SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Galois theory in higher algebra

av

Julián Komáromy

2026 - No M6

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Galois theory in higher algebra

Julián Komáromy

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Professor Gregory Arone

2026

Abstract

In this thesis, we introduce Galois theory in higher algebra and show that it arises naturally from a reformulation of classical Galois theory. To motivate the definition of the Galois category of an $\mathbb{E}_\infty$-ring, we study the Galois theory of classical rings and introduce Mathew's axiomatic Galois theory. This itself is a reformulation of Grothendieck's original Galois theory for schemes.

Higher algebra, especially $\mathbb{E}_\infty$-rings and modules over them, are briefly introduced. Using this language, we define (weak) finite covers of an $\mathbb{E}_\infty$-ring, which, together with the axiomatic Galois theory introduced previously, allows us to assign a Galois group(oid) to an $\mathbb{E}_\infty$-ring. We compare the resulting theory to Rognes' original theory of G-Galois extensions and show that G-torsors in the category of weak finite covers are precisely the faithful G-Galois extensions. We reprove some of the key theorems of Rognes using ∞-categorical language. Finally, real and complex K-theory spectra are introduced as $\mathbb{E}_\infty$-rings and parts of Rognes' classic proof that complexification $KO \to KU$ is a Galois extension of ring spectra are proven. This provides a non-algebraic example of a C_2-Galois extension.

Sammanfattning

I denna uppsats introducerar vi Galois-teorin inom högre algebra och visar att den uppstår naturligt från en omformulering av den klassiska Galois-teorin. För att motivera definitionen av Galois-kategorin för en $\mathbb{E}_\infty$-ring studerar vi Galois-teorin för klassiska ringar och introducerar Mathews axiomatiska Galois-teori. Detta är i sig en omformulering av Grothendiecks ursprungliga Galois-teori för scheman.

Högre algebra, särskilt $\mathbb{E}_\infty$-ringar och moduler över dessa, presenteras kortfattat. Med hjälp av detta språk definierar vi (svaga) ändliga täckningar av en $\mathbb{E}_\infty$-ring, vilket tillsammans med den axiomatiska Galois-teorin som presenterades tidigare, gör det möjligt för oss att tilldela en Galois-grupp(oid) till en $\mathbb{E}_\infty$-ring. Vi jämför den resulterande teorin med Rognes ursprungliga teori om G-Galois-utvidgningar och visar att G-torsorer i kategorin av svaga ändliga täckningar är exakt de trogna G-Galois-utvidgningarna. Vi bevisar om några av Rognes centrala satser med ∞-kategoriskt språk. Slutligen introduceras reella och komplexa K-teorispektrum som $\mathbb{E}_\infty$-ringar, och delar av Rognes klassiska bevis för att komplexifieringen $KO \to KU$ är en Galois-utvidgning av ringspektrum bevisas. Detta ger ett icke-algebraiskt exempel på en C_2-Galois-utvidgning.

Contents

Introduction

In this thesis, we provide an exposition to the foundations required to study the Galois theory of $\mathbb{E}_\infty$-ring spectra. The study of Galois extensions of ring spectra was initiated by Rognes [Rog08] in order to provide new tools for the study of ring spectra such as (algebraic) K-theories and the Morava E-theories in chromatic homotopy theory. The definition of a G-Galois extension of ring spectra is a direct generalization of the same notion for ordinary rings as presented in e.g. [Gre92]. More recently, Rognes' work has been expanded by Mathew [Mat16] and presented from a category-theoretical point of view resembling the axiomatic Galois theory of Grothendieck introduced in SGA1 [71]. Even more recently, Naumann and Pol [NP24] have compared Mathew's axiomatic approach to a theory based on Balmer's separable commutative algebras in tensor-triangular geometry.

In the context of Mathew's paper, one computes the Galois group (or fundamental group) of a suitable ∞-category. Examples are the ∞-category of modules over the topological real and complex K-theories KU and KO (these can be given the structure of $\mathbb{E}_\infty$-rings), the category of $K(n)$-local spectra as well as certain stable module ∞-categories [Mat16]. This reflects that such a Galois theory has potential to be a powerful invariant in various applications of stable homotopy theory.

Broadly speaking, ring spectra are homotopical objects which allow a theory very similar to algebra and especially homological algebra. In fact, the term "brave new algebra" was introduced by Waldhausen to refer to the program of building algebra using spectra in the place of abelian groups [Gre07]. As always when working with spectra, one faces the choice of either working with a specific model of spectra (symmetric or orthogonal spectra, S-modules, etc.) or working model-independently by utilizing the language of ∞-categories. The seminal works of Lurie on ∞-categories and higher algebra [Lur09; Lur17] now provide extensive foundations for working with ring spectra using ∞-categorical language. This is what we use in this thesis.

Mathew's theory draws heavily on properties enjoyed by finite étale algebras (in ordinary algebra), especially their descent properties. This notion is algebro-geometric, but we only ever work with affine schemes. In section 1, we prove a few basic theorems about finite étale algebras with a view towards the generalization to the higher algebraic setting.

In section 2, we follow Mathew's axiomatization of Galois categories and provide the necessary background in categorical descent theory. We show that given an ordinary ring R, the G-torsors in the opposite category of finite étale R-algebras are precisely the G-Galois extensions of R. This will show that G-torsors provide a bridge between the categorical language of SGA1 and the notion of G-Galois extensions of rings as in [Gre92].

We then provide a brief and necessarily incomplete introduction to the most important objects of higher algebra in section 3. In particular, we define symmetric monoidal ∞-categories, $\mathbb{E}_\infty$-ring spectra and modules over these. We will mainly follow Lurie's Higher Algebra [Lur17].

Equipped with this background, we define weak finite covers of $\mathbb{E}_\infty$-ring spectra following [Mat16], which are a candidate for the higher categorical analog of finite étale algebras. These will form a Galois category in the sense of Mathew, and we show that the G-torsors in the opposite category of weak finite covers are precisely the (opposite) G-Galois extensions of Rognes. The proof of this fact relies on Rognes' proof that G-Galois extensions are dualizable modules, which we prove using ∞-categorical language.

Finally, to provide a nontrivial, homotopical example of a Galois extension, we define the $\mathbb{E}_\infty$-rings KU and KO and show (assuming certain background on Bott periodicity and some computations) that complexification $KO \to KU$ is a C_2-Galois extension.

Throughout the work, some familiarity with ∞-categories, general algebraic topology and some stable homotopy theory is assumed, although we do sketch and define the most important properties of ∞-categories in an informal, axiomatic manner in appendix A.

1 Ordinary Galois theory

In this section, we briefly sketch the Galois theory of fields and commutative rings in order to provide motivation for the more general theory in chapters 2 and 4. In particular, we will not prove the main theorems in this section as these follow from the general machinery of section 2. Our starting point is the rich analogy between Galois theory and finite covering spaces of a (sufficiently nice) topological space X.

Construction 1.1. Let X be a manifold for simplicity. Recall that once we have chosen a base point $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ acts on the fibers of a cover $f\colon E \to X$ via the *monodromy action*, with the class of a loop $[\gamma\colon S^1 \to X]$ sending $y \in f^{-1}(x_0)$ to the endpoint of the unique lift $\widetilde{\gamma}\colon I \to E$ of γ starting at y. This defines a functor (covers of X) $\to \mathrm{Set}_{\pi_1(X,x_0)}$ where the target category is the category of sets with a (left) action of $\pi_1(X, x_0)$.

For the analogy with Galois theory, we are only interested in *finite* covers, so we require a slight modification. Before stating the "main theorem of Galois theory for finite covers", we introduce profinite groups. One only has to think about profinite groupoids for the Galois theory of fields, but in order for the presentation of the theorems to be maximally symmetric, we take a brief detour now.

Definition 1.2 (Profinite group)**.** A topological group G is *profinite* if it is the cofiltered limit $\lim_{i\in\Lambda} G_i$ of a diagram of finite discrete groups $\{G_i \mid i \in \Lambda\}$. If we only consider the group structure of a profinite group, then these are exactly the pro-objects in the category FinGroup of finite groups. We refer to appendix B for a definition of cofiltered limits and pro-objects.

Definition 1.3. Given a group G, we can form its *profinite completion* $\widehat{G}$ by forming the limit over the cofiltered system of G/N for normal subgroups N of finite index. More formally, let the indexing category Ω_G have as objects the normal subgroups of finite index of G and one morphism $N_\alpha \to N_\beta$ whenever $N_\alpha \supseteq N_\beta$. The category Ω_G is filtered since the intersection of normal subgroups is normal. Then there is a functor $Q\colon {\Omega_G}^{\mathrm{op}} \to \mathrm{TopGroup}$ defined by $N_\alpha \mapsto G/N_\alpha$. Then $\widehat{G} := \lim_{{\Omega_G}^{\mathrm{op}}} Q = \lim_{N \in {\Omega_G}^{\mathrm{op}}} G/N$[1] is a profinite group. The system of quotient projections $G \to G/N$ defines a canonical map $G \to \widehat{G}$.

Lemma 1.4. *Let G be any group and equip it with the topology generated by the normal subgroups of finite index (these form a neighborhood basis of the identity). Then, given any profinite group H and continuous group homomorphism $\varphi\colon G \to H$, there is a unique lift*

$$\begin{array}{ccc} \widehat{G} & & \\ \uparrow & \overset{\overline{\varphi}}{\searrow} & \\ G & \xrightarrow[\varphi]{} & H. \end{array}$$

Proof. We construct $\overline{\varphi}$. Let $U \subseteq H$ be any open subgroup and consider the open set $\varphi^{-1}(U) \subseteq G$. Since φ is continuous, $\varphi^{-1}(U)$ is open and there is a normal subgroup of finite index $N \subseteq \varphi^{-1}(U)$. For any open subgroup $U \subseteq H$, we can consider therefore define the composition $\widehat{G} \to G/N \to H/U$. Since H is profinite, we have $H \cong \lim_{U \in {\Omega_H}^{\mathrm{op}}} H/U$ and the maps we have just defined give a map $\widehat{G} \to H$. For the rest of the statement, we refer to [RZ10, Lemma 3.2.1]. □

Corollary 1.5. *If G is any group, H a (discrete) finite group and $\varphi\colon G \to H$ a group homomorphism, then there is a unique lift*

$$\begin{array}{ccc} \widehat{G} & & \\ \uparrow & \overset{\overline{\varphi}}{\searrow} & \\ G & \xrightarrow[\varphi]{} & H. \end{array}$$

where $\overline{\varphi}$ is a continuous *group homomorphism from $\widehat{G}$.*

Proof. To apply lemma 1.4, we need to show tat φ is continuous when G is given the topology generated by the normal subgroups of finite index. Since this makes G into a topological group, it suffices to check continuity of φ at $1 \in G$. As H is discrete, we just have to check that $\ker(\varphi)$ is open. This is clear, since the kernel is normal and of finite index (because H is finite). □

We can now state the following result from topology [Sza09, Corollary 2.3.9].

[1] note that Q lands in TopGroup, the category of topological groups, where cofiltered limits of discrete finite groups exist

Theorem 1.6. *Given a connected manifold X, let Cov_X denote the category of finite covers $E \to X$ and morphisms continuous maps $f\colon E \to E'$ such that*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes. Further, choose a base point $x_0 \in X$ and let $\mathrm{FinSet}_G^{\mathrm{cts}}$ denote the category of finite sets with a continuous action of the profinite group $G := \widehat{\pi_1(X, x_0)}$. Then there is an equivalence of categories

$$\mathrm{Cov}_X \xrightarrow{\sim} \mathrm{FinSet}_G^{\mathrm{cts}}$$

given by sending a finite cover $f\colon E \to X$ to the fiber $f^{-1}(x_0)$ with the monodromy action of $\pi_1(X, x_0)$ extended to a continuous action of G by the preceding discussion.

Example 1.7. The degree n cover $f\colon S^1 \to S^1$ corresponds to the set $\{1, \ldots, n\}$ with $\hat{\mathbb{Z}}$ acting by first projecting to $\mathbb{Z}/n\mathbb{Z}$ and then permuting elements.

Then, Galois theory can be seen as a vast generalization of this result. Indeed, *Grothendieck's Galois theory* and its variant discussed in [Mat16] which we will introduce in chapter 2 both characterize categories of the form $\mathrm{FinSet}_G^{\mathrm{cts}}$. We now show how this relates to Galois theory in the classical algebraic sense.

1.1 Over fields

Given K, we want to think of Galois extensions L/K as covering spaces, with the absolute Galois group $G := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ playing the role of the fundamental group. Since field extensions are maps $K \to L$, we will more often work in the opposite category of K-algebras, ${\mathrm{CAlg}_K}^{\mathrm{op}}$, or equivalently the category $\mathrm{AffSch}_{\mathrm{Spec}\,K}$ of affine schemes over $\mathrm{Spec}\,K$.

In order for the analogy between covering spaces and Galois theory to hold, we should be able to take finite disjoint unions of Galois extensions and still remain in the category of "finite covers" of K. For this, we must leave the category of fields, since the disjoint union of L, L' in ${\mathrm{CAlg}_K}^{\mathrm{op}} \simeq \mathrm{AffSch}_{\mathrm{Spec}\,K}$ is $\mathrm{Spec}(L \times L') \to \mathrm{Spec}\,K$, but the product of two fields is never a field! The key definition to remedy this is:

Definition 1.8 (Separable R-algebra)**.** Given a commutative ring R, A *separable R-algebra* is a commutative algebra $R \to S$ whose multiplication $S \otimes_R S \to S$ admits a section as a $S \otimes_R S$-module homomorphism. Equivalently we ask the exact sequence of $S \otimes_R S$-modules

$$0 \longrightarrow \ker(\mu) \longrightarrow S \otimes_R S \xrightarrow{\mu} S \longrightarrow 0.$$

to be split exact.

The following theorem justifies the definition in light of our wanting disjoint unions of Galois extensions.

Theorem 1.9. *([For17, Corollary 4.5.8 (1)]) Given a field K, a K-algebra A is separable if and only if it is a finite product of finite separable field extensions.*

Separable K-algebras are sometimes called *finite étale algebras* in the literature, see e.g. [Sza09, Definition 1.5.3]. This will be become clear in light of definition 1.12. Now we have a direct analog of theorem 1.6, which is [Sza09, Theorem 1.5.4]:

Theorem 1.10 (Main theorem of Galois theory over a field)**.** *Let K be a field, and denote by $\mathrm{Cov}_{\mathrm{Spec}\,K}$ the opposite category of separable commutative K-algebras. Further, choose a separable closure K^{sep} of K and let $\mathrm{FinSet}_G^{\mathrm{cts}}$ denote the category of finite sets with a* continuous *action of the profinite group $G := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Then there is an equivalence of categories*

$$\mathrm{Cov}_{\mathrm{Spec}\,K} \xrightarrow{\sim} \mathrm{FinSet}_G^{\mathrm{cts}}$$

given by sending a separable commutative K-algebra A to the set $\mathrm{Hom}_K(A, K^{\mathrm{sep}})$ with G acting by postcomposition.

Remark 1.11. If we do not require the action of G on finite sets to be continuous, this theorem fails. Let $K \to L$ be some field extension and $X := \mathrm{Hom}_K(L, K^{\mathrm{sep}})$. Then, given a continuous action $\rho\colon\ \mathrm{Gal}(K) \to \mathrm{Aut}(X)$, the stabilizer of any point $f : L \to K^{\mathrm{sep}}$ is open, hence closed and of finite index [Sza09, Corollary 1.5.9]. The stabilizer G_f is exactly $\mathrm{Gal}(K^{\mathrm{sep}}/L)$. Now we will show why these subgroups must be closed for the Galois correspondence to work.

L $K = \mathbb{F}_p$. Then $\mathrm{Gal}(K) \cong \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ since $\mathbb{F}_p$ is perfect. Because the finite field extension of $\mathbb{F}_p$ are $\mathbb{F}_{p^n}$, and $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$, one computes that $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$, the profinite completion of the integers. Consider the Frobenius automorphism $\sigma(x) := x^p$ defined on $\overline{\mathbb{F}_p}$. This generates a cyclic group $\langle\sigma\rangle \subsetneq \hat{\mathbb{Z}}$, where the inequality comes from the fact that $\hat{\mathbb{Z}}$ is isomorphic to $\prod_p \mathbb{Z}_p$ where p ranges over all primes p and $\mathbb{Z}_p$ denotes the p-adic integers. On the other hand, the fixed field of $\langle\sigma\rangle$ is $\mathbb{F}_p$ (this is because σ generates the Galois groups of all finite field extensions), so we cannot recover $\langle\sigma\rangle$ by taking the Galois group over $\mathbb{F}_p$. This fails because $\langle\sigma\rangle$ is not closed inside $\hat{\mathbb{Z}}$.

Notice that in topology, we had to choose a base point. Here, this choice becomes the choice of a separable closure K^{sep} of K. From the scheme-theoretic point of view, this is the choice of a *geometric point*, which is a morphism $\mathrm{Spec}\,\Omega \to X$ for an algebraically closed field Ω. In our case, $X = \mathrm{Spec}\,K$, so the choice of a geometric point is the choice of an algebraic closure of K, which uniquely determines a separable closure.

1.2 Over commutative rings

Grothendieck's Galois theory over schemes showed that we can work more generally than just over fields. Our task is to define what it means to be a "finite cover" in the algebraic setting. These are the so called *finite étale morphisms* of schemes, which admit a very large number of equivalent characterizations and are very well studied. Here, we give the definition for the case where the base scheme is affine, but we keep the terminology from scheme theory.

Definition 1.12. Given a commutative ring R, a *finite étale algebra* over R is a commutative algebra $R \to S$ such that

(1) S is a finitely presented R-module,

(2) S is a flat R-module, and

(3) S is a separable R-algebra.

Our goal is to show that we can characterize finite étale algebras as those algebras which can be trivialized by a very well-behaved base change (this will be theorem 1.17). This will provide the foundation for Mathew's definition of weak finite covers in higher algebra (this will be definition 4.1). For now, we study finite étale algebras.

