic potential is present.

In Section 5.1 we introduce a transformation U that transforms the magnetic Schrödinger operator into the Laplacian, albeit with altered vertex conditions. The Lasso graph is transformed in Section 5.2, and its transformed vertex conditions at v_0 are shown to be scaling invariant and fulfill the controllability condition of Theorem 4.6 if the magnetic potential fulfills a non-resonance condition (see Equation (55)). When this non-resonance condition is fulfilled, the Lasso graph with a magnetic potential turns out to be controllable at $T = 3\ell$, as shown in Section 5.3.

5.1 Eliminating the magnetic potential

The Schrödinger operator with identically zero electric potential, L_a, has the differential expression

$$\tau_a = (i\partial_x + a(x))^2 \qquad a(x) \in C(\Gamma \backslash V, \mathbb{R})$$

and instead of the normal derivates defined in (2), we define the **extended normal derivates** of a function u on the graph Γ as

$$\partial u(x_j) = \begin{cases} (\partial_x - ia(x_j))u(x_j) & \text{if } x_j \text{ is a left endpoint} \\ -(\partial_x - ia(x_j))u(x_j) & \text{if } x_j \text{ is a right endpoint} \end{cases}$$

We will eliminate the magnetic potential using the following unitary transformation:

$$U : u(x,t) \longmapsto e^{-i\theta(x)}u(x,t) \qquad \text{where } \theta \text{ satisfies } \theta'(x) = a(x)$$

An admissible θ would be

$$\theta(x) = \int_{x_{2n-1}}^{x} a(y)\,dy \qquad x \in E_n$$

This transformation is chosen such that $U^{-1} \circ \tau_a \circ U = \tau_0$;

$$\begin{aligned} & (i\partial_x + a(x)) \circ U = e^{i\theta(x)}(i\partial_x + a(x) + \underbrace{i \cdot i\theta'(x)}_{=-a(x)}) = U \circ (i\partial_x) \\ \Rightarrow \quad & (i\partial_x + a(x))^2 \circ U = \underbrace{(i\partial_x + a(x)) \circ U}_{=U\circ(i\partial_x)} \circ (i\partial_x) = U \circ (-\partial_x^2) \\ \Rightarrow \quad & U^{-1} \circ \tau_a \circ U = \tau_0 = -\partial_x^2 \end{aligned}$$

The corresponding magnetic wave operator is analogously transformed into a regular wave operator

$$U^{-1}\left(\partial_t^2 + \tau_a\right)U = \partial_t^2 + \tau_0$$

Let us denote the transformed operator as $\tilde{L} = U^{-1}L_aU$. if $u \in \text{Dom}(\tilde{L})$ then $U(u) \in \text{Dom}(L_a^S)$. Let S_n denote the scattering matrix of L_a at the vertex v_n. To determine the vertex conditions of $\tilde{L}$ we first look at how U transforms the vertex states of a function u

$$U : \vec{u}(v_n) = (u(x_j))_{x_j \in v_n} \longmapsto (\underbrace{e^{i\theta(x_j)}}_{\Theta_j :=} u(x_j))_{x_j \in v_n} = \underbrace{\text{diag}(e^{i\theta(x_j)})_{x_j \in v_n}}_{:=U_n} \vec{u}(v_n)$$

Where Θ_j will be called a **vertex phase**. For the sake of clarity, let $\tilde{u} \in \mathrm{Dom}(\tilde{L})$ and let $u = U(\tilde{u}) \in \mathrm{Dom}(L_a^S)$. We now show that the extended normal derivate of u is equal to the transformed normal derivate of $\tilde{u}$.

$$\begin{aligned}
&\partial_x U(\tilde{u})) = e^{i\theta(x)}(\partial_x \tilde{u} + ia\tilde{u}) \\
&(\partial_x - ia(x))u = \partial_x(U(\tilde{u})) - iaU(\tilde{u}) = e^{i\theta(x)}(\partial_x \tilde{u} + \cancel{ia\tilde{u} - ia\tilde{u}}) = e^{i\theta(x)}\partial_x \tilde{u} \\
\Rightarrow \quad &\partial u(x_j) = e^{i\Theta_j}\partial \tilde{u}(x_j) \\
\Rightarrow \quad &\partial \vec{u}(v_n) = U_n \partial \vec{\tilde{u}}(v_n)
\end{aligned}$$