Construction 1.13 (Degree)**.** In general, one can define the degree of a finite locally free map of schemes $X \to Y$ at any point of Y, but we follow [Len08] and define the degree only for $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ where S is a finitely generated projective R-module (e.g. a finite étale R-algebra S, since flat and finitely presented implies projective [Rot09, Theorem 3.56]). In this case, the *degree* $\deg_f(\mathfrak{p})$ of f at a point $\mathfrak{p} \in \operatorname{Spec} R$ is given by the rank of the free $R_\mathfrak{p}$-module $S_\mathfrak{p}$, where $S_\mathfrak{p}$ is free by [Len08, Theorem 4.6]. Thus, we obtain a well-defined function function $\deg_f\colon \operatorname{Spec} R \to \mathbb{Z}_{\geq 0}$. This is locally constant, since [Len08, Theorem 4.6(iii)] implies the existence of elements $(f_i)_{i\in I} \subset R$ such that $\sum_{i \in I}(f_i) = R$, so that the distinguished opens $D(f_i)$ cover $\operatorname{Spec} R$ and $\deg_f$ is constant on each $\operatorname{Spec} R\big|_{D(f_i)} = \operatorname{Spec} R_{f_i}$. Finally, since $\deg_f$ is locally constant, it is continuous.

The idea behind the definition is the same as in topology: we look at the size of the fiber. For a map $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$, the *geometric fiber* at a geometric point $x\colon \operatorname{Spec} \Omega \to \operatorname{Spec} R$ is $\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} \Omega \cong \operatorname{Spec}(S \otimes_R \Omega)$, where Ω is an algebraic closure of the residue field $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of $\mathfrak{p} := x(0)$. By theorem 1.9, if S is a finite étale R-algebra, $S \otimes_R \Omega$ is a finite product of the form $\Omega \times \cdots \times \Omega$, so degree of f at x is actually the cardinality of the geometric fiber. The next two results will characterize connected components of affine schemes, which we will need for the main theorem in this section.

Lemma 1.14. $\operatorname{Spec} R$ *is connected if and only if* R *has no nontrivial idempotents.*

Proof. Assume that $e \in R$ is such that $e^2 = e$. Then $R \cong Re \times R(1-e)$ and we have $\operatorname{Spec} R \cong \operatorname{Spec} Re \sqcup \operatorname{Spec} R(1-e)$ (notice that Re is a ring since $re \cdot se = rse$). On the other hand, assume that $\operatorname{Spec} R = U \sqcup V$ for open and closed U, V which are not empty. Then we can define $s \in \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ by letting it be 1 on U and 0 on V. For this to work, we use that since $U \neq \emptyset$, we must have $\Gamma(U, \mathcal{O}_{\operatorname{Spec} R}) \neq 0$ because the stalk $R_\mathfrak{p} = \mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}}$ is nonempty for any $\mathfrak{p} \in U$. Then, since U and V are disjoint, we can glue $s_U = 1 \in \Gamma(U, \mathcal{O}_{\operatorname{Spec} R})$ and $s_V = 0 \in \Gamma(V, \mathcal{O}_{\operatorname{Spec} R})$ to construct s.

Now clearly $s^2 = s \in \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \cong R$, and $s \neq 0$ since this would restrict to 0 on both U and V. Similarly, $s \neq 1$ and the image of s is a nontrivial idempotent of R. □

If $e \in R$ is an idempotent, R splits as $R \cong Re \times R(1-e)$. In this case, $R[e^{-1}] \cong Re[e^{-1}] \times R(1-e)[e^{-1}] \cong Re$ since e is the unit in Re and zero in $R(1-e)$. Therefore, R is a product of the form $R[e_1^{-1}] \times R[e_2^{-1}]$ for idempotents $e_1, e_2 \in R$. To say even more, one requires some nontrivial results from algebraic geometry, but given the previous lemma, this is not very surprising:

Proposition 1.15. *The closed and open subsets of an affine scheme* $\operatorname{Spec} R$ *are affine and of the form* $\operatorname{Spec} R[e^{-1}]$ *for an idempotent* $e \in R$*.*

Proof. Let $U \subseteq \operatorname{Spec} R$ be closed and open, so that $\operatorname{Spec} R = U \sqcup V$ for $V := \operatorname{Spec} R \setminus U$ is a decomposition of $\operatorname{Spec} R$ into disjoint closed and open subsets. Then U, V are naturally subschemes of R with structure sheaves $\mathcal{O}_{\operatorname{Spec} R \,|\, U}$ and $\mathcal{O}_{\operatorname{Spec} R \,|\, V}$. By [Har77, Proposition 5.9], a closed subscheme of $\operatorname{Spec} R$ corresponds exactly to quasi-coherent sheaf of ideals on $\mathcal{O}_{\operatorname{Spec} R}$, that is, an ideal $I \subseteq R$. Therefore, we have $U \cong \operatorname{Spec} R/I$ and $V \cong \operatorname{Spec} R/J$ for suitable ideals I, J. But then by the equivalence between $\mathrm{CRing}^{\mathrm{op}}$ and AffSch, we have

$$R \cong R/I \times R/J$$

and the idempotent $e = (1, 0)$ is now such that $\operatorname{Spec} R[e^{-1}] = \operatorname{Spec} U$. □

We will need one more lemma.

Lemma 1.16. *If* $f\colon R \to S$ *is a finite étale algebra such that* $\deg_f(\mathfrak{p}) \geq 1$ *at every point* $\mathfrak{p} \in \operatorname{Spec} R$*, then* S *is a faithfully flat* R*-module.*

Proof. By definition, S is a flat R-module. To show faithfulness, we need to show that $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective (this condition is equivalent to asking for faithful flatness by some commutative algebra [Mat80, (4.D)]), so let $\mathfrak{p} \in \operatorname{Spec} R$. By assumption the fiber $\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}}$ is nonzero, where we pull back along $\operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$. Thus, there must be at least one $\mathfrak{q} \in \operatorname{Spec} S$ that is sent to $\mathfrak{p}$. □

The following theorem is (roughly) an algebraic version of the following statement for topological spaces: given a finite cover $Y \to X$, we can find another cover $Y' \to Y$ such that $Y \times_X Y' \to Y'$ is a trivial cover, i.e. of the form $\coprod Y' \to Y'$.

Theorem 1.17. *Let* $R \to S$ *be a finite étale algebra. Then there is another finite étale algebra* $R \to S'$ *such that*

(1) S' *is a faithfully flat* R*-module*

(2) $S \otimes_R S' \cong \prod_{i=1}^n S'[e_i^{-1}]$ *for some idempotents* $e_i \in S'$*.*

Proof. Assume first that the degree $\deg_f\colon \operatorname{Spec} R \to \mathbb{Z}_{\geq 0}$ is constant, where $f\colon R \to S$ is the R-algebra in question. In this case, we will prove a stronger statement, which is that $S \otimes_R S' \cong \prod_{i=1}^n S'$. We use scheme-theoretic language and proceed by induction on the degree

n, following [Len08, Theorem 5.10]. If $n = 0$, then $S = 0$ and we can take $S' = R$. Let $n \geq 1$ and note that since S is a separable R-algebra, the exact sequence

$$0 \longrightarrow T \longrightarrow S \otimes_R S \xrightarrow{\mu} S \longrightarrow 0$$

where $T := \ker \mu$, splits. Therefore, there is an isomorphism $\operatorname{Spec} T \sqcup \operatorname{Spec} S \to \operatorname{Spec} S \otimes_R S$. The composition

$$\operatorname{Spec} T \longrightarrow \operatorname{Spec} T \sqcup \operatorname{Spec} S \cong \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \xrightarrow{p_2} \operatorname{Spec} S$$

gives us a finite étale S-algebra T since $T \times S \to T$ and $S \to S \otimes_R S$ can be seen to be finite étale, and then so is their composition by standard results [Sza09, Remark 5.2.3]. Since $(S \otimes_R S) \otimes_S R_{\mathfrak{p}} \cong S \otimes_R R_{\mathfrak{p}} \cong S_{\mathfrak{p}}$ at any point $\mathfrak{p} \in \operatorname{Spec} R$, the map $\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \to \operatorname{Spec} S$ has degree n. This implies that the fibers of $\operatorname{Spec} T \to \operatorname{Spec} S$ have dimension $n-1$, so we can apply the inductive hypothesis to find a $\operatorname{Spec} S' \to \operatorname{Spec} S$ such that S' is a faithfully flat S-module and $T \otimes_S S' \cong \prod_{i=1}^{n-1} S'$. Consider S' as an R-module. Then:

(1) S' is a faithfully flat R-module since it is faithfully flat over S by construction, and S is faithfully flat over R by lemma 1.16 and our assumption that $n \geq 1$.

(2) We have the chain of isomorphisms

$$S' \otimes_R S \cong (S \otimes_R S) \otimes_S S' \cong (T \times S) \otimes_S S' \cong (T \otimes_S S') \times (S \otimes_S S') \cong \prod_{i=1}^{n} S'$$

finishing the case that $\deg_f$ is constant.

Now consider the case that $\deg_f$ is possibly non-constant. Since $\operatorname{Spec} R$ is affine, it is quasi-compact and every open cover has a finite subcover. Therefore, the locally constant function $\deg_f$ can take only finite many values. For any given degree n, we have that $\deg_f^{-1}(n) \subseteq \operatorname{Spec} R$ is closed and open, so this is of the form $\operatorname{Spec} R[e_n^{-1}] \subseteq \operatorname{Spec} R$ by proposition 1.15. Since $S \otimes_R R[e_n^{-1}]$ is again a finite étale $R[e_n^{-1}]$ algebra (finite presentation and flatness both follow from the localization $R[e_n^{-1}]$ being a flat R-module and separability is always preserved under base change) we can apply the constant degree case to the $R[e_n^{-1}]$-algebra $S \otimes_R R[e_n^{-1}]$ to obtain a faithfully flat $R[e_n^{-1}]$-module S'_n satisfying

$$S \otimes_R S'_n \cong (S \otimes_R R[e_n^{-1}]) \otimes_{R[e^{-1}]} S_n \cong \prod_{i=1}^{k_n} S'_n \tag{1}$$

Set $S' := \prod_{i=1}^{m} S'_{n_i}$, where $\{n_1, \ldots, n_m\} = \operatorname{im} \deg_f$. Then S' is faithfully flat since all S'_i are, and

$$S \otimes_R S' \cong \prod_{i=1}^{m} S \otimes_R S'_{n_i} \cong \prod_{i=1}^{m} \prod_{i=1}^{k_{n_i}} S'_{n_i} \cong \prod_{i=1}^{m} \prod_{i=1}^{k_{n_i}} S'[e_{n_i}^{-1}].$$

□

Remark 1.18. One can show that theorem 1.17 is an if and only if: If $R \to S$ is an algebra such that there is $R \to S'$ satisfying (1) and (2), then $R \to S$ is finite étale [Sza09, Proposition 5.2.9].

Remark 1.19. Stated somewhat more algebro-geometrically, theorem 1.17 implies that after base change along a scheme $\operatorname{Spec} S'$ which admits descent (see section section 2.1 for a definition), a finite étale cover becomes a trivial covering consisting only of connected components of the base space. This result also directly shows that Mathew's definition of finite covers (this will be definition 4.1) is a direct generalization of a fact that is true in the case of ordinary commutative rings.

The next lemma will be useful to connect the usual Galois theory of fields to the axiomatic approach explored in the next chapter.

Lemma 1.20. *If K is a field, then the finite étale K-algebras are precisely the separable commutative K-algebras.*

Proof. A finite étale algebra is separated by definition, so there is nothing to show in one direction. On the other hand, over a field, every module is flat, so we only need to argue that a separable K-algebra is finitely presented. By theorem 1.9, any separable K-algebra is a finite product of finite separable field extensions, and it follows that a separable K-algebra is in particular a finite K-vector space. Alternatively, finite generation is asserted directly by [For17, Proposition 4.4.5]. □

Remark 1.21. Grothendieck's approach to Galois theory is to provide a characterization of categories equivalent to categories of the form $\mathrm{FinSet}_G^{\mathrm{cts}}$ for a profinite group G. These are called *Galois categories* (we will define a variant in definition 2.8). In the context of Galois theory over a connected scheme X, the Galois category of concern is the category of finite étale covers of X. For $X = \operatorname{Spec} R$ affine, this is precisely the opposite category of finite étale R-algebras, and the "finite étale" terminology really stems from algebraic geometry. The resulting theory gives us a way to assign a profinite group $\pi_1^{\text{ét}}(X, x)$ to a connected scheme with a chosen geometric point $x\colon \operatorname{Spec} \Omega \to X$ and in particular a group $\pi_1^{\text{ét}}(R, x)$ to a commutative ring which does not have any idempotent elements (this is equivalent to requiring that the scheme $\operatorname{Spec} R$ be connected).

Definition 1.22. Given a ring R, denote by Cov_R the opposite category of finite étale R-algebras. We can view this as a subcategory of the category of affine schemes over $\operatorname{Spec} R$, $\mathrm{AffSch}_{\operatorname{Spec} R} \simeq {\mathrm{CRing}_{R/}}^{\mathrm{op}}$.

In this setting, the main theorem of Galois theory is [Sza09, Theorem 5.4.2]:

Theorem 1.23. *Let R be a ring with no nontrivial idempotents, so that $\operatorname{Spec} R$ is connected. Choose a geometric point $x\colon \operatorname{Spec}(\Omega) \to \operatorname{Spec} R$ and let $G := \pi_1^{\text{ét}}(R, x)$. Then there is an equivalence of categories*

$$\mathrm{Cov}_R \xrightarrow{\sim} \mathrm{FinSet}_G^{\mathrm{cts}}.$$

2 Axiomatic Galois theory

The idea is to characterize which categories look like $\mathrm{FinSet}_G^{\mathrm{cts}}$, the category of finite sets with a continuous action of a profinite group G. We will follow the exposition of [Mat16], which itself is a modified version of Grothendieck's original axiomatic Galois theory. The modification in Mathew's theory is we do not choose a "basepoint" (e.g. a literal basepoint in the case of covering spaces, or the choice of separable closure in the algebraic case) since such a notion may not be available in the use cases considered in [Mat16]. As a result, we do not obtain an (étale) fundamental group but a *profinite groupoid*, similar to how we obtain a fundamental groupoid instead of fundamental group when we do not choose a basepoint of a topological space. Mathew's formulation of the abstract theory of Galois categories uses some notions from *descent theory*, which we briefly introduce.

2.1 Descent

Since Mathew's Galois theory is based on descent, it will be useful to have a basic understanding of the theory. In particular, we will need the notion of effective descent morphisms. The setup is that we have some morphism $Y \to X$ which satisfies nice properties making it into a "cover", meaning that "objects" over X (e.g. covering spaces) are objects over Y together with some gluing data. A systematic exposition is contained in [BJ01], but we mostly follow [Mat16]. First we need to define some categorical terminology which will allow us to define categories of objects together with gluing data in high generality.

Definition 2.1 (Monadic adjunction)**.** Let $L : C \rightleftarrows D : R$ be an adjunction of ordinary 1-categories C, D with unit η and counit ε. Let $T := R \circ L$ be an endofunctor $C \to C$.

(1) T has the structure of a *monad*, that is, there are natural transformations $\eta\colon \mathrm{id}_C \to T$ (given by the unit of the adjunction $L \dashv R$) and $\mu\colon T^2 \to T$ (given by applying the counit to the middle part of $T^2 = R(LR)L$) such that η serves as the unit and μ as the multiplication of a monoid in the category of endofunctors on C.

(2) There is a category C^T with objects (X, h) where X is an object of C and $h\colon TX \to X$ is a morphism in C such that the diagrams

$$\begin{array}{ccc} T^2X & \xrightarrow{Th} & TX \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle h} \\ TX & \xrightarrow{h} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta} & TX \\ & \searrow{=} & \downarrow{\scriptstyle\mu} \\ & & X \end{array}$$

commute. The morphisms $(X, h) \to (Y, g)$ are arrows $\alpha\colon X \to Y$ of C such that $T\alpha$ commutes over α. This category is called the category of algebras over the monad T, or the *Eilenberg-Moore category* of T.

(3) The natural transformation $R\varepsilon\colon R(LR) = TR \to R$ defines a functor

$$D \to C^T$$

by sending $X \mapsto (RX, (R\varepsilon)_X)$.

(4) If the above functor $D \to C^T$ is an equivalence of categories, we say that the adjunction is *monadic*.

The following definition is as in [Mat16, Definition 5.14] and [BJ01, Definition 4.4.1].

Definition 2.2 (Effective descent morphism)**.** In an ordinary 1-category C with finite limits, let $f\colon Y \to X$ be a morphism and consider the adjunction $C_{/Y} \rightleftarrows C_{/X}$ given by

$$\Sigma_f\colon C_{/Y} \to C_{/X}, \qquad f^*\colon C_{/X} \to C_{/Y}$$

where $\Sigma_f(g) := f \circ g$ and $f^*(g)$ is the pullback of g along f:

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ {\scriptstyle f^*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\ f\ } & X. \end{array}$$

Then, f is an *effective descent morphism* if the adjunction $C_{/Y} \rightleftarrows C_{/X}$ is *monadic*.

Example 2.3. For our purpose, we wish to think of the functor $C_{/X} \to (C_{/Y})^T$ as comparing the category $C_{/X}$ against the category $C_{/Y}$, together with *gluing data* (the data of a T-algebra). To give an example, let $C :=$ Top be some category of topological spaces, let X be a topological space and $\{U_i\}$ an open cover of X. Then there is a morphism $p\colon \coprod_i U_i \to X$ given by all of the inclusions $U_i \hookrightarrow X$.

In this situation, the monad $T := p^*p_*\colon C_{/\coprod_i U_i} \to C_{/\coprod_i U_i}$ sends $f\colon E \to \coprod_i U_i$ to the pullback

$$\begin{array}{ccc} \coprod_i U_i \times_X E & \longrightarrow & E \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle p\circ f} \\ \coprod_i U_i & \xrightarrow{\ p\ } & X \end{array}$$

where $\coprod_i U_i \times_X E \to \coprod_i U_i$ "spreads $E \to X$ out" over every open set in the cover $\{U_i\}$ of X. T-algebras are then maps $f\colon E \to \coprod_i U_i$ together with the data of a diagram

$$\begin{array}{ccc} \coprod_i U_i \times_X E & \longrightarrow & E \\ & \searrow \quad \swarrow{\scriptstyle f} & \\ & \coprod_i U_i & \end{array}$$

satisfying the properties needed to make this pair into a T-algebra. To ask that p is an effective descent morphism is to ask that $C_{/X}$ is equivalent to $(C_{/\coprod_i U_i})^T$ and we see that the structure of a T-algebra is encoding the compatibility conditions needed to glue pieces $f^{-1}(U_i) \to U_i$ together to a map $E \to X$. Therefore, C^T can be viewed as a category of objects together with "gluing data".