Suppose u at v_n fulfills the vertex conditions given by S_n and let $\tilde{S}_n := U_n^{-1} S_n U_n$. Then the conditions on u impose corresponding conditions on $\tilde{u}$

$$\begin{aligned}
u \in \mathrm{Dom}(L_a^S) \quad &\Rightarrow \quad i(S_n - I)\vec{u}(v_n) = (S_n + I)\partial \vec{u}(v_n) \\
&\Rightarrow \quad i(S_n - I)U_n \vec{\tilde{u}}(v_n) = (S_n + I)U_n \partial \vec{\tilde{u}}(v_n) \\
&\Rightarrow \quad U_n^{-1} i(S_n - I)U_n \vec{\tilde{u}}(v_n) = U_n^{-1}(S_n + I)U_n \partial \vec{\tilde{u}}(v_n) \\
\tilde{u} \in \mathrm{Dom}(\tilde{L}) \quad &\Rightarrow \quad i(\tilde{S}_n - I)\vec{\tilde{u}}(v_n) = (\tilde{S}_n + I)\partial \vec{\tilde{u}}(v_n)
\end{aligned}$$

5.2 Transforming the Lasso graph

Consider the Lasso graph Γ with a magnetic potential a with standard conditions at v_0;

$$L = L_a^{S_{\mathrm{st}}} = (i\partial_x + a(x))^2 \qquad S = S_{\mathrm{st}} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

We want to eliminate the magnetic potential, and accordingly select a function θ for the unitary transform U

$$\theta(x) = \begin{cases} \theta_1(x) = \int_{0_1}^x a_1(y)\,dx & \text{if } x \in E_1 \\ \theta_2(x) = \mathrm{sgn}(x) \cdot \int_{0_2}^x a_2(y)\,dx & \text{if } x \in E_2 \end{cases}$$

This yields the vertex phases $\Theta_j = e^{i\theta(x_j)}$ and unitary matrix U_0 of v_0

$$\Theta_1 = e^{i\theta_1(0)} = 0 \qquad \Theta_2 = e^{i\theta_2(-\ell)} \qquad \Theta_3 = e^{i\theta_2(\ell)} \qquad U_0 = \begin{bmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{bmatrix}$$

Furthermore, note that $U_0^{-1} = U^{\dagger} = U_0^*$, which yields the transformed vertex conditions

$$\tilde{S}_0 = U_0^{-1} S_{st} U_0 = 2U_0^* v \cdot v^* U_0 - I = 2(U_0^* v) \cdot (U_0^* v)^* - I$$

Since U_0 is unitary, the vector $U_0^* v$ also has unit norm, and hence $\tilde{S}$ describes scaling invariant vertex conditions (since it is of the form described in Theorem 4.8). Let us compute the transformed vertex conditions:

$$U_0^* v = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \Theta_2^* \\ \Theta_3^* \end{bmatrix} \qquad \tilde{S} = \frac{2}{3} \begin{bmatrix} 1 & \Theta_2^* & \Theta_3^* \\ \Theta_2 & 1 & \Theta_3^* \Theta_2 \\ \Theta_3 & \Theta_2^* \Theta_3 & 1 \end{bmatrix} - I = \frac{2}{3} \begin{bmatrix} -\frac{1}{2} & \Theta_2^* & \Theta_3^* \\ \Theta_2 & -\frac{1}{2} & \Theta_3^* \Theta_2 \\ \Theta_3 & \Theta_2^* \Theta_3 & -\frac{1}{2} \end{bmatrix}$$

Hence, by the Theorem 4.6, the Lasso graph with the Laplacian and vertex conditions $\tilde{S}$ is controllable at $T = 3\ell$ precisely when

$$\Theta_2 = e^{-i\theta_2(-\ell)} \neq \pm e^{-i\theta_2(\ell)} = \Theta_3 \tag{54}$$

which precisely corresponds to when

$$\theta_2(-\ell) = \int_{-\ell}^{0} a(x)\,dx \not\equiv_{\pi} \int_{0}^{\ell} a(x)\,dx = \theta_2(\ell) \tag{55}$$

Could this make the Lasso graph with the magnetic operator controllable?

5.3 Controllability of Lasso graph with a magnetic potential

The (transformed) PDE-problem looks like this

$$\begin{aligned} \left(\partial_t^2 + \tau_0\right) \tilde{u}(x,t) &= 0 \qquad (x,t) \in (\Gamma \backslash V) \times [0,T] \\ \tilde{u}(x,0) = \partial_t \tilde{u}(x,0) &\equiv 0 \\ \tilde{u}(v_1,t) &= \tilde{f}(t) \\ i(\tilde{S} - I)\vec{\tilde{u}}(v_0,t) &= (\tilde{S}+I)\partial\vec{\tilde{u}}(v_0,t) \end{aligned} \tag{56}$$

And has a unique solution, so its corresponding control operator is well-defined.

If $\tilde{u}$ satisfies the PDE-problem above for some input $\tilde{f}$, what can we say about $u = U(\tilde{u})$? First, the corresponding PDE is the magnetic wave equation instead of the regular one

$$\begin{aligned} \left(\partial_t^2 + \tau_0\right) \tilde{u}(x,t) = \underbrace{e^{-i\theta(x)}}_{\neq 0 \ \forall x}(\partial_t^2 + \tau_a)u(x,t) = 0 \qquad (x,t) \in (\Gamma \backslash V) \times [0,T] \\ \Rightarrow \qquad (\partial_t^2 + \tau_a)u(x,t) = 0 \qquad (x,t) \in (\Gamma \backslash V) \times [0,T] \end{aligned} \tag{57}$$

Also, u satisfies standard conditions at v_0, by construction of $\mathrm{Dom}(\tilde{L})$. At v_1, we have that $u(v_1,t) = U^{-1}(\tilde{u}(v_1,t)) = e^{-i\theta_1(\ell)}\tilde{f}(t)$.

The initial conditions remain unchanged

$$\begin{aligned} u(x,0) = U^{-1}\tilde{u}(x,0) = e^{-i\theta(x)} \cdot 0 \equiv 0 \\ \partial_t \tilde{u}(x,0) = \partial_t(U^{-1}(u(x,0))) = U^{-1}\partial_t u(x,t) = \underbrace{e^{-i\theta(x)}}_{\neq 0 \ \forall x} u_t(x,0) \equiv 0 \\ \Rightarrow \qquad \partial_t u(x,0) \equiv 0 \end{aligned} \tag{58}$$

The magnetic PDE-problem corresponding to the transformed one can be phrased as follows

Problem 5.1 [Magnetic PDE-problem on the Lasso graph]
Given an input $\tilde{f}(t) \in X$ and a magnetic potential a on the Lasso graph Γ, find a solution

$$u \in Y := \left\{ u(x,t) \in C^2(\Omega_\Gamma \times [0,T], \mathbb{C}) \ \middle| \ i(S_{\mathrm{st}} - I)\vec{u}(v_0,t) = (S_{\mathrm{st}} + I)\partial\vec{u}(v_0,t) \quad \forall t \right\}$$

fulfilling

$$\left(\partial_t^2 + \tau_a\right) u(x,t) = 0 \qquad (x,t) \in (\Gamma \backslash V) \times [0,T] \tag{PDE}$$

$$u(x,0) = \partial_t u(x,0) \equiv 0 \tag{IC}$$

$$u(v_1,t) = e^{-i\theta_1(\ell)}\tilde{f}(t) =: f(t) \tag{BC}$$

If $\tilde{u}(x,t)$ is the solution to the transformed PDE-problem, then $u(x,t) = e^{-i\theta(x)}\tilde{u}(x,t)$ is the solution to the magnetic PDE-problem.

The control operators of the transformed and the magnetic systems at $T = 3\ell$ are, respectively

$$\tilde{W}^{3\ell} : \tilde{f}(t) \longmapsto \tilde{u}(x,3\ell) \qquad W_a^{3\ell} : f(t) \longmapsto u(x,3\ell) \qquad t \in (0,3\ell)$$

Using the relations $f = e^{-i\theta_1(\ell)}\tilde{f}$ and $u(x,t) = e^{-i\theta(x)}\tilde{u}(x,t)$, we arrive at

$$W_a^{3\ell} = e^{i\theta_1(\ell)}\, U^{-1} \circ \tilde{W}^{3\ell}$$

Since U is a bijection, $W_a^{3\ell}$ is surjective exactly when $\tilde{W}^{3\ell}$ is - i.e. when Equation (55) holds. This yields the following theorem.

Theorem 5.2 [Exact controllability conditions: magnetic operator]
The Lasso graph with a magnetic potential a, and standard vertex conditions at v_0 is controllable at $T = 3\ell$ precisely when the magnetic potential fulfills the following non-resonance condition

$$\int_{-\ell}^{0} a(x)\,dx \not\equiv_{\pi} \int_{0}^{\ell} a(x)\,dx \tag{59}$$

The inverse control operator $(W_a^{3\ell})^{-1} = e^{-i\theta_2(\ell)} \cdot (\tilde{W}^{3\ell})^{-1} \circ U$ can be computed using Theorem 4.6.

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Errata

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1. On page 6 it says:
   This section exclusively concerns Laplacians on metric graphs, i.e. Schrödinger operators with $q \equiv 0$ *but* $a \not\equiv 0$
   Although it should say $a \equiv 0$

2. In the proof of theorem 4.3 on page 25: all instances of $3\ell - t$ (or $3\ell - T$) should be $t - 2\ell$ (or $T - 2\ell$). Likewise, $t - 3\ell$ should be $2\ell - t$.