The following famous theorem is a useful tool when working effective descent morphisms and therefore monadic adjunctions. It is also often called the Barr-Beck criterion [BJ01, Section 4.4].

Theorem 2.4 (Beck's monadicity theorem). *A functor $R\colon D \to C$ is* monadic *(i.e. R admits a left adjoint $L\colon C \to D$ such that the adjunction $L \dashv R$ is monadic) if and only if the following conditions are met:*

(1) U has a left adjoint,

(2) U is conservative (i.e. U reflects isomorphisms), and

(3) whenever $f, g\colon x \to y$ in D are such that the pair (Rf, Rg) has a split coequalizer, then (f, g) has a split coequalizer which is created by R. This means that whenever there is a diagram

$$Rx \underset{Rg}{\overset{Rf}{\rightrightarrows}} Ry \xrightarrow{q} z, \qquad s\colon Ry \to Rx, \quad r\colon z \to Ry$$

such that $q \circ Rf = q \circ Rg$, $q \circ r = \mathrm{id}_z$, $Rf \circ s = \mathrm{id}_{Ry}$, $Rg \circ s = r \circ q$, then f creates the coequalizer of (f, g).

Example 2.5. It is a key result in the algebro-geometric theory of descent that *faithfully flat* morphisms of schemes $Y \to X$ are effective descent morphisms. The special case we are interested in is when both schemes are affine. Then, if $f\colon R \to S$ is a morphism of rings exhibiting S as a faithfully flat R-module, the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is an effective descent morphism [BJ01, Proposition 4.4.3].

In one direction, this can be seen as a consequence of the Barr-Beck criterion theorem 2.4: If $\operatorname{Spec} C \to \operatorname{Spec} A$ is en effective descent morphism in the category of finite étale covers of $\operatorname{Spec} A$, then base change $\mathrm{Cov}_{\operatorname{Spec} A} \to \mathrm{Cov}_{\operatorname{Spec} C}$ given by $- \times_A \operatorname{Spec} C$ is conservative. In terms of algebras, this implies that $- \otimes_A C$ is conservative, i.e. C is a *faithfully* flat A-module.

Example 2.6. Consider the category FinSet_G of finite sets with an action of a finite group G. Then, all maps $f\colon Y \to *$ where $Y \neq \emptyset$ turn out to be effective descent morphisms. This can be seen by elementary reasoning: The functor f^* is given on objects by $X \mapsto X \times Y$ with the right hand side given the diagonal action of G. This functor is conservative as long as $Y \neq \emptyset$ by checking conservativity on the level of sets. Moreover, coequalizers $\phi, \psi\colon X \rightrightarrows Y \to Z$ in FinSet_G always exists and are computed on the level of sets as $Z = \{x \in X \mid \phi(x) = \psi(x)\}$ with G-action inherited from X. One can also check that $- \times Y$ preserves these, so by theorem 2.4, f^* is monadic.

2.2 Galois categories

Definition 2.7. Let C be an ordinary 1-category with finite coproducts.

(1) If C has an empty initial object $\emptyset$ (that is, any $x \to \emptyset$ is an isomorphism), we say that *coproducts are disjoint* if the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & x \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \sqcup y \end{array}$$

is a pullback square for all objects x and y.

(2) If C has finite limits, we say that *coproducts are distributive* if for every morphism $y \to x$, the pullback functor $C_{/x} \to C_{/y}$ preserves finite coproducts.

Define exactly as in [Mat16, Definition 5.15]:

Definition 2.8 (Galois category)**.** A *Galois category* is an ordinary 1-category $\mathcal{C}$ such that:

(1) $\mathcal{C}$ admits finite limits and coproducts, and the initial object $\emptyset$ is empty,

(2) coproducts are disjoint and distributive, and

(3) given $x \in \mathcal{C}$, there is an effective descent morphism $x' \to *$ to the terminal object such that x' has a decomposition $x' \cong x'_1 \sqcup \cdots \sqcup x'_n$ such that each $x \times x'_i \to x'_i$ decomposes as the fold map $x \times x'_i \cong \coprod_{S_i} x'_i \to x'_i$ for a finite set S_i.

Note that although the third condition on $\mathcal{C}$ in definition 2.8 is somewhat lengthy, it is of the form of theorem 1.17 (finite étale algebras are trivialized by some faithfully flat base change). As such, it states that for any object x of $\mathcal{C}$, are able to find a descendable object x' such that after pulling back to the connected components of x', we see x as a trivial cover. This motivates the following definitions.

Definition 2.9 (Elementary form)**.** In a Galois category $\mathcal{C}$, an object x is in *elementary form* if it is isomorphic to a disjoint union of the form $\coprod_S *$ for a finite set S.

An object x is in *mixed elementary form* if there is a decomposition $* \cong y_1 \sqcup \cdots \sqcup y_n$ such that each pullback $x \times y_i \to y_i$ is in elementary form in $\mathcal{C}_{/y_i}$ (this is again a Galois category by [Mat16, Corollary 5.18]).

Example 2.10. The category FinSet_G of finite sets with an action of a finite group G forms a Galois category. The first two properties are purely categorical and hold since they hold for the category of finite sets. For the third condition, let X be a finite set with a G-action. we choose $X' = G$ with G acting on X' by translation, now the (diagonal) action of G on $X \times G$ is free. Therefore, given $x \in X$, the orbit of (x, e) has size G, and the orbits of all

$\{(x,e) \mid x \in X\}$ are disjoint since $(x,e) = (gy,g)$ is possible only for $g = e$ and therefore $x = y$. Thus, we have a decomposition $X \times G \cong \coprod_X G$ as sets with G-action. Therefore, the pullback $X \times G \to G$ is in fact a fold map $\coprod_X G \to G$. We can also see that in FinSet_G, all objects in elementary form as isomorphic to $\coprod_G * \cong G$ as finite sets with G-action.

We can easily extend this example to consider finite groupoids (see definition 2.13) by noting that by definition, such groupoids are equivalent to disjoint unions $\coprod_{i=1}^n BG_i$ for finite groups G_i.

Example 2.11. The category Cov_R of finite étale covers of $\operatorname{Spec} R$ is a Galois category. Here, we have finite limits since we have pullbacks (fiber product) and $\operatorname{Spec} R$ is terminal. We have coproducts (given by tensor product on the algebra level) and the initial object is $\operatorname{Spec} 0 = \emptyset$. The third condition is provided exactly by theorem 1.17, since $\operatorname{Spec} S' \to \operatorname{Spec} R$ is effective descent if S' is a faithfully flat R-module by example 2.5. That is, algebras in elementary form are the diagonal maps $S \to \prod_{i=1}^n S$ and every finite étale algebra is in mixed elementary form.

Example 2.12. Let X be a manifold and Cov_X be the category of *finite* covers $f\colon Y \to X$ of X. We show that for any fixed cover $f\colon Y \to X$ of constant degree $d < \infty$, there is another finite cover $X' \to X$ such that the pullback $X' \times_X Y \to X'$ is of the form $\coprod_{i=1}^d X' \to X'$. To this end, define

$$X' := \coprod_{x \in X} \{\text{bijections}[d] \to f^{-1}(\{x\})\}$$

where $[d] = \{1, \dots, d\}$. Let $\pi\colon X' \to X$ denote the projection $(x,\sigma) \mapsto x$.[2] We equip X' with a topology as follows: if $U \subset X$ is an open subset such that $f^{-1}(U) \cong \coprod_d U$, then there is a bijection $\pi^{-1}(U) \to U \times S_d$ since we can choose a consistent labeling of all fibers $f^{-1}(\{x\})$ for $x \in U$. Then, we give X' the finest topology such that all of the inverses $U \times S_d \to \pi^{-1}(U) \subseteq X'$ are continuous. This turns $\pi\colon X' \to X$ into a continuous map by properties of the final topology.

Now, we need to show that the pullback $f'\colon X' \times_X Y \to X'$ is a trivial cover (of degree d). The total space of the covering is $X' \times_X Y = \{(x,\sigma,y) \mid y \in f^{-1}(\{x\})\}$, and (x,σ,y) is projected to (x,σ). We define

$$g\colon X' \times_X Y \to \coprod_d X', \qquad (x,\sigma,y) \mapsto (\sigma^{-1}(y), x, \sigma).$$

Since the preimages of each $\pi^{-1}(U)$ are of the form $\pi^{-1}(U) \times U$, g is continuous. Finally, g admits an inverse given by $(k,x,\sigma) \mapsto (x,\sigma,\sigma(k))$ which can also be shown to be continuous, and we have shown that $X' \to X$ is the trivializing cover as desired. (In the case that X is not connected, d may not be constant, but in this case, we can do this construction for all connected components and obtain $X' = X'_1 \sqcup \cdots \sqcup X'_n$ such that each pullback $X'_i \times_X Y \to X'_i$

[2] This construction is entirely motivated by the construction of the *frame bundle* for a vector bundle; we just translate it to the discrete setting by replacing $\mathrm{GL}_n(\mathbb{R})$ by S_n.

is trivial.) It remains to check that $X' \to *$ (where we mean by $*$ the trivial cover $\mathrm{id}_X \colon X \to X$ which is the terminal object of Cov_X) is effective descent.

Denote by $\pi \colon X' \to X$ the morphism in Cov_X corresponding to the triangle

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & \searrow \quad \swarrow_{\mathrm{id}_X} & \\ & X & \end{array}$$

and by π^* the functor $\mathrm{Cov}_X \to \mathrm{Cov}_{X'}$ given by pullback along π. We need to show that this functor is monadic (note that we have used that $(\mathrm{Cov}_X)_{/X'} \simeq \mathrm{Cov}_{X'}$, which follows from general properties of covering spaces). By the Barr-Beck criterion (see theorem 2.4), it suffices to show that π^* is conservative and that if (ϕ, ψ) is a pair $\phi, \psi \colon U \rightrightarrows V$ in Cov_X so that $(\pi^*\phi, \pi^*\psi)$ has a coequalizer[3], then (ϕ, ψ) also has a coequalizer and this is preserved by π^*.

To see that π^* is conservative, note that if a cover $Z \times_X X' \to X'$ is an isomorphism, it has constant degree 1. But the degree is invariant under pullback, so $Z \to X$ must have had degree 1 to begin with, hence it was also an isomorphism. Next, note that Cov_X admits all coequalizers, with the coequalizer of $\phi, \psi \colon U \rightrightarrows V$ being $W := \{x \in U \mid \phi(x) = \psi(x)\}$ which covers X via the covering $U \to X$. The pullback f^*W is then $f^*W = \{(x, (y, \sigma)) \in U \times X' \mid x = y, \phi(x) = \psi(x)\}$, which is isomorphic to the coequalizer of $(f^*\phi, f^*\psi)$ constructed as W was.

2.3 Galois correspondence

In this section, we state the main theorem of this abstract version of Galois theory, which shows that Galois categories correspond to certain well-behaved groupoids, their *fundamental groupoids.* Note that we briefly introduce the theory of ind- and pro-objects of a category in appendix B, which we will need for the rest of this section.

Definition 2.13 (Finite groupoid)**.** A groupoid $\mathcal{G}$ (i.e. an ordinary 1-category in which all morphisms are isomorphisms) is *finite* if the following conditions hold:

(1) $\mathcal{G}$ has finitely many isomorphism classes of objects, and

(2) for every object $x \in \mathcal{G}$, the automorphism group $\mathrm{Aut}_{\mathcal{G}}(x)$ is finite.

We can build a (2, 1)-category FinGpd with objects the finite groupoids, morphisms the functors between them and 2-morphisms the natural isomorphisms.[4]

Theorem 2.14. *For a finite groupoid* $\mathcal{G}$*, the category* $\mathrm{Fun}(\mathcal{G}, \mathrm{FinSet})$ *is a Galois category, and this defines a functor* $(\mathrm{FinGpd})^{\mathrm{op}} \to \mathrm{GalCat}$.

[3] the actual criterion requires a weaker statement, but we show this stronger version for simplicity

[4] Since we do not use any (2, 1)-category theory in any meaningful way, we omit defining them. For our purposes, it suffices to think of them as categories enriched in groupoids. One can look at [RV22, Appendix B] for a detailed exposition.

This theorem follows from example 2.11. To do more, we need a structural result.

Theorem 2.15. *([Mat16, Proposition 5.34])* GalCat *admits filtered colimits, and these are computed at the level of underlying categories (i.e. the forgetful functor* GalCat $\to$ Cat *reflects filtered colimits).*

A consequence of this theorem is that the functor $(\mathrm{FinGpd})^{\mathrm{op}} \to \mathrm{GalCat}$ extends to a functor $\mathrm{Pro}(\mathrm{FinGpd})^{\mathrm{op}} \simeq \mathrm{Ind}((\mathrm{FinGpd})^{\mathrm{op}}) \to \mathrm{GalCat}$[5]. This allows us to state the abstract *Galois correspondence* [Mat16, Proposition 5.34]:

Theorem 2.16. *The functor* $\mathrm{Pro}(\mathrm{FinGpd})^{\mathrm{op}} \to \mathrm{GalCat}$ *is an equivalence of* $(2,1)$*-categories. In other words, given any Galois category* $\mathcal{C}$*, we can find a profinite groupoid* $\mathcal{G}$*, represented by the limit* $\lim_{i\in\Lambda} \mathrm{Hom}(\mathcal{G}_i, -)$ *where each* $\mathcal{G}_i$ *is a finite groupoid, such that* $\mathcal{C}$ *is equivalent to the filtered colimit of categories* $\mathrm{colim}_{i\in\Lambda^{\mathrm{op}}} \mathrm{Fun}(\mathcal{G}_i, \mathrm{FinSet})$.

Remark 2.17. To check that the functor is fully faithful, one shows that for finite groupoid $\mathcal{G}, \mathcal{G}'$, the functor

$$\mathrm{Fun}(\mathcal{G}, \mathcal{G}') \to \mathrm{Fun}^{\mathrm{Gal}}(\mathrm{Fun}(\mathcal{G}', \mathrm{FinSet}), \mathrm{Fun}(\mathcal{G}, \mathrm{FinSet}))$$

is an equivalence of groupoids (that these are groupoids follows from the (2, 1)-categorical structure). This can be reduced to computing the torsors in the category FinSet_G for any finite group G, which we have done in example 2.26. Essential surjectivity is a more involved argument.

We come now to the central definition of Mathew's Galois theory.

Definition 2.18 (Fundamental profinite groupoid)**.** Given a Galois category $\mathcal{C}$, the *fundamental profinite groupoid* $\pi_{\leq 1}(\mathcal{C})$ is the profinite groupoid associated to $\mathcal{C}$ under the equivalence of theorem 2.16. We may also call this the *Galois groupoid*.

Example 2.19. Let K be a field, and $\mathrm{Cov}_{\mathrm{Spec}\,K}$ the category of finite étale K-algebras. Any finite étale K-algebra $K \to R$ is of the form $R \cong \prod_{i=1}^{d} F_i$ for finite separable field extensions F_i of K. Recall from theorem 1.10 that $\mathrm{Cov}_{\mathrm{Spec}\,K}$ is equivalent to the category $\mathrm{FinSet}_G^{\mathrm{cts}}$ of finite sets with a continuous action of $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Theorem 2.16 offers a slightly different version of this theorem, since it talks talks not about a fundamental *group* but a *groupoid*. We will see in section 2.6 that since $\mathrm{Spec}\,K$ is connected, the profinite groupoid $\pi_{\leq 1}(\mathrm{Cov}_{\mathrm{Spec}\,K})$ has just a single isomorphism class of objects. Therefore, it is equivalent as a category to a group, i.e. a groupoid with one object. It follows that $\pi_{\leq 1}(\mathrm{Cov}_{\mathrm{Spec}\,K}) \cong B\,\mathrm{Gal}(K^{\mathrm{sep}}/K)$, where $B\,\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is the limit of the 1-object groupoids $B\,\mathrm{Gal}(F/K)$ where F ranges over all finite separable field extensions of K ordered by inclusion, and $B\,\mathrm{Gal}(F/K) \to B\,\mathrm{Gal}(F'/K)$ is the inclusion whenever $F' \supseteq F \supseteq K$. This can also be understood in terms of the later definition 2.30, where we define the functor $B(-)$ for profinite groups.

[5] see appendix B for an overview of Pro and Ind-objects and categories

2.4 Galois contexts

The above theory is entirely 1-categorical (or at most (2, 1)-categorical), but we will need to produce Galois categories from ∞-categories (say, suitable subcategories of modules over an $\mathbb{E}_\infty$-ring). The following tool introduced in [Mat16, Definition 5.26] is what achieves this.

Definition 2.20 (Galois context)**.** A *Galois context* is an ∞-category $\mathcal{C}$ with a class of morphisms $\mathcal{E}$ such that:

(1) $\mathcal{C}$ has finite limits and coproducts, and the initial object $\emptyset$ is empty (any map $x \to \emptyset$ is an isomorphism).

(2) Coproducts are disjoint and distributive.

(3) $\mathcal{E}$ is closed under composition and base change and contains all isomorphisms.

(4) Every morphism in $\mathcal{E}$ is an *effective descent morphism*[6].

(5) If the lower horizontal map of a pullback diagram is in $\mathcal{E}$, then the left vertical map is in $\mathcal{E}$ if and only if the right vertical map is in $\mathcal{E}$.

(6) $x \to y = y_1 \sqcup y_2$ is in $\mathcal{E}$ if and only if the projections $x \times_y y_1 \to y_1$ and $x \times_y y_2 \to y_2$ are in $\mathcal{E}$.

(7) Finite folds $\coprod_S x \to x$ belong to $\mathcal{E}$.

To show that a map is an effective descent morphism in an ∞-category, we will use the following generalization of the Barr-Beck criterion.

Remark 2.21. The Barr-Beck-Lurie criterion [Mat16, Theorem 3.3] states that an adjunction $F : C \rightleftarrows D : G$ between ∞-categories C and D is comonadic (this term is obtained by dualizing the definition of a monadic adjunction) if and only if

(1) F is conservative, and

(2) For any cosimplicial object $X^\bullet$ of C such that $F(X^\bullet)$ admits a splitting, then $\mathrm{Tot}(X^\bullet)$ exists and $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is an equivalence.

In a Galois context, one looks at the objects $x \in \mathcal{C}$ for which there is a map $y \to *$ such that $x \times y \to y$ is in "mixed elementary form". We call such objects *Galoisable.* The category of Galoisable objects is actually an ordinary 1-category, which is how we go from the ∞-categorical world to the 1-categorical world.

Theorem 2.22. *([Mat16, Proposition 5.28]) The full subcategory formed by the "Galoisable objects" in a Galois context is a Galois category.*

[6] since we are now in an ∞-category, this would require a new definition; we sketch a possible approach in section A.6

2.5 Torsors

Torsors are objects of a category with an action of some finite group G satisfying nice properties. It will turn out that torsors identify the Galois covers in categories of covers (i.e. Galois categories), so they will be of importance throughout our work.

Definition 2.23. Given a discrete group G, let BG be the groupoid with one object $*$ and a morphism $f_g\colon * \to *$ for every $g \in G$. Define composition by $f_g \circ f_h := f_{gh}$.

Remark 2.24. The notation BG suggests a strong relation to the classifying space of G. In fact, the geometric realization of the nerve $|N(BG)|$ is a model of the classifying space (whenever G is discrete) [Lur26, Tag 0038].

Definition 2.25 (G-torsor)**.** Let a finite group G and a suitable nice category $\mathcal{C}$ (we will always consider Galois categories defined in the following section) be given. A G-torsor in $\mathcal{C}$ is an object $x \in \mathcal{C}$ with a G-action (a functor $BG \to \mathcal{C}$ sending $*$ to x). Additionally, we require the existence of an effective descent morphism $y \to *$ such that $x \times y \in (\mathcal{C}_{/y})^{BG}$ is given by

$$x \times y \cong \coprod_G y$$

where the action of G on the right hand side is given by permutation.

Example 2.26. Let G be a finite group and $\mathcal{C} := \mathrm{FinSet}_G$. Then the set G with the left action of G is a G-torsor since $G \cong \sqcup_G *$. We can go one step further an examine all G'-torsors in FinSet_G. We have seen in example 2.6 that all $Y \to *$ such that $Y \neq \emptyset$ are effective descent morphisms, so by definition, the G'-torsors in FinSet_G are finite G-sets X with a G-equivariant action of G' such that there is a G-set $Y \neq \emptyset$ satisfying

$$X \times Y \cong \coprod_{G'} Y. \tag{2}$$

In particular, $|X| = G'$. Moreover, the G' action on X has to be transitive for the isomorphism in eq. (2) to hold. We conclude that $X = G'$ has to carry the permutation of action of G'. Thus, any G'-torsor is the set G' together with action of G that commutes with the permutation action of G', i.e. group homomorphisms $\phi\colon G \to G'$. Moreover, if $\psi\colon G \to G'$ is a homomorphism such that $\phi = g^{-1}\psi g$ for some $g \in G'$, then ϕ and ψ determine isomorphic torsors.

The following corollary [Mat16, Corollary 5.41] can be a useful result once we characterize the G-torsors in some given Galois category.

Corollary 2.27. *If x is an object in a Galois category $\mathcal{C}$, there exists a G-torsor y for some* finite group *G such that $x \times y \to y$ is in mixed elementary form in $\mathcal{C}_{/y}$.*

Proof. By theorem 2.16, x arises as the image of some object of $\mathrm{Fun}(\mathcal{G}, \mathrm{FinSet})$ for a finite groupoid $\mathcal{G}$ by proposition B.2. We will work inside $\mathrm{Fun}(\mathcal{G}, \mathrm{FinSet}) \subset \mathcal{C}$ for the remainder of the proof. By definition, $\mathcal{G}$ has a finite set of representatives of isomorphism classes, $x_1, \ldots, x_n$ with finite automorphism groups $G_1, \ldots, G_n$. Thus, we have an equivalence of categories $\mathcal{G} \simeq \coprod_{i=1}^n BG_i$ where BG_i is the one-object groupoid associated to G_i. In particular,

$$\mathrm{Fun}(\mathcal{G}, \mathrm{FinSet}) \simeq \prod_{i=1}^{n} \mathrm{Fun}(BG_i, \mathrm{FinSet}) = \prod_{i=1}^{n} \mathrm{FinSet}_{G_i} .$$

We also write G_i for the universal torsor in FinSet_{G_i}, which is the set G_i with the permutation action of G_i, $g \cdot h = gh$. We define a $G_1 \times \cdots \times G_n$-torsor $y \in \prod_{i=1}^n \mathrm{FinSet}_{G_i}$ as follows:

- as an object of $\prod_{i=1}^n \mathrm{FinSet}_{G_i}$, it is the tuple $(Y_1, \ldots, Y_n)$, where $Y_i = G_1 \times \cdots \times G_n$, with G_i acting only on the i-th component by permutation.
- y carries an action of $G_1 \times \cdots \times G_n$ by letting the group act by permutation on each Y_i.

Now $x \times y \to y$ is the collection $\{X_i \times Y_i \to Y_i\}_{i=1}^n$, so it suffices to show that each $X_i \times Y_i \to Y_i$ is in mixed elementary form in FinSet_{G_i}. This follows exactly as in example 2.10. □

2.6 Profinite groups

As with ordinary Galois theory, we are able to choose a basepoint and produce a profinite group instead of a profinite groupoid. In particular, in good cases, we will be able to talk about the Galois *group* of a Galois category, which is what we will do for the example in section 5. This is related to the Galois category being *connected* (think of the base space being connected, or the base ring being free from idempotents).

Definition 2.28. A Galois category is *connected* if there exists no nontrivial decomposition of the terminal object, and if the terminal object is not the initial (hence empty) object (i.e if the category is not just a single object).

The connectedness of the Galois category is equivalent to connectedness of its fundamental profinite groupoid (this means that $\pi_0\mathcal{G}$ is a single point, i.e. a single isomorphism class of objects).

Proposition 2.29. *([Mat16, Proposition 5.45]): A Galois category $\mathcal{C}$ is connected if and only if $\pi_{\leq 1}(\mathcal{C})$ is a connected profinite groupoid.*

We extend definition 2.23 to cover the case where G is not necessarily discrete but profinite.

Definition 2.30. Let $G \in \mathrm{Pro}(\mathrm{FinGroup})$ be a profinite group. Recall the category Ω_G with objects the normal subgroups of finite index of G, ordered by reverse inclusion from definition 1.3. Then we let $BG \in \mathrm{Pro}(\mathrm{FinGpd})_{*/}$ be given by the formal limit $\lim_{N \in \Omega_G{}^{\mathrm{op}}} B(G/N)$. This gives us a pro-object by remark B.4.

Remark 2.31. Given a profinite group, any continuous action of G factors through G/N, where N is normal subgroup of finite index (take the kernel of the action). Conversely, any action of some G/N can be extended to an action of G by precomposing with $G \to G/N$. This gives an equivalence of categories

$$\mathrm{FinSet}_G^{\mathrm{cts}} \simeq \mathrm{colim}_{N \in \Omega_G{}^{\mathrm{op}}} \mathrm{Fun}(B(G/N), \mathrm{FinSet}).$$

There is a functor $\pi_1\colon \mathrm{FinGpd}_{*/} \to \mathrm{FinGroup}$ given by taking the automorphism group of the chosen basepoint. This can be extended to a functor $\pi_1\colon \mathrm{Pro}(\mathrm{FinGpd})_{*/} \to \mathrm{FinGroup}$ which commutes with filtered limits, see [Mat16, Definition 5.47].

Proposition 2.32. *([Mat16, Proposition 5.48]) Let $\mathcal{G}$ be a pointed profinite groupoid, i.e. an object of* $\mathrm{Pro}(\mathrm{FinGpd})_{*/}$. *Then $\mathcal{G}$ is connected if and only if $B\pi_1\mathcal{G} \to \mathcal{G}$ is an isomorphism in* $\mathrm{Pro}(\mathrm{FinGpd})_{*/}$. *It follows that $B\colon \mathrm{Pro}(\mathrm{FinGroup}) \to \mathrm{Pro}(\mathrm{FinGpd})_{*/}$ is is fully faithful with essential image the* connected *pointed profinite groupoids.*

Thus, if we have a connected Galois category at hand, we obtain from it a connected profinite groupoid. Finally, to produce a profinite group, we need to choose a basepoint for the profinite groupoid.

Example 2.33. Recall from example 2.11 the Galois category Cov_R given a ring R. By definition, if $\mathrm{Spec}\, R$ is connected, then so is the Galois category, and we obtain a connected profinite groupoid $\pi_{\leq 1}(R) = \pi_{\leq 1}(\mathrm{Cov}_{\mathrm{Spec}\, R})$.

Choosing a basepoint $* \to \pi_{\leq 1}(R)$ gives us a profinite group G by taking the automorphism group of the basepoint (i.e. applying the functor π_1), and $\pi_{\leq 1}(R) \simeq BG$. The Galois correspondence (theorem 2.16) then states that the Galois category $\mathrm{Fun}(BG, \mathrm{FinSet})$ is equivalent to the category $\mathrm{Cov}_{\mathrm{Spec}\, R}$. By comparing with Grothendieck's original Galois theory, we see that choosing a basepoint of $\pi_{\leq 1}(R)$ (at least loosely) corresponds to choosing a geometric point of $\mathrm{Spec}\, R$, which is required to obtain a fundamental group in Grothendieck's formulation of Galois theory [Len08, Section 5]. We also see that the profinite group G is the étale fundamental group $\pi_1^{\text{ét}}(R)$ studied in [Sza09], [Len08] and originally in [71]. This is because $\pi_1^{\text{ét}}(R)$ is the unique profinite group up to isomorphism such that $\mathrm{Cov}_{\mathrm{Spec}\, R}$ is isomorphic to $\mathrm{Fun}(BG, \mathrm{FinSet})$ by [Len08, Theorem 1.11].

2.7 Galois extensions

To finish this section, we briefly return to the world of finite étale algebras and "classical" Galois theory. Recall that Galois extensions of fields K are field extensions L such that $\mathrm{Aut}(L/K)$ fixes K. It is not difficult to generalize this notion to Galois extensions of rings, which we do now, before introducing abstract machinery to extract this concept from the underlying Galois category. We show that the two agree in proposition 2.36. The following definition is from [Rog08, Definition 2.3.1], but a more expansive treatment is in [Gre92].

Definition 2.34. Given a finite group G, an algebra $R \to S$ such that G acts on S via R-algebra homomorphisms, $R \to S$ is called a *G-Galois extension* if

(1) The canonical map $R \to S^G$ is an isomorphism, and

(2) $S \otimes_R S \to \prod_G S$ is a G-equivariant isomorphism, where G acts on the second component of $S \otimes_R S$ and by permutation on $\prod_G S$.

Lemma 2.35. *A G-Galois extension $A \to B$ exhibits B as a finite étale algebra over A. Moreover, B is a faithfully flat A-module.*

Proof. We follow various proofs in [Gre92, Chapter 0]. First note that multiplication $\mu\colon B \otimes_A B \to B$ factors as

$$B \otimes_A B \xrightarrow{\cong} \prod_G B \xrightarrow{p_e} B$$

where p_e denotes projection onto the factor of the identity $e \in G$. Thus, multiplication is in fact projection onto a component, and B is a separable A-algebra.

To proceed, let $\sum_i x_i \otimes y_i$ be the preimage of δ_e for under $B \otimes_A B \to \prod_G B$ (note that i runs over a finite index set). Since this isomorphism is G-invariant, we have

$$\sum_i x_i g(y_i) = \mu\left(g \cdot \sum_i x_i \otimes y_i\right) = p_e(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

Further, define $\mathrm{tr}\colon B \to A$ by $x \mapsto \sum_{g \in G} g(y)$, which is well-defined since $\mathrm{tr}(x)$ lands in $B^G \cong A$. Define $x_i^*\colon B \to A$ by $z \mapsto \mathrm{tr}(zy_i)$. Then, for any $z \in B$, we have

$$\sum_i x_i^*(z) x_i = \sum_i \sum_{g \in G} g(zy_i) x_i = z \sum_{g \in G} \sum_i g(y_i) x_i = z \sum_{g \in G} \delta_e(g) = z$$

meaning that (x_i, x_i^*) is a finite dual basis of B over A, so B is finitely generated and projective (hence flat) over A. Thus, there is an exact sequence

$$0 \longrightarrow \ker \phi \longrightarrow A^n \xrightarrow{\phi} B \longrightarrow 0$$

and since B is projective, this is split, so $\ker \phi$ is a direct summand of A^n and is therefore finite. This shows that B is a finitely presented A-module, showing that $A \to B$ is finite étale.

It remains to show that B is a faithfully flat A-module. For this, We need to show that for any maximal ideal $\mathfrak{m} \subset A$, the quotient $B/\mathfrak{m}B$ is nonzero. If $\mathfrak{m}B = B$ for some maximal ideal $\mathfrak{m}$, then $\mathfrak{m}B_\mathfrak{m} = B_\mathfrak{m}$ by tensoring with $A_\mathfrak{m}$. Now $B_\mathfrak{m}$ is a finitely generated local ring, so Nakayma's lemma implies in this case that $B_\mathfrak{m} = 0$. However, since $A \to B$ is the inclusion of $A \cong B^G$ into a subring, the map $A_\mathfrak{m} \to B_\mathfrak{m}$ is injective and since A is not zero, neither is $A_\mathfrak{m}$ (being zero is a local property), so $B_\mathfrak{m} \neq 0$. □

For the following proposition, we need to take care with opposite categories. A G-torsor in $\mathcal{C}$ is potentially quite different to a G-torsor in $\mathcal{C}^{\mathrm{op}}$: for example, G with its natural self-action is a G-torsor in FinSet_G (take $y := *$) but not in ${\mathrm{FinSet}_G}^{\mathrm{op}}$. It appears that [Mat16, Proposition 6.13] has to be enjoyed under appropriate dualization.

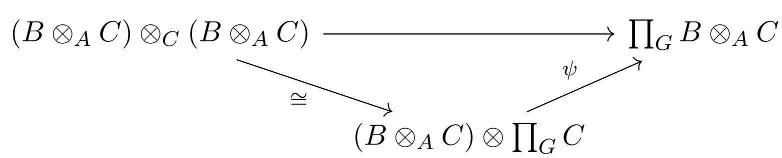
Proposition 2.36. *The G-torsors in the category $(\mathrm{CAlg}_R^{\text{ét}})^{\mathrm{op}}$ of (opposite) finite étale algebras over R are precisely the (opposite) G-Galois extensions of R.*

Proof. Let a G-Galois extension $A \to B$ be given. Then by definition, $B \otimes_A B \cong \prod_G B$ in the category $(\mathrm{CAlg}_R^{\text{ét}})^{BG} \subset (\mathrm{CAlg}_R)^{BG}$ since B is finite étale over A by lemma 2.35 (i.e. the isomorphism is G-equivariant with respect to the action of G on the second component on the left hand side, and by permutation on the right hand side). Passing to the opposite category shows the second property in the definition of a G-torsor (definition 2.25). The map $A \to B$ is effective descent by example 2.5 since B is faithfully flat as an A-module by [Rog08, Proposition 2.3.4(a)].

Conversely, given a G-torsor B, we obtain $C \otimes_A B \cong \prod_G C$ for some C immediately from the definition. We apply base change $A \to C$ to the map $B \otimes_A B \to \prod_G B$ and get (note that base change is a right adjoint and therefore commutes with fiber products)

$$(B \otimes_A C) \otimes_C (B \otimes_A C) \to \prod_G B \otimes_A C \tag{3}$$

given as a C-algebra morphism by $b \otimes b' \mapsto (b(gb'))_{g \in G}$. We apply the the G-equivariant isomorphism $B \otimes_A C \cong \prod_G C$ to see that this map factors as



where ψ is just a shift of the coordinates of $\prod_G C$ and then applying the canonical isomorphism $(B \otimes_A C) \otimes \prod_G C \cong \prod_G B \otimes_A C$. Thus, the map in eq. (3) is a G-equivariant isomorphism of C-algebras and we are done by faithfulness of C.

Now we show that $A \to B^G$ is an isomorphism. Note that C is a finite étale A-algebra, so in particular a flat A-module. Since $A \to C$ is effective descent by definition of G-torsors, theorem 2.4 implies that base change is conservative and we see that the map $B \otimes_A B \to \prod_G B$ must have been an equivalence.

Note first that B^G is defined as an A-algebra by the limit $\lim_{* \in BG} B$ where BG is the groupoid with one object $*$ and morphisms corresponding to elements $g \in G$. Since C is in particular a flat A-module, the functor $- \otimes_A C$ is exact and therefore preserves finite limits. In our case, $B^G = \lim_{* \in BG} B$ is finite, so

$$B^G \otimes C \cong (B \otimes_A C)^G \cong \left(\prod_G C \right)^G \cong C$$

is an isomorphism of C-algebras, where the last isomorphism is given by the diagonal $C \to \prod_G C, c \mapsto (c)_{g \in G}$. Since $- \otimes_A C$ is conservative, $A \to B^G$ must also be an isomorphism. $\square$

3 Higher algebra

Before continuing with Galois theory, it is necessary to introduce basic concepts in higher algebra. We will focus on defining $\mathbb{E}_\infty$-rings, which are spectra with the additional data of a homotopy coherent commutative ring structure. One of the reasons why spectra are interesting from an algebraic point of view is *stability*: (homotopy) fiber and cofiber sequences of spectra coincide up to stable equivalence, allowing us to work with *exact sequences* of spectra much as we would do with abelian groups or inside general abelian categories. We will first define stable ∞-categories, which are the categories in which there is such a good theory of exact sequences. The universal example is the ∞-category of spectra Sp. Then, we define symmetric monoidal ∞-categories which will allow us to define $\mathbb{E}_\infty$-rings as commutative monoid objects in the category of spectra, much as rings are commutative monoid objects in the category of abelian groups. We will assume the language of ∞-categories from now on, but no knowledge of any concrete model (e.g. quasicategories) is required. A collection of central ideas in model-independent ∞-category theory can be found in appendix A.

3.1 Stable ∞-categories

Stable ∞-categories are in the ∞-categorical world what abelian categories are in the classical world. Moreover, there are deep connections between stable ∞-categories and (derived) algebra. For example, the homotopy category of a stable ∞-category is triangulated. We will define the minimum of tools we will need, which is essentially just to be able to work with *exact sequences* of spectra.

Definition 3.1 (Stable ∞-categories)**.** An ∞-category C is *stable* if:

(1) C is pointed, that is, there is a zero object 0 which is both initial and terminal.

(2) C admits fibers and cofibers, that is, for any X, there are objects $\mathrm{fib}(X)$ and $\mathrm{cofib}(X)$ such that the diagrams

$$\begin{array}{ccc} \mathrm{fib}(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}, \qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(X) \end{array}$$

are pullback and pushout diagrams, respectively.

(3) A sequence $X \to Y \to Z$ such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

commutes is a pullback square if and only if it is a pushout square. In this case, we can equivalently say that $X \to Y \to Z$ is a *cofiber*, a *fiber* or an *exact* sequence.

Remark 3.2. Instead of $\mathrm{fib}(X)$ and $\mathrm{cofib}(X)$, we may also write ΩX and ΣX, especially when inside a category of spectra[7]. Instead of the iterates $\Omega^n X$ and $\Sigma^n X$, we may also write $X[-n]$ and $X[n]$, respectively. As justification for the notation, one can show that $X[n][m] \cong X[n+m]$.

Definition 3.3 (Exact functor)**.** A functor $F\colon C \to D$ between stable ∞-categories is *exact* if and only if it preserves the zero object and sends short exact sequences $X \to Y \to Z$ to exact sequences, i.e. if the left square below is a pullback (or pushout) square, then so is the right:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array} \quad \overset{F}{\rightsquigarrow} \quad \begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F(Z). \end{array}$$

Proposition 3.4. *([Lur17, Proposition 1.1.4.1]) Let C, D be stable ∞-categories, and $F\colon C \to D$ a functor. Then, the following are equivalent:*

(1) F is exact,

(2) F is left exact*, that is, it preserves terminal objects and pullbacks,*

(3) F is right exact*, that is, it preserves initial objects and pushouts.*

Remark 3.5. Stable ∞-categories are *additive* in the following sense:

(1) there is a zero object, there are finite products and coproducts, and the canonical map $X \sqcup Y \to X \times Y$ is an isomorphism[8] (hence we can write $X \oplus Y$ for the *biproduct*),

[7] recall that since we are in the ∞-categorical setting, (co)limits correspond to homotopy (co)limits in the classical world, so for example the pushout $\mathrm{cofib}(X)$ indeed has the underlying spectrum ΣX (in the classical sense).

[8] it is worth remarking that we will speak of *isomorphisms* in an ∞-category where other authors may speak of *equivalences*; this is due to the fact that what we call isomorphisms are, in $\mathcal{S}$ or Sp, classically called weak and stable equivalences

(2) the shear map $\sigma\colon X \oplus Y \to X \oplus Y$ given by the matrix

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix}$$

is an isomorphism.

This follows from having pullbacks and pushouts and checking that the two diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id}_X,0)} \\ Y & \xrightarrow[(0,\mathrm{id}_Y)]{} & X \times Y \end{array}, \qquad \begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id}_X,\mathrm{id}_X)} \\ X & \xrightarrow[(0,\mathrm{id}_X)]{} & X \times X \end{array}$$

are both pushout (equivalently, pullback) squares. We can then add maps $f, g\colon X \to Y$ by forming the following composition

$$f + g\colon X \xrightarrow{(\mathrm{id}_X,\mathrm{id}_X)} X \oplus X \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)} Y \oplus Y \xrightarrow{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} Y.$$

Here, the rightmost map is the fold map $X \oplus Y \cong Y \sqcup Y \to Y$. Moreover, if $\sigma\colon X \oplus Y \to X \oplus Y$ is the shear map as above, then we can form $-f$ by the following composition

$$-f\colon X \xrightarrow{(0,f)} X \oplus X \xrightarrow{\sigma^{-1}} X \oplus X \xrightarrow{\mathrm{pr}_1} X.$$

As an example, we can consider the split exact sequence $X \xrightarrow{i_1} X \oplus X \xrightarrow{\mathrm{pr}_2} X$. Using the shear map, we can shift this sequence:

$$\begin{array}{ccccc} X & \xrightarrow{(1,0)} & X \oplus X & \xrightarrow{(0,1)^T} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle -1} \\ X & \xrightarrow[(1,1)]{} & X \oplus X & \xrightarrow[(1,-1)^T]{} & X \end{array} \tag{4}$$

to obtain a new exact sequence in the bottom row, which we will denote $X \xrightarrow{\Delta} X \oplus X \xrightarrow{\delta} X$.

3.2 The ∞-category of spectra

Spectra are the basic objects of higher algebra. They can be thought of as homotopy coherent versions of abelian groups (indeed, there is a fully faithful inclusion from abelian groups to spectra, called taking the *Eilenberg-Maclane* spectrum HA). The difficulty with defining spectra is that one really wants to define a category Sp of spectra which is symmetric monoidal under an operation called the *smash product*. Having such an operation allows one to really begin to do algebra with spectra and e.g. define ring spectra as commutative algebra objects in Sp.

There are many "models" of spectra as 1-categories (model categories), but the most universal way to work with spectra is by working in the ∞-category of spectra. Then, all of the model categories (symmetric spectra, orthogonal spectra, $\mathbb{S}$-modules, etc.) are presentations[9] of the ∞-category Sp.

Definition 3.6 (Spectra)**.** The ∞-category of spectra is defined as

$$\mathrm{Sp} := \lim(\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$

taken in the ∞-category Cat_∞ of small ∞-categories (this definition is equivalent to others by [Lur17, Remark 1.4.2.25]). where $\mathcal{S}_*$ denotes the ∞-category of pointed spaces, and Ω the loop space functor.

The above definition is quite nice, because it relates the ∞-category of spectra directly to the ordinary 1-category of Ω-spectra, which are sequences $(X_n)_{n=1}^\infty$ of spaces with *bonding maps* $\Sigma X_n \to X_{n+1}$ such that the adjoint bonding maps $X_n \xrightarrow{\sim} \Omega X_{n+1}$ are weak homotopy equivalences. The objects of Sp are also of this form, only that the X_n are no longer pointed spaces in the classical sense but pointed homotopy types or ∞-groupoids.

Remark 3.7. Let $\mathcal{S}_*^{\mathrm{fin}}$ be the smallest subcategory of $\mathcal{S}_*$ containing S^0 and which is closed under finite colimits. Then Sp is equivalent to the subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*)$ consisting of those functors which:

(1) are *excisive*, that is, they send pushout squares to pullback squares, and

(2) are *reduced*, that is, they send the terminal object to the terminal object.

Example 3.8. For any pointed space X, one can construct a (sequential) spectrum $\Sigma^\infty X$ by considering the sequence $(\Sigma^n X)_{n=0}^\infty$ of pointed spaces. We define bonding maps $\xi_n \colon \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ via the homeomorphism $S^1 \wedge S^n \cong S^{n+1}$. This defines what is called a sequential spectrum, but not an Ω-spectrum since the adjoints of ξ_n are not weak homotopy equivalences in general. However, given a sequential spectrum X, one can always form an Ω-spectrum RX stably homotopy equivalent to X. Examples of spectra obtained in this way are:

(1) The *sphere spectrum* is $\mathbb{S} := \Sigma^\infty S^0$. We will sketch an alternative construction which showcases more of the algebraic structure of $\mathbb{S}$ in example 5.6.

(2) Given a finite (discrete) group G, we let $\mathbb{S}[G] := \Sigma^\infty G_+$ where G_+ is G together with a disjoint basepoint.

We will not go into further detail on spectra in this thesis. Mostly, we will be working on a more abstract categorical level rather than dealing with concrete constructions on spectra.

[9] simplicial model categories are *presentations* of an ∞-category in a precise sense [BGH22, Chapter 1]

Construction 3.9. A stable ∞-category C is naturally enriched in spectra. To see this, consider the Yoneda embedding

$$C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}), \qquad Y \mapsto \mathrm{Hom}_C(-, Y)$$

(see theorem A.18 or [Lur09] for a precise definition of the ∞-categorical Yoneda embedding). All functors of the form $\mathrm{Hom}_C(-, Y)$ preserves limits, so they are in particular left exact. There is an equivalence $\mathrm{Fun}^{\mathrm{ex}}(C^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{op}}, \mathcal{S}_*)$ due to the fact that Sp is the *stabilization* of $\mathcal{S}_*$ [Lur17, Corollary 1.4.2.23]. Thus, we can upgrade the Yoneda embedding to a functor $C \to \mathrm{Fun}^{\mathrm{ex}}(C^{\mathrm{op}}, \mathrm{Sp})$, giving rise to hom-spectra $\underline{\mathrm{Hom}}_C(X, Y)$. It also follows that *exact* functors between stable ∞-categories give rise to maps $\underline{\mathrm{Hom}}_C(X, Y) \to \underline{\mathrm{Hom}}_D(F(X), F(Y))$ between hom-spectra.

We will also need to speak of spectra with an action of G. Recall from definition 2.23 the groupoid BG associated to a finite group G. We can view this as an ∞-category $N(BG)$ and define the category of spectra with a G-action as follows. We will often drop the $N(-)$ from the notation and write BG even when we need ∞-categories.

Definition 3.10. If G is a finite group, then the *category of spectra with G-action*[10] is $\mathrm{Sp}^{BG} := \mathrm{Fun}(BG, \mathrm{Sp})$. If $X\colon BG \to \mathrm{Sp}$ is a spectrum with G-action, we will often also denote its underlying spectrum $X(*)$ by X.

Definition 3.11. Let C be an ∞-category admitting (co)limits of diagrams indexed by BG. The *homotopy fixpoints* of $B \in C^{BG}$ are $B^{hG} := \lim_{BG} B$. Dually, the *homotopy orbits* of $B \in C^{BG}$ are $B_{hG} := \mathrm{colim}_{BG} B$.

Remark 3.12. For finite G, there is a *norm map* $N\colon X_{hG} \to X^{hG}$. The general construction of the norm map is detailed in [Lur17, Proposition 6.1.6.19]. We give a sketch of the definition here. Let $f\colon BG \to *$ be a map of Kan complexes and C any stable ∞-category. Note that BG is a finite Kan complex since G is finite. Furthermore, let $\delta\colon BG \to BG \times BG$ be the diagonal map. Then, it turns out that the functor $\delta^*\colon C^{BG\times BG} \to C^{BG}$ given by precomposition with δ admits a left adjoint $\delta_!$ and a right adjoint δ_*. Moreover, one can argue that $\delta\colon BG \to BG \times BG$ has discrete homotopy fibers, and it follows from [Lur17, Proposition 6.1.6.12] that there is a natural isomorphism $\mathrm{Nm}_\delta\colon \delta_! \to \delta_*$. Let $p_0, p_1\colon BG \times BG \to BG$ be the two projections, giving a natural transformation

$$p_0^* \to \delta_*\delta^* p_0^* \cong \delta_* \overset{\mathrm{Nm}_\delta^{-1}}{\cong} \delta_! \cong \delta_!\delta^* p_1^* \to p_1^*$$

where the left and rightmost maps are the unit of $\delta^* \dashv \delta_*$ and the counit of $\delta_! \dashv \delta^*$, respectively. The unlabeled maps come from the isomorphisms $p_0\delta \cong p_1\delta \cong \mathrm{id}_{BG}$. This gives rise to a

[10] In the literature, G-spectrum often refers to a related but different concept, so we are careful to only speak of spectra with G-action.

natural transformation $\mathrm{id}_{C^{BG}} \to (p_0)_* p_1^*$. The pullback diagram

$$\begin{array}{ccc} BG \times BG & \xrightarrow{p_0} & BG \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ BG & \xrightarrow{f} & * \end{array}$$

gives rise to a natural isomorphism $f^* f_* \cong (p_0)_* p_1^*$ by [Lur17, Lemma 6.1.6.3], so there is a natural transformation $\mathrm{id}_{C^{BG}} \to f^* f_*$ and from this a $\mathrm{Nm}_f \colon f_! \to f_*$.

We had defined $f_! \colon C^{BG} \to C$ as the left adjoint of $f^* \colon C \to C^{BG}$, which sends an object X of C to an object of C^{BG} with a trivial action of G. If Y has trivial G-action, and X is any object with G-action, then $\mathrm{Hom}_{C^{BG}}(X, Y)$ is isomorphic to $\mathrm{Hom}_C(X_{hG}, Y)$ since any $\phi \colon X \to Y$ must be constant on every orbit by Y having the trivial action. Therefore, $f_! \cong (-)_{hG}$ by uniqueness of left adjoints. Similarly, $f_* \cong (-)^{hG}$. Therefore, we get a natural transformation $\mathrm{Nm} \colon (-)_{hG} \to (-)^{hG}$. By [Lur17, Remark 6.1.6.23], $X_{hG} \to X^{hG}$ is given informally by $[x] \mapsto \sum_{g \in G} gx$.

Example 3.13. Recall from example 3.8 that we defined a spectrum $\mathbb{S}[G] = \Sigma^\infty G_+$ for a finite group G. As a pointed space, $G_+ \cong \bigvee_G S^0$, so by properties of the functor Σ^∞, there is an isomorphism of spectra $\mathbb{S}[G] \cong \bigvee_G \Sigma^\infty S^0 \cong \prod_G \mathbb{S}$ (note that both $\bigvee$ and $\prod$ denote the biproduct in the ∞-category of spectra; however, $\bigvee$ is more common when working with a concrete model of spectra). Then, we can let G act on $\mathbb{S}[G]$ by permuting the coordinates of $\prod_G \mathbb{S}$.

Moreover, since the finite products coincide with finite coproducts in Sp, we also see that for any spectrum X

$$\underline{\mathrm{Hom}}(\mathbb{S}[G], X) \cong \underline{\mathrm{Hom}}\left(\prod_G \mathbb{S}, X\right) \cong \prod_G \underline{\mathrm{Hom}}\,(\mathbb{S}, X) \cong \prod_G X$$

compatible with the action of G on $\mathbb{S}[G]$ on the left hand side and the permutation action of G on the right hand side.

Example 3.14. Assume that $X = Y \otimes \mathbb{S}[G]$ with G acting freely on the $\mathbb{S}[G]$ component. Then, the norm map is of the form $x \otimes g \mapsto x \otimes (\sum_{h \in G} hg) = x \otimes (\sum_{g \in G} g)$. More precisely, there is a commutative diagram

$$\begin{array}{ccc} (Y \otimes \mathbb{S}[G])_{hG} & \xrightarrow{\mathrm{Nm}_X} & (Y \otimes \mathbb{S}[G])^{hG} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ Y \otimes \mathbb{S}[G]_{hG} & \xrightarrow{\mathrm{id}_Y \otimes \mathrm{Nm}_{\mathbb{S}[G]}} & Y \otimes \mathbb{S}[G]^{hG}. \end{array}$$

One now has to show that $\mathbb{S}[G]_{hG} \to \mathbb{S}[G]^{hG}$ is an isomorphism. This is not straightforward with our definition, so we just give the idea here: the map $\mathbb{S}[G]_{hG} \to \mathbb{S}[G]^{hG}$ is informally given by $[g_i] \mapsto \sum_{h\in G} hg_i$, which factors as

$$\begin{array}{ccc} \mathbb{S}[G]_{hG} & \longrightarrow & \mathbb{S}[G]^{hG} \\ & \searrow \quad \nearrow & \\ & \mathbb{S} & \end{array}$$

where $\mathbb{S}[G]_{hG} \to \mathbb{S}$ is $[g_i] \mapsto 1$ and $\mathbb{S} \to \mathbb{S}[G]^{hG}$ is $1 \mapsto \sum_{g\in G} g$. Both of these maps are isomorphisms. Accepting these facts shows that the norm map $\mathrm{Nm}_X \colon X_{hG} \to X^{hG}$ is an isomorphism in this case.

Moreover, Rognes shows that even in the case where Y has an action of G, then the norm map on X is an isomorphism: In the notation of [Rog08, Section 3.6], the map $N\colon ((Y \wedge G_+) \wedge S^{adG})_{hG} \to (Y \wedge G_+)^{hG}$ is an equivalence, where S^{adG} is called the *dualizing spectrum*. When G is finite, $S^{adG} = S$ by [Rog08, Section 3.5], implying the claim.

3.3 Symmetric monoidal ∞-categories

Heuristically, a symmetric monoidal ∞-category should be an ∞-category with an operation $\otimes$ which is associative and symmetric and has a unit *up to coherent homotopy*. For example, we should have a homotopy equivalence $s_{X,Y} \colon X \otimes Y \to Y \otimes X$ such that amongst others, the two maps $(X \otimes Y) \otimes Z \to Z \otimes (Y \otimes X)$ commute up to homotopy (and we have to provide the data of this homotopy). This is intractable to directly write down, and we have to introduce the terminology of cocartesian fibrations.

Definition 3.15. Let $p\colon C \to D$ be a functor of ∞-categories. Then:

(1) A morphism $f\colon x \to y$ in C is *p-cocartesian* if, for every object z of C,

$$\begin{array}{ccc} \mathrm{Hom}_C(y,z) & \xrightarrow{\circ f} & \mathrm{Hom}_C(x,z) \\ \downarrow p & & \downarrow p \\ \mathrm{Hom}_D(p(y),p(z)) & \xrightarrow{\circ p(f)} & \mathrm{Hom}_D(p(x),p(z)) \end{array}$$

is a pullback square (in the category of spaces $\mathcal{S}$).

(2) p is a *cocartesian fibration* if for every $g\colon x' \to y'$ in D, and every object x of C such that $p(x) = x'$, there exists a p-cocartesian morphism $f\colon x \to y$ such that $p(f) = g$. In this case, we call f a *p-cocartesian lift of g*.

(3) Given cocartesian fibrations $p\colon C \to D$, $p'\colon C' \to D$, a functor $F\colon C \to C'$ is *cocartesian over D* if there is a commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{F} & C' \\ & {}_p\searrow \quad \swarrow_{p'} & \\ & D & \end{array}$$

such that F sends p-cocartesian morphisms in C to p'-cocartesian morphisms in C'.

(4) If C is a small ∞-category, let $\mathrm{Cocart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$ be the ∞-category of cocartesian fibrations over C, with morphisms the cocartesian functors over C.

A fundamental result in ∞-category theory is the straightening/unstraightening equivalence.

Theorem 3.16. *(Straightening/unstraightening, [Lur09, p. 3.2.0.1]) Let C be a small ∞-category. There is an equivalence of ∞-categories*

$$\mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty)$$

given by sending a cocartesian fibration $p\colon E \to C$ to its fibers*, given on objects by the pullbacks $E_x := \{x\} \times_C E$.*

Remark 3.17. This equivalence is the ∞-categorical version of the *Grothendieck construction* relating functors $F\colon C^{\mathrm{op}} \to \mathrm{Cat}$ to *Grothendieck fibrations* $\int F \to C$ (we will not define this notion). The idea is that given such a functor F, we can construct a category $\int F$ where

- objects are pairs (c, x) with c an object of C and x an object of the category $F(c)$, and
- morphisms are pairs $(f, \phi)\colon (c, x) \to (d, y)$ where $f\colon c \to d$ is a morphism in C and $\phi\colon F(f)(x) \to y$ a morphism in the category $F(d)$.

Many fibered objects one often works with fit into this construction. For example, taking (real or complex) vector bundles on manifolds forms a functor $F\colon \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Cat}$, $M \mapsto \mathrm{Vect}(M)$ which we can equivalently see as a Grothendieck fibration $V \to \mathrm{Mfd}$, where V is now the category of pairs $(M, \phi\colon E \to M)$ where M is a manifold and ϕ a vector bundle on M. The fibration is given by projection $(M, \phi) \mapsto M$.

We will make a special choice of base category C to encode what a symmetric monoidal structure should be.

Definition 3.18. Let Fin_* be the 1-category with objects $\langle n \rangle = \{0, 1, \dots, n\}$ for all $n \geq 0$ and morphisms $\alpha\colon \langle n \rangle \to \langle m \rangle$ functions such that $\alpha(0) = 0$ (i.e. 0 is treated as the base point of every object). Sometimes, we will use the notation $\langle n \rangle^\circ = \langle n \rangle \setminus \{0\} = \{1, \dots, n\}$. Furthermore, let $f\colon \langle n \rangle \to \langle m \rangle$ be a morphism. Then:

(1) f is *inert* if $f^{-1}(i)$ has one element for each $i \geq 1$,

(2) f is *active* if $f^{-1}(0) = \{0\}$.

Under straightening/unstraightening, cocartesian fibrations $p\colon C^\otimes \to N(\mathrm{Fin}_*)$ correspond to functors $F\colon N(\mathrm{Fin}_*) \to \mathrm{Cat}_\infty$. We write $C^\otimes_{\langle n \rangle}$ to mean to the value of F at $\langle n \rangle$, and we refer to this as the *fiber over* $\langle n \rangle$.

Definition 3.19 (Symmetric monoidal ∞-category)**.** A *symmetric monoidal* ∞*-category* $C^{\otimes}$ is a cocartesian fibration $p\colon C^{\otimes} \to N(\mathrm{Fin}_*)$ such that the maps $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ given by $\rho^i(i) = 1$ and $\rho^i(j) = 0$ for $j \neq i$ induce maps

$$\rho^i_!\colon C^{\otimes}_{\langle n\rangle} \to C^{\otimes}_{\langle 1\rangle}$$

which together induce an equivalence $C^{\otimes}_{\langle n\rangle} \simeq (C^{\otimes}_{\langle 1\rangle})^n$.

The fiber $C^{\otimes}_{\langle 1\rangle}$ over $\langle 1\rangle$ is called the *underlying* ∞*-category* of $C^{\otimes}$ and is often just denoted C. From the definition, we can recover a pairing $\otimes$ by considering the morphism $\alpha\colon \langle 2\rangle \to \langle 1\rangle$ given by $\alpha(1) = \alpha(2) = 1$. Since p is cocartesian, this gives a map

$$C^{\otimes}_{\langle 2\rangle} \cong C \times C \to C = C^{\otimes}_{\langle 1\rangle}$$

which we denote $\otimes\colon C \times C \to C$ (observe that the other, morphisms $\langle 2\rangle \to \langle 1\rangle$ are all inert and would give rise to the two projections $C \times C \to C$ and to the functor constant at the unit). The category C is a very different category from $C^{\otimes}$, which we will see in the following remark.

Remark 3.20. To give an idea of why this definition is sensible, Lurie [Lur17] offers the following intuition: building a tensor product functor $C \times C \to C$ usually involves making some choices (e.g. choosing the vector space $V \otimes W$), and hence all identities such as $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ only hold up to canonical isomorphism instead of equality. If we introduce homotopy coherence into this setting, we end up with the problem mentioned at the beginning of this section. However, notice that it is easy to describe maps *out* of both $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ by just using the universal property, so we can define a symmetric monoidal structure by specifying maps *out* of tuples $[V_1, \ldots, V_n]$ without actually ever constructing a functor $C \times C \to C$. Then, the category $C^{\otimes}$ consists of such tuples, and the relevant maps between n and m-tuples are identified by the morphisms of Fin_*. In this context, inert morphisms correspond to projections, e.g. $V \times W \to V$ and $V \times W \to W$, whereas active morphisms can "combine" inputs to a tensor product, as in $(V, W) \mapsto V \otimes W$.

Example 3.21. In the context of higher algebra, the most important symmetric monoidal ∞-category is the category of spectra Sp with the *smash product* symmetric monoidal structure. Following the introduction to [Lur17, Section 4.8.2], we outline potential approaches to defining this symmetric monoidal structure:

(1) The classical approach is to choose some good model category of spectra, such as symmetric spectra or the S-modules of [EKMM97], which have an ordinary 1-categorical symmetric monoidal structure.

(2) There is an equivalence of categories $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$ given by evaluation at $\mathbb{S}$. It is not surprising that $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, \mathrm{Sp})$ admits a *monoidal* structure by composition. This turns out to be symmetric, but this is not easy.

(3) Let Pr^L be the ∞-category of all presentable ∞-categories and colimit preserving functors between them. Then, taking products of ∞-categories gives a symmetric monoidal structure on Pr^L which also descends to a symmetric monoidal structure on the full subcategory spanned by the *stable* presentable ∞-categories. One then shows that Sp can be given the structure of a commutative algebra object with respect to this symmetric monoidal structure, which implies exactly that Sp is a presentable symmetric monoidal stable ∞-category such that $\otimes\colon \mathrm{Sp}\times\mathrm{Sp}\to\mathrm{Sp}$ preserves colimits in each variable.

Example 3.22. Recall the spectrum $\mathbb{S}[G]$ from example 3.8. This enjoys the universal property that a map $\mathbb{S}[G]\to X$ is given by a map $G\to\Omega^\infty X$ of unbased spaces. Then, we can obtain a map $\mathbb{S}[G]\otimes B\to B$ as the adjoint of $\mathbb{S}[G]\to\underline{\mathrm{Hom}}(B,B)$ which itself is the adjoint of $G\to\Omega^\infty\underline{\mathrm{Hom}}(B,B)$ in $\mathcal{S}$. Given a functor $F\colon BG\to\mathrm{Sp}$, such a map can be obtained by constructing the map on *spaces* $G\cong\mathrm{Hom}_{BG}(*,*)\to\mathrm{Hom}(B,B)$. Note that the infinite loop space of the (internal) hom *spectrum* is the hom space in the ∞-category of spectra.

Definition 3.23. Given two symmetric monoidal ∞-categories $p\colon C^\otimes\to N(\mathrm{Fin}_*)$ and $q\colon D^\otimes\to N(\mathrm{Fin}_*)$, a functor $F\colon C^\otimes\to D^\otimes$ over $N(\mathrm{Fin}_*)$ is *lax symmetric monoidal* if it preserves cocartesian lifts of inert morphisms.

Given a lax symmetric monoidal functor F, we get a map $F(X)\otimes F(Y)\to F(X\otimes Y)$, but this is not necessarily an isomorphism since we do not preserve the non-inert morphism $\alpha\colon\langle 2\rangle\to\langle 1\rangle$.

Definition 3.24 (Commutative algebra object)**.** A *commutative algebra object* of a symmetric monoidal ∞-category $C^\otimes$ is a section $s\colon N(\mathrm{Fin}_*)\to C^\otimes$ of p which sends inert morphisms to cocartesian morphisms in $C^\otimes$. These form a category $\mathrm{CAlg}(C^\otimes)$. For the case where $C^\otimes=\mathrm{Sp}^\otimes$, we often write $\mathrm{CAlg}=\mathrm{CAlg}(\mathrm{Sp}^\otimes)$ and call this the category of $\mathbb{E}_\infty$-rings[11].

3.4 Modules

To define modules over commutative algebra objects of a symmetric monoidal category $C^\otimes$, we expand the category Fin_* slightly. What follows is a specialization of [Lur17, Notation 4.2.1.6] to the case where we care not about associative algebras but only commutative algebras.

Definition 3.25. Let FinM_* denote the 1-category with

(1) Objects pairs $(\langle n\rangle, S)$ with $\langle n\rangle$ an object of Fin_* and $S\subseteq\langle n\rangle^\circ$ (recall from definition 3.18 that $\langle n\rangle^\circ=\{1,\dots,n\}$).

(2) Morphisms $(\langle n\rangle,S)\to(\langle m\rangle,T)$ consist of a morphism $\alpha\colon\langle n\rangle\to\langle m\rangle$ in Fin_* such that $\alpha(S\cup\{0\})\subseteq T\cup\{0\}$ and so that for all $t\in T$, we have that $\alpha^{-1}(\{t\})$ contains exactly one element of S.

[11] The $\mathbb{E}$ is referring to "everything", i.e. associative and commutative, and the ∞ is referring to the fact that these properties hold up to coherent homotopy.

This definition can be explained as follows: the elements of the set S correspond to the module elements of a multiplication operation taking inputs from an algebra and a module over that algebra. Then, any input module element must be either projected away (sent to the basepoint 0) or sent to an output module element. Moreover, to every output module element, there has to be a unique input module element. That is, if y is an output module element, it must have arisen as $a_1 \cdots a_i x a_{i+1} \cdots a_n$ for a_j algebra elements. The construction of [Lur17, Notation 4.2.1.6] achieves precisely this, but for the case where we care about the relative order of multiplicands.

Note that there is forgetful functor $N(\mathrm{FinM}_*) \to N(\mathrm{Fin}_*)$ sending $(\langle n \rangle, S)$ to $\langle n \rangle$.

Definition 3.26. Given a symmetric monoidal ∞-category C (i.e. a cocartesian fibration $p\colon C^{\otimes} \to N(\mathrm{Fin}_*)$) the category of modules $\mathrm{Mod}(C)$ is defined to the category of diagrams

$$\begin{array}{ccc} N(\mathrm{FinM}_*) & \xrightarrow{\quad f \quad} & C^{\otimes} \\ & \searrow \quad \swarrow_{p} & \\ & N(\mathrm{Fin}_*) & \end{array}$$

where f takes inert morphisms (i.e. those where the underlying $\alpha\colon \langle n \rangle \to \langle m \rangle$ is inert) to inert morphisms.

Example 3.27. As an example, note that the image of $(\langle 1 \rangle, \emptyset)$ gives the *underlying algebra object* A of the module in question, and $(\langle 1 \rangle, \{1\})$ is the underlying object of the module M itself. The map $(\langle 2 \rangle, \{2\}) \to (\langle 1 \rangle, \{1\})$ given by $\alpha(0) = 0$ and $\alpha(2) = \alpha(1) = 1$ gives a multiplication map

$$A \otimes M \to M \ .$$

We can also restrict the category to modules over some fixed algebra object A by forming the pullback

$$\begin{array}{ccc} \mathrm{Mod}_A & \longrightarrow & \mathrm{Mod}(C) \\ \downarrow & & \downarrow \\ \{A\} & \longrightarrow & \mathrm{CAlg}(C) \end{array}$$

where the right vertical map is induced by the forgetful functor $N(\mathrm{FinM}_*) \to N(\mathrm{Fin}_*)$. In our situation, we are interested in the case where $C = \mathrm{Sp}$, so that $\mathrm{CAlg}(C) = \mathrm{CAlg}$ is the category of $\mathbb{E}_\infty$-rings. Then, given an $\mathbb{E}_\infty$-ring R, we write Mod_R for the category of modules over R.

3.5 Relative tensor product

We can define a symmetric monoidal structure on the ∞-category Mod_R. On the level of underlying objects, this is given by the *bar construction*: let $\mathrm{Bar}(M, N)_n := M \otimes A^{\otimes n} \otimes N$ be a simplicial object, then

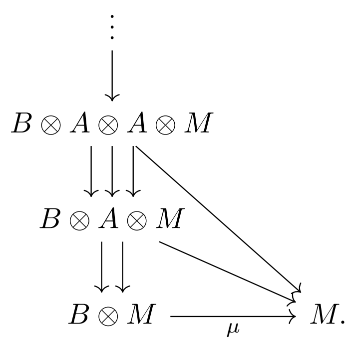
$$N \otimes_A M \cong |\mathrm{Bar}_A(M, N)| = \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Bar}_A(M, N)_n$$

is the geometric realization of this simplicial spectrum. Of course, this is not an actual definition, since to provide the actual symmetric monoidal structure, we would need to define a cocartesian fibration $p\colon \mathrm{Mod}_A^{\otimes} \to N(\mathrm{Fin}_*)$. This is the content of [Lur17, Sections 4.4 and 4.5], and in particular the bar construction [Lur17, Construction 4.4.2.7]. The functor $- \otimes_A -$ we define is therefore "only" a functor, and we assume that it comes from a full symmetric monoidal structure.

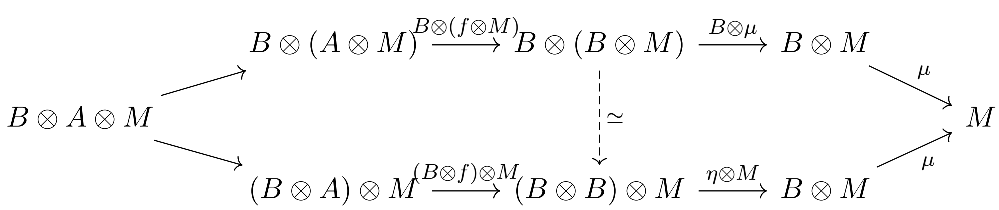
We will need the following lemma, which we prove using the bar construction heuristic.

Lemma 3.28. *Let A, B be $\mathbb{E}_\infty$-rings and $f\colon A \to B$ a morphism in* CAlg. *Let M be a B-module, so that M also inherits an A-module structure along f. Then the module multiplication $\mu\colon B \otimes M \to M$ extends to a map $\overline{\mu}\colon B \otimes_A M \to M$.*

Proof. To get a map from $B \otimes_A M \cong |\mathrm{Bar}_A(M, N)|$, we need to build a tower



Let $B \otimes A \otimes M \to M$ be given by multiplying $B \otimes A \to A$ using the map f. We then need to show that this commutes (up to homotopy) with first applying $A \otimes M \to M$ and then multiplying by B. That is,



should commute up to homotopy, where η comes from the $\mathbb{E}_\infty$-ring structure of B. We split the diagram into two at the dashed arrow. Then, the left part is (homotopy) commutative by the symmetry of the tensor product. The right part is commutative by the fact that M is a B module on the right. Proceeding in the same fashion, we may climb up the tower and finally obtain the desired map $\overline{\mu}\colon B \otimes_A M \to M$. $\square$

Lemma 3.29. *Let A, B be $\mathbb{E}_\infty$-rings, M a spectrum with the structure of an A-module as well as a B-module, X an A-module and Y a B-module. Then*

$$X \otimes_A (M \otimes_B Y) \cong (X \otimes_A M) \otimes_B Y$$

where $M \otimes_B Y$ inherits the structure of an A-module by multiplying M, and similar for $X \otimes_A M$.

Proof. We will use the heuristic of treating the relative tensor product as the bar construction $M \otimes_B Y \cong \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Bar}_B(M, Y)_n$. Note next that $X \otimes_A -$ is a left adjoint (to $\mathrm{Hom}_A(X, -)$) so it preserves colimits. Consider first the case $A = \mathbb{S}$, where we have

$$\begin{aligned} X \otimes (M \otimes_B Y) &\cong X \otimes \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Bar}_B(M, Y)_n \\ &\cong \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} X \otimes \mathrm{Bar}_B(M, Y)_n \\ &\cong \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} X \otimes (M \otimes B^{\otimes n} \otimes Y) \\ &\cong \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Bar}_B(X \otimes M, Y)_n \\ &\cong (X \otimes M) \otimes_B Y. \end{aligned}$$

The case for general A follows the same calculation, except we have to apply the result just proven to see $X \otimes_A (M \otimes B^{\otimes n} Y) \cong (X \otimes_A M) \otimes B^{\otimes n} \otimes Y$. □

We follow the approach of [Cno26] to define an internal hom functor in the module categories Mod_R for an $\mathbb{E}_\infty$-ring R.

Theorem 3.30. *There is an internal-hom functor*

$$\underline{\mathrm{Hom}}_R \colon {\mathrm{Mod}_R}^{\mathrm{op}} \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

such that $- \otimes_R M : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_R : \underline{\mathrm{Hom}}_R(M, -)$ is an adjunction.

Proof. Let N, P be fixed R-modules, and $F \colon {\mathrm{Mod}_R}^{\mathrm{op}} \to \mathcal{S}$ be the functor $F := \mathrm{Hom}_R(- \otimes_R N, P)$. This preserves limits since

$$\begin{aligned} F(\lim_{i \in I} M(i)) &\cong \mathrm{Hom}_R((\lim_{i \in I} M(i)) \otimes_R N, P) \\ &\cong \mathrm{Hom}_R((\mathrm{colim}_{i \in I} M(i) \otimes_R N), P) \\ &\cong \lim_{i \in I} \mathrm{Hom}_R(M(i) \otimes_R N, P) \end{aligned}$$

where we have used that the limit in ${\mathrm{Mod}_R}^{\mathrm{op}}$ is a colimit in Mod_R. Since Mod_R is presentable (see section A.5), [Lur09, Corollary 5.5.2.2] (a functor from a presentable ∞-category is representable if and only if it preserves small limits) implies that F is representable by an object of C. We denote this object by $\underline{\mathrm{Hom}}_R(N, P)$.

Recall that the Yoneda lemma now states that

$$\mathrm{Hom}_{\mathrm{Fun}({\mathrm{Mod}_R}^{\mathrm{op}}, \mathcal{S})}(\mathrm{Hom}_R(-, \underline{\mathrm{Hom}}_R(N, P)), X) \xrightarrow{\sim} X(\underline{\mathrm{Hom}}_R(N, P)) \tag{5}$$

is an isomorphism for any $X\colon \mathrm{Mod}_R{}^{\mathrm{op}} \to \mathcal{S}$. To see functoriality, let $f_{N'}\colon N' \to N$ and $f_P\colon P \to P'$ be two morphisms and F, F' be the functors as above corresponding to (N, P) and (N', P'). We can define a natural transformation $\eta\colon F \to F'$ by considering $\eta_M\colon F(M) \to F(M')$ given as

$$\mathrm{Hom}_R(M \otimes_R N, P) \to \mathrm{Hom}_R(M \otimes_R N', P'), \qquad g \mapsto f_P \circ g \circ (\mathrm{id}_M \otimes f_{N'}).$$

To be more precise, we can also construct the above natural transformation by using the Yoneda lemma (5) few times:

(1) We can obtain an evaluation map $\mathrm{ev}\colon \underline{\mathrm{Hom}}_R(N, P) \otimes_N N \to P$ by applying the Yoneda lemma for F to F itself:

$$\varphi\colon\ \mathrm{Nat}(F, F) \xrightarrow{\sim} \mathrm{Hom}_R(\underline{\mathrm{Hom}}_R(N, P) \otimes_R N, P).$$

Taking the image of id_F yields a map corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}_R(N,P)}$, which justifies calling $\varphi(\mathrm{id}_F)$ the evaluation map ev.

(2) To get a natural transformation $F \to F'$, the Yoneda lemma tells us that

$$\mathrm{Nat}(F, F') \sim \to \mathrm{Hom}_R(\underline{\mathrm{Hom}}_R(N, P) \otimes_R N', P')$$

and it suffices to provide a map $\underline{\mathrm{Hom}}_R(N, P) \otimes_R N' \to P'$ of R-modules. This is the composition

$$\underline{\mathrm{Hom}}_R(N, P) \otimes_R N' \xrightarrow{1 \otimes f_{N'}} \underline{\mathrm{Hom}}_R(N, P) \otimes_R N \xrightarrow{\mathrm{ev}} P \xrightarrow{f_P} P'.$$

Thus we have defined a map $\eta\colon F \to F'$, which corresponds to a morphism $\mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(N', P')$. This shows functoriality of $\underline{\mathrm{Hom}}_R(-, -)$. Moreover, the fact that F is representable gives us a natural isomorphism

$$\mathrm{Hom}_R(- \otimes_R N, P) \cong \mathrm{Hom}_R(-, \underline{\mathrm{Hom}}_R(N, P))$$

showing that the *functor* $\underline{\mathrm{Hom}}_R(N, -)$ is right adjoint to $- \otimes_R N$ (see definition A.21). $\square$

3.6 Localization of ring spectra

There is an analog to the localization of rings for ring spectra. We follow [Gep20], which itself follows [Lur17]. To define localizations, we can state the familiar universal property.

Definition 3.31. Let A be an $\mathbb{E}_\infty$-ring and $S \subseteq \pi_*(A)$ a multiplicatively closed subset of homogenous elements of the graded commutative ring $\pi_*(A)$. A map $\eta\colon A \to A'$ exhibits A' as a *localization of* A at S if for every $B \in \mathrm{CAlg}$, the map

$$\eta^*\colon\ \mathrm{Map}_{\mathrm{CAlg}}(A', B) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$$

is a fully faithful map of ∞-groupoids (viewed as ∞-categories or Kan complexes) with essential image consisting of those ring maps $f\colon A \to B$ which invert S at the level of homotopy groups (i.e. $f_*(s)$ is invertible for all $s \in S$, where $f_*\colon \pi_*(A) \to \pi_*(B)$).

Fix some $\mathbb{E}_\infty$-ring R. The most important example of such a localization for us is the case where $S = \{1, x, x^2, \ldots\}$ for some element $x \in \pi_d(R)$, which we will study for the remainder of this section. Note that x is represented by a map $\mathbb{S}^d \to R$, so we can form a "multiplication by x" map $M[d] \to M$ by taking the composition

$$M[d] \cong \mathbb{S}^d \otimes M \xrightarrow{x \otimes 1} R \otimes M \xrightarrow{\mu} M.$$

Definition 3.32. Let $S \subseteq \pi_*(R)$ be a multiplicatively closed subset. An R-module M is *S-local* if any of the following equivalent properties are satisfied [Lur17, Proposition 7.2.3.14]:

(1) For every element $s \in S$: multiplication by s induces an isomorphism $m_s \colon \pi_* M \to \pi_* M$ of graded abelian groups.

(2) For every element $s \in S$ and integer $n \in \mathbb{Z}$: the hom-spectrum $\mathrm{Hom}(R/Rs[n], M)$ is contractible.

Lemma 3.33. *Let $S = \{1, x, x^2, \ldots\}$ where $x \in \pi_d(R)$, M be an R-module and set $M' := \mathrm{colim}_{n \geq 0} M[-dn]$ with maps given by multiplication $m_x \colon M[d] \to M$. Then M' is S-local.*

Proof. We can define multiplication by x on M' by the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\cdot x} & M[-d] & \xrightarrow{\cdot x} & M[-2d] & \xrightarrow{\cdot x} & \cdots \\ \downarrow{\scriptstyle \cdot x} & \nearrow & \downarrow{\scriptstyle \cdot x} & \nearrow & \downarrow{\scriptstyle \cdot x} & \nearrow & \\ M[-d] & \xrightarrow{\cdot x} & M[-2d] & \xrightarrow{\cdot x} & M[-3d] & \xrightarrow{\cdot x} & \cdots \end{array}$$

giving a map $m'_x \colon M' \to M'$. By taking the diagonal maps to be the identities, we see that this is invertible and therefore induces an isomorphism $\pi_* m'_x \colon \pi_* M' \to \pi_* M'$. □

Lemma 3.34. *Let $S = \{1, x, x^2, \ldots\}$ where $x \in \pi_d(R)$, and let $M' = \mathrm{colim}_{n \geq 0} M[-dn]$. Then, for every S-local R-module N, postcomposition by the canonical map $\eta \colon M \to M'$ induces an isomorphism $\mathrm{Map}_R(M', N) \cong \mathrm{Map}_R(M, N)$.*

Proof. Under the isomorphism

$$\mathrm{Hom}_R(M', N) \cong \lim_n \mathrm{Hom}_R(M[-dn], N),$$

the map $\eta^* \colon \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$ is given by the morphism from the limit of the diagram

$$\begin{array}{ccccccc} & & & & \lim_n \mathrm{Hom}_R(M[-dn], N) & & \\ & \swarrow{\scriptstyle \eta^*} & & \swarrow & \downarrow & \searrow & \\ \mathrm{Hom}_R(M, N) & \xleftarrow[x^*]{} & \mathrm{Hom}_R(M[-d], N) & \xleftarrow[x^*]{} & \mathrm{Hom}_R(M[-2d], N) & \xleftarrow[x^*]{} & \cdots \end{array} \tag{6}$$

to the leftmost term. Now consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_R(M[-d], N) & \xrightarrow{x^*} & \mathrm{Hom}_R(M, N) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_R(M, N[d]) & \longrightarrow & \mathrm{Hom}_R(M, N) \end{array}$$

where the bottom map is given by *postcomposition* with the map $N[d] \to N$ given by multiplication by x (this is okay, since N is an R-module). This is an isomorphism of spectra (i.e. a stable homotopy equivalence) since N is S-local, so x^* is an isomorphism. Therefore, the entire limit in the diagram (6) collapses, and η^* is an isomorphism. □

Combining lemma 3.33 with lemma 3.34 shows that the colimit $M' = \mathrm{colim}_{n\geq 0} M[-dn]$ is in fact an explicit model for the localization $S^{-1}M$, where $S = \{1, x, x^2, \dots\}$ with $x \in \pi_d(R)$.

Corollary 3.35. *In the above setting,* $S^{-1}M \cong \mathrm{colim}_{n\geq 0} M[-dn]$.

When considering the case $M = R$, we will want to have a ring structure on $S^{-1}R$. For this, consider the *endomorphism ring* $\mathrm{End}_R(S^{-1}R, S^{-1}R)$, which is equivalent to $\mathrm{Hom}_R(R, S^{-1}R) \cong S^{-1}R$ by lemma 3.33 and lemma 3.34. We can therefore put the endomorphism ring structure on $S^{-1}R$ (see [Lur17, Remark 7.2.3.26]). Finally, [Lur17, Proposition 7.2.3.27] assures that the ring map $R \to S^{-1}R$ satisfies the condition of definition 3.31. Therefore, we are now able to speak of the ring $R[x^{-1}]$ for $x \in \pi_d(R)$.

3.7 Infinite loop spaces and the recognition principle

In this section, we describe a tool to generate examples of *connective* spectra. It turns out that these correspond to pointed spaces with the structure of a group *up to coherent homotopy*, so called E_∞-spaces, or *infinite loop spaces*.

Definition 3.36. A spectrum $X \in \mathrm{Sp}$ is *connective* if $\pi_k(X) = 0$ for $k < 0$.

Definition 3.37 (Commutative monoid object)**.** A *commutative monoid object* of an ∞-category C with finite products is a commutative algebra object of $C^\times$ which is C together with the cartesian symmetric monoidal structure given by the product. If $C = \mathcal{S}$, one can call these objects $\mathbb{E}_\infty$-spaces. We denote the category of all such objects by $\mathrm{CMon}(C)$.

Definition 3.38 (Commutative group object)**.** If $M\colon N(\mathrm{Fin}_*) \to C^\times$ is a commutative monoid, we say it is *grouplike* or a *commutative group object*, if any of the following conditions are satisfied:

(1) The commutative monoid structure on the image of M in $\mathrm{Ho}(C)$ is a group.

(2) There is an inversion map $i\colon M \to M$ such that

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{id}\times i} M \times M \xrightarrow{\mu} M$$

is homotopic to the identity.

(3) The *shear map* $(p_1, \mu)\colon M \times M \to M \times M$ is an isomorphism in C.

A grouplike $\mathbb{E}_\infty$-space is called an *infinite loop space* and denote the category of all such objects with morphisms which preserve their monoidal structure (i.e. morphisms under $N(\mathrm{Fin}_*)$) by $\mathrm{CGroup}(\mathcal{S})$.

We can produce these $\mathbb{E}_\infty$-spaces from spectra by applying the functor Ω^∞. In fact, we have the following theorem, called the *May recognition theorem*.

Theorem 3.39. *The functor $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$ has a unique lift to a functor $\mathbf{\Omega}^\infty\colon \mathrm{Sp} \to \mathrm{CGroup}(\mathcal{S})$. Furthermore, there is a fully faithful left adjoint $\mathbf{B}^\infty\colon \mathrm{CGroup}(\mathcal{S}) \hookrightarrow \mathrm{Sp}$ whose image is $\mathrm{Sp}^{\geq 0}$, the full subcategory of connective spectra.*

At the level of Ω-spectra, $\mathbf{B}^\infty G$ is given by spaces $\mathbf{B}^n G$ and structure maps $\mathbf{B}^n G \to \Omega\mathbf{B}^{n+1}G$ (the units of adjunction). This also gives an abstract way to define the Eilenberg-MacLane spectra $HA := \mathbf{B}^\infty A$ by choosing the discrete topology on an abelian group A.

Remark 3.40. We can also similarly define (not-necessarily-commutative) monoid and group objects in $\mathcal{S}$. There is an adjunction $\mathbf{B}\colon \mathrm{Mon}(\mathcal{S}) \rightleftarrows \mathcal{S}_* : \mathbf{\Omega}$. Moreover, this restricts to an equivalence of ∞-categories $\mathrm{Grp}(\mathcal{S}) \xrightarrow{\sim} \mathcal{S}_*^{\geq 1}$ [Lur17, Theorem 5.2.6.10]. If we begin with a monoid M, $\mathbf{B}M$ is connected, so $\mathbf{\Omega B}M$ is a group object of $\mathcal{S}$. In particular, we can also apply $\mathrm{CMon}(-)$ to the adjunction and obtain a *commutative group completion* functor $(-)^{\mathrm{grp}} := \mathbf{\Omega B}\colon \mathrm{CMon}(\mathcal{S}) \to \mathrm{CGroup}(\mathcal{S})$.

3.8 Dualizable and compact objects

Definition 3.41. A symmetric monoidal ∞-category $C^\otimes$ is *closed* if for any object A of C, the functor $-\otimes A\colon C \to C$ admits a right adjoint $\underline{\mathrm{Hom}}(A, -)$. Whenever it exists, this adjunction is called the tensor-hom adjunction.

In a closed symmetric monoidal ∞-category, the counit of the tensor-hom adjunction gives an evaluation map $\varepsilon\colon \underline{\mathrm{Hom}}(A, \mathbf{1}) \otimes A \to \mathbf{1}$. We define $DA := \mathrm{Map}(A, \mathbf{1})$.

Definition 3.42. An object A in a closed symmetric monoidal ∞-category C is *dualizable* if there is a coevaluation map $\eta\colon \mathbf{1} \to DA \otimes A$ such that the compositions

$$A \cong A \otimes \mathbf{1} \stackrel{1\otimes\eta}{\to} A \otimes DA \otimes A \stackrel{\varepsilon\otimes 1}{\to} \mathbf{1} \otimes A \cong A, \quad DA \cong \mathbf{1} \otimes DA \stackrel{\eta\otimes 1}{\to} DA \otimes A \otimes DA \stackrel{1\otimes\varepsilon}{\to} A \otimes \mathbf{1} \cong A$$

are homotopic to the identity, that is, they are equal to the identity after passing to the homotopy category. This is [Gep20, Definition 4.5.5].

4 Finite covers and Galois extensions

We are now ready to define in higher algebra what we defined in ordinary algebra:

(1) A notion of "finite étale" R-algebra where R is an $\mathbb{E}_\infty$-ring. This allows us to define a Galois category and hence a profinite fundamental groupoid.

(2) A notion of G-Galois extension of an $\mathbb{E}_\infty$-ring R.

These two definitions will behave much as they did in the case of ordinary algebra, and none of the results should come as a surprise given our treatment of the 1-categorical case.

4.1 Galois categories for ring spectra

Mathew defines analogs of finite étale covers in any presentable, symmetric monoidal stable ∞-category (which he calls *stable homotopy theories*) where the tensor product commutes with all colimits (see [Mat16, Definition 2.1]). In this exposition, we are interested in the case where our category is $C = \mathrm{Mod}_R$, the category of modules over an $\mathbb{E}_\infty$-ring spectrum R. Note that in this case, $\mathrm{CAlg}(C) = \mathrm{CAlg}_{R/}$ by [Lur17, Corollary 3.4.1.7].

Definition 4.1 (Weak finite cover)**.** Let an $\mathbb{E}_\infty$-ring R be given. In Mod_R, an object A of $\mathrm{CAlg}(\mathrm{Mod}_R) \simeq \mathrm{CAlg}_{R/}$ is called a *weak finite cover* if there is another object A' such that

(1) $- \otimes A'$ commutes with all limits,

(2) $- \otimes A'$ is conservative, and

(3) $A \otimes A' \cong \prod_{i=1}^{n} A'[e_i^{-1}]$ in the category $\mathrm{CAlg}(\mathrm{Mod}_C(A'))$ where each e_i is an idempotent in A' (i.e. an idempotent element of $\pi_0(A')$).

We denote the category of weak finite covers of R by $\mathrm{CAlg}_R^{\mathrm{w.cov}}$.

Remark 4.2. The above definition is exactly the characterization of finite étale algebras given in theorem 1.17. Therefore, it is a reasonable candidate for "finite étale" in the higher algebraic setting. However, this author knows of no deep reason why there should not be a different candidate for generalization. That this definition is reasonable will follow from theorem 4.25, which states that the G-torsors in the opposite category of weak finite covers are precisely the *faithful* G-Galois extensions in the sense of [Rog08].

Mathew also introduces the notion of *finite covers*, but they coincide for the case of Mod_R by [Mat16, Theorem 6.5]. It requires more higher algebraic terminology to define these finite covers, so we omit their definition. However, for the case we consider, which is (weak) finite covers of an $\mathbb{E}_\infty$-ring R, the two notions coincide by [Mat16, Theorem 6.5] since R is a compact object of Mod_R.

Theorem 4.3. $(\mathrm{CAlg}_R^{\mathrm{w.cov}})^{\mathrm{op}}$ *is a Galois category.*

Proof. We show that $C := (\mathrm{CAlg}_R)^{\mathrm{op}}$ together with the *opposite* of the collection $\mathcal{E}$ of maps $A \to B$ such that $- \otimes_A B \colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ commutes with limits and is conservative is a Galois context. We verify the axioms of definition 2.20 one-by-one.

(1) To see that C has finite limits, it suffices by the dual of [Lur09, Corollary 4.4.2.5] to show that C admits pullbacks and a terminal object. The terminal object is the zero ring, and the pullbacks are given by the tensor product $S \otimes_R T$ (the pullback in C is the pushout in CAlg_R). Finally, any morphism $0 \to S$ of R-algebras is an isomorphism, since such a morphism shows that $R \to S$ is the zero map.

(2) Coproducts are disjoint in CAlg_R, and distributive since if $S \to T$ is a map of R-algebras, then $C_{/T} \to C_{/S}$ is equivalently a functor $\mathrm{CAlg}_S \to \mathrm{CAlg}_T$ given by $- \otimes_S T$ and this preserves finite coproducts since it is a left adjoint.

(3) The class $\mathcal{E}$ is closed under composition and base change by part of the argument in (1). If $f \colon A \to B$ is an isomorphism in C^{op}, then the induced $- \otimes_A B \colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ is an equivalence and is therefore in $\mathcal{E}$.

(4) By the Barr-Beck-Lurie criterion of remark 2.21, it follows immediately that for $f \colon A \to B$ in $\mathcal{E}$, the functor $- \otimes_A B$ is comonadic, so f^{op} is of effective descent in C.

(5) We have to show that given a pushout diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ T & \longrightarrow & S' \otimes_S T \end{array}$$

if $S \to S'$ is in $\mathcal{E}$, then the right vertical map is in $\mathcal{E}$ if and only if the left vertical map is in $\mathcal{E}$. We have already seen in (1) that if the left vertical map is in $\mathcal{E}$, then so is the right vertical map. Let us assume that $S' \to S' \otimes_S T$ is in $\mathcal{E}$. First, we show that $- \otimes_S T$ is conservative. It suffices by remark 4.18 to show that if $M \otimes_S T \cong 0$, then so is $M \cong 0$. We have

$$0 \cong (M \otimes_S T) \otimes_S S' \cong (M \otimes_S S') \otimes_{S'} (S' \otimes_S T) \tag{7}$$

by properties of extension of scalars (see [Lur17, Section 4.5.3] for the higher algebraic treatment of this functor). Since $S' \to S' \otimes_S T$ belongs to $\mathcal{E}$, it is conservative and $M \otimes_S S'$ must have been zero. Finally, $- \otimes_S S'$ is also conservative, so X must have been zero.

To check that $- \otimes_S T$ commutes with limits, we need to show

$$T \otimes_S (\lim_{i \in I} M_i) \cong \lim_{i \in I} (T \otimes_S M_i). \tag{8}$$

We can check this equation after applying the conservative functor $- \otimes_S S'$:

$$S' \otimes_S T \otimes_S (\lim_{i\in I} M_i) \to S' \otimes_S (\lim_{i\in I} T \otimes_S M_i).$$

Applying the extension of scalars isomorphism (7) again, and using that $- \otimes_S S'$ commutes with limits, this is the same as checking that

$$(S' \otimes_S T) \otimes_{S'} (\lim_{i\in I} M_i \otimes_S S') \to \lim_{i\in I}(S' \otimes_S T) \otimes_{S'} (M_i \otimes_S S').$$

is an isomorphism. This holds because $- \otimes_{S'} (S' \otimes_S T)$ commutes with limits by assumption.

(6) The property that $S \oplus T \to U$ is in $\mathcal{E}$ if and only if $S \to S \otimes_{S\oplus T} U$ and $T \to T \otimes_{S\oplus T} U$ are holds by considering

$$X \otimes_{S\otimes T} U \cong X \otimes_{S\otimes T}((S\otimes_{S\oplus T} U)\oplus(T\otimes_{S\oplus T} U)) \cong (X\otimes_S(S\otimes_{S\oplus T} U))\oplus(X\otimes_T(T\otimes_{S\oplus T} U)).$$

(7) Finite folds $S \to S^n$ are naturally in $\mathcal{E}$.

To finish the proof, note that the full subcategory of Galoisable objects in $(C, \mathcal{E}^{\mathrm{op}})$ is precisely $(\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C}))^{\mathrm{op}}$. This follows essentially by the definition of Galoisable objects. ☐

Proposition 4.4. *If R is an $\mathbb{E}_\infty$-ring, there is a fully faithful inclusion*

$$(\mathrm{Cov}_{\mathrm{Spec}\,\pi_0 R})^{\mathrm{op}} \to \mathrm{CAlg}_R^{\mathrm{w.cov}}$$

given by lifting *a finite étale algebra $\pi_0 R \to S_0$ to a finite cover S of R.*

Proof idea. A result of Lurie [Mat16, Theorem 2.32] states that the ∞-category consisting of those R-algebras S which satisfy[12]:

(1) $\pi_0 R \to \pi_0 S$ is a finite étale map, and

(2) $\pi_n R \otimes_{\pi_0 R} \pi_0 S \to \pi_n R$ is an isomorphism for all $n \in \mathbb{Z}$.

is equivalent under π_0 to the category of finite étale π_0-algebras. In particular, we are able to lift finite étale $\pi_0 R$-algebras to the $\mathbb{E}_\infty$-world. We need to show that any such R-algebra is in fact a weak finite cover. Let $R_0 = \pi_0$ and S_0 be a finite étale R_0-algebra. Then by theorem 1.17, there is S_0' such that

(1) S_0' is a faithfully flat R_0-module,

(2) $S_0 \otimes_{R_0} S_0' \cong \prod_{i=1}^n S_0'[e_i^{-1}]$ for idempotents $e_i \in S_0'$.

[12] such maps may be called *Lurie-étale* [BGH22, Definition 6.8.3]

The proof now proceeds by observing that lifting S_0, S_0' to $\mathbb{E}_\infty$-rings S, S' under the aforementioned equivalence of categories yields that S' is an R-algebra witnessing the fact that S is a weak finite cover of R [Mat16, Proposition 6.10]. □

Remark 4.5. Using some of the results we will show in section 4.3, one can show that any weak finite cover of an $\mathbb{E}_\infty$-ring R is dualizable [NP24, Lemma 8.4]. Therefore, Mathew's Galois theory actually only depends on the *perfect* R-algebras (see [Gep20] for a definition of perfect modules; and note that these coincide with the dualizable modules).

Remark 4.6. Proposition 4.4 suggests a strong relationship between finite étale $\pi_0 R$-algebras and weak finite covers. In fact, Mathew shows that for two classes base $\mathbb{E}_\infty$-rings R, the two categories do coincide. These are:

(1) R a connective $\mathbb{E}_\infty$-ring, and

(2) R even-periodic with $\pi_0 R$ regular Noetherian.

If the category of weak finite covers coincides with the category of finite étale $\pi_0 R$-algebras, we say that the Galois theory of R is *algebraic* [Mat16, Definition 6.11]. The example of a Galois extension in section 5 will be an example of a *non-algebraic* extension.

4.2 G-Galois extensions

To emulate the definition of G-Galois extensions of rings (definition 2.34) we use the notion of orbits and fixed points in the homotopy coherent setting, introduced in definition 3.11.

Definition 4.7. Let G be a finite group. Then a map of $\mathbb{E}_\infty$-ring spectra $A \to B$, where B is endowed with an action of G via A-algebra homomorphisms (i.e. we have the data of a functor of ∞-categories $F\colon BG \to \mathrm{CAlg}_A$) is called a *$G$-Galois extension* if

(1) the canonical map $A \to B^{hG}$ is an isomorphism,

(2) the map $B \otimes_A B \to \prod_G B$ given on A-algebras by the collection of maps

$$\{B \otimes_A B \xrightarrow{\mathrm{Id}_B \otimes F(g)} B \otimes_A B \xrightarrow{\mu} B\}_{g \in G}$$

and viewed as a G-equivariant map with G acting on the right B in $B \otimes_A B$ and by permutation on $\prod_G B$ is an isomorphism in $(\mathrm{CAlg}_A)^{BG}$.

Unsurprisingly, the first examples of G-Galois extensions come from G-Galois extensions of ordinary rings [Rog08, Proposition 4.2.1].

Proposition 4.8. *Let G be a finite group. If $R \to S$ is a G-Galois extension of rings, then $HR \to HS$ is a G-Galois extension of spectra.*

Remark 4.9. For any B-module M, there is a natural map $j_M\colon M \otimes \mathbb{S}[G] \to \underline{\mathrm{Hom}}_A(B, M)$ defined as the adjoint of the map

$$(M \otimes \mathbb{S}[G]) \otimes_A B \cong \mathbb{S}[G] \otimes (M \otimes_A B) \cong (\mathbb{S}[G] \otimes B) \otimes_A M \xrightarrow{\mu_{\mathbb{S}[G]} \otimes M} B \otimes M \xrightarrow{\mu_M} M$$

where we use for the first isomorphisms that M and B are both B-modules as well as A-modules and apply lemma 3.29 twice. The map $\mu_{\mathbb{S}[G]}\colon \mathbb{S}[G] \otimes B \to B$ comes from the $\mathbb{S}[G]$-algebra structure of $B \in \mathrm{CAlg}^{BG}$, see example 3.22.

One of the central results about G-Galois extensions is that they are dualizable. This is a finiteness property which is quite useful in higher algebra[13]. The remainder of this section is devoted to proving that for a finite group G, G-Galois extensions are dualizable objects.

Construction 4.10 (Assembly map)**.** Let C, D be stable ∞-categories and $F\colon C \to D$ an exact functor. Recall from construction 3.9 that exact functors give us maps between hom-spectra

$$\underline{\mathrm{Hom}}_C(x, y) \to \underline{\mathrm{Hom}}_D(Fx, Fy).$$

By the tensor-hom adjunction, we get a map $x \to \underline{\mathrm{Hom}}_C(y, x \otimes y)$ for any y, and applying F yields

$$x \to \underline{\mathrm{Hom}}(y, x \otimes y) \to \underline{\mathrm{Hom}}(Fy, F(x \otimes y)) \quad \rightsquigarrow \quad x \otimes F(y) \to F(x \otimes y).$$

We call the map $x \otimes F(y) \to F(x \otimes y)$ obtained in this way an *assembly map.*

Remark 4.11. Recall from proposition 3.4 that when working between stable ∞-categories, a functor is exact if and only if it is left exact (i.e. it preserves finite limits) if and only if it is right exact (i.e. it preserves finite colimits). Hence, any functor which is either a right or left adjoint is exact. In particular, all of the functors $(-)_{hG}$, $(-)^{hG}$, $\mathrm{Hom}_R(M, -)$, $- \otimes_R M$, as well as their various compositions are exact.

Remark 4.12. For $A \to B$ a G-Galois extension, there is a variant of the map $h\colon B \otimes B \to \prod_G B$ which works for A-modules M. This is a map

$$h_M\colon M \otimes_A B \to \underline{\mathrm{Hom}}(\mathbb{S}[G], M)$$

which is equivariant with respect to the action of G on B on the left and the permutation action of G on $\underline{\mathrm{Hom}}(\mathbb{S}[G], M) \cong \prod_G M$ on the right (we have discussed this isomorphism in example 3.13). Therefore, h_M is a map in Mod_A^{BG}. We construct this map by taking the following composition:

$$(M \otimes_B B) \otimes_A B \cong M \otimes_B (B \otimes_A B) \xrightarrow{h} M \otimes_B \prod_G B \to \prod_G M.$$

This is an isomorphism since all maps above are isomorphisms, or alternatively by [Rog08, Lemma 6.1.2].

[13] in ordinary algebra, the dualizable objects in Mod_A are precisely the finitely generated and projective modules

Remark 4.13. Recall from theorem 3.30 that the ∞-category Mod_R of modules over an $\mathbb{E}_\infty$-ring R is not only spectrally enriched, but there is a natural R-linear structure on the hom-spectra. Similarly, the category Mod_R^{BG} also admits R-linear hom-spectra by using the forgetful functor $\mathrm{Fun}(BG, \mathrm{Mod}_R) \to \mathrm{Mod}_R$.

In the upcoming proof of dualizability (theorem 4.16), we also need certain functors to preserve the module structure on spectra. Since we constructed R-linear spectra hom-spectra in theorem 3.30 using the Yoneda lemma and representability of $\mathrm{Hom}_R(- \otimes_R X, Y)$, to show that a functor $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves the R-linear structure, we have to provide a natural transformation

$$\mathrm{Hom}_R(- \otimes_R X, Y) \to \mathrm{Hom}_R(- \otimes_R F(X), F(Y))$$

and similarly for functor $F\colon \mathrm{Mod}_R^{BG} \to \mathrm{Mod}_R$.

Next, we provide an alternative characterization of dualizability. Construct a map $\nu\colon DB \otimes_A M \to \mathrm{Hom}(B, M)$ as follows. Let $\varepsilon_X\colon B \otimes_A \mathrm{Hom}_A(B, X) \to X$ and $\eta_X\colon X \to \mathrm{Hom}_A(B, B \otimes_A X)$ be the counit and unit of adjunction. Then $\nu\colon DB \otimes_A M \to \mathrm{Hom}(B, M)$ is given by the composition

$$DB \otimes_A M \xrightarrow{\eta_{DB \otimes_A M}} \mathrm{Hom}(B, B \otimes_A DB \otimes_A M) \xrightarrow{(\varepsilon_{DB} \otimes 1)_*} \mathrm{Hom}(B, A \otimes_A M).$$

This map allows us to state an equivalent characterization of dualizability (see definition 3.42).

Lemma 4.14. *Let X be an object in some closed symmetric monoidal ∞-category. If $\nu : DX \otimes X \to \underline{\mathrm{Hom}}(X, X)$ is an isomorphism, where ν is the adjoint of the map $\varepsilon \otimes 1\colon X \otimes DX \otimes X \to X$, then X is dualizable in the usual sense.*

Proof. Given the isomorphism $\nu\colon DX \otimes X \to \underline{\mathrm{Hom}}(X, X)$, we define a coevaluation $\eta\colon \mathbf{1} \to DX \otimes X$ by taking the composition

$$\mathbf{1} \xrightarrow{\mathrm{id}_X} \underline{\mathrm{Hom}}(X, X) \xrightarrow{\cong} DX \otimes X.$$

Then, the composite

$$X \cong X \otimes \mathbf{1} \xrightarrow{1 \otimes \eta} X \otimes DX \otimes X \xrightarrow{\varepsilon \otimes 1} \mathbf{1} \otimes X \cong X$$

factors as

$$X \otimes \mathbf{1} \xrightarrow{1 \otimes \mathrm{id}_X} X \otimes \underline{\mathrm{Hom}}(X, X) \xrightarrow{\varepsilon} \mathbf{1} \otimes X$$

which is isomorphic to the identity on X. The other triangle identity holds similarly. ☐

Lemma 4.15. *If $A \to B$ is an A-algebra and M a B-module, there is a natural map $M \otimes_A B^{hG} \to (M \otimes_A B)^{hG}$. Moreover, if $A \to B$ is a G-Galois extension, then this map is an isomorphism.*

Proof. Let $F\colon \mathrm{Mod}_A^{BG} \to \mathrm{Mod}_A$ be the functor $F := (-)^{hG}$. This is exact by remark 4.11, so for any two A-modules B, M, we get an assembly map

$$\nu'\colon M \otimes_A B^{hG} \to (M \otimes_A B)^{hG}.$$

Now assume that $A \to B$ is actually an A-algebra and a G-Galois extension. There is an isomorphism $\varphi\colon M \to \underline{\mathrm{Hom}}(\mathbb{S}[G], M)^{hG}$. We construct φ by letting each $g \in G$ act on M via the identity, giving us a map $\mathbb{S}[G] \to \underline{\mathrm{Hom}}(M, M)$, and by symmetry a map $M \to \underline{\mathrm{Hom}}(\mathbb{S}[G], M)$. This map can be extended to the homotopy fixpoints since it is invariant under the action of G by definition.

Recall from remark 4.12 that there is a G-equivariant isomorphism $h_M\colon M \otimes_A B \to \mathrm{Hom}(\mathbb{S}[G], M)$, and applying $(-)^{hG}$ yields an isomorphism

$$h_M^{hG}\colon (M \otimes_A B)^{hG} \to \underline{\mathrm{Hom}}(\mathbb{S}[G], M)^{hG}.$$

Then we can factor the isomorphism $\varphi\colon M \to \underline{\mathrm{Hom}}(\mathbb{S}[G], M)^{hG}$ by using this map:

$$M \xrightarrow{\sim} M \otimes_A B^{hG} \xrightarrow{\nu'} (M \otimes_A B)^{hG} \xrightarrow{h_M^{hG}} \underline{\mathrm{Hom}}(\mathbb{S}[G], M)^{hG}$$

showing that ν' is an isomorphism. That this factoring is valid follows from the structure of the map $h\colon B \otimes_A B \to \underline{\mathrm{Hom}}(\mathbb{S}[G], B)$ used to define h_M; this sends the identity of the ring $B \otimes_A B$ to the map $\mathbb{S}[G] \to B$ sending each $g \in G$ to the identity of B. □

Theorem 4.16. *Let G be a finite group. If $A \to B$ is a G-Galois extension, then B is a dualizable A-module.*

Proof. We build a diagram as in [Rog08, Proposition 6.2.1] out of horizontal maps which are assembly maps and the vertical maps come from natural transformations. By using this method, commutativity of every diagram is automatically assured. We work with the category of A-modules Mod_A, as well as with Mod_A^{BG}, the category of A-modules with an A-linear action of G. Each functor below has the signature $F_i\colon \mathrm{Mod}_A^{BG} \to \mathrm{Mod}_A$.

(1) Let $F_1(X) := \underline{\mathrm{Hom}}_A(B, X^{hG})$ be a functor $\mathrm{Mod}_A^{BG} \to \mathrm{Mod}_A$. To show that F_1 preserves the A-linear structure on hom-spectra, we use remark 4.13. By using evaluation and the inclusion $X^{hG} \to X$, we define maps

$$\underline{\mathrm{Hom}}_A(Z \otimes_A X, Y) \to \underline{\mathrm{Hom}}_A(Z \otimes_A \underline{\mathrm{Hom}}_A(B, X^{hG}), \underline{\mathrm{Hom}}_A(B, Y^{hG}))$$

by taking an $f\colon Z \otimes_A X \to Y$ to

$$Z \otimes_A \underline{\mathrm{Hom}}_A(B, X^{hG}) \otimes B \to Y^{hG} \to Z \otimes_A X^{hG} \xrightarrow{f^{hG}} Y^{hG}$$

which is well-defined since f is G-equivariant. By the assembly map construction, we get a map

$$M \otimes_A \underline{\mathrm{Hom}}_A(B, X^{hG}) \to \underline{\mathrm{Hom}}_A(B, (M \otimes_A X)^{hG}).$$

(2) Let $F_2(X) := \underline{\mathrm{Hom}}_A(B, X)^{hG}$. Since this also preserves A-linear structure by the same argument, we get a map

$$M \otimes_A \underline{\mathrm{Hom}}_A(B, X)^{hG} \to \underline{\mathrm{Hom}}_A(B, M \otimes_A X)^{hG}.$$

(3) Let $F_3(X) := (X \otimes \mathbb{S}[G])^{hG}$ where G acts diagonally on $X \otimes \mathbb{S}[G]$ (i.e. simultaneously on both components). This preserves the A-linear structure on hom-spectra, since given a G-equivariant $f\colon Z \otimes_R X \to Y$, we can define a map

$$Z \otimes_R (X \otimes \mathbb{S}[G])^{hG} \cong (Z \otimes_R (X \otimes \mathbb{S}[G]))^{hG} \xrightarrow{f^{hG} \otimes 1} (Y \otimes \mathbb{S}[G])^{hG}$$

which is G-equivariant under the diagonal action of G on the target since f is and the $\mathbb{S}[G]$ component stays unchanged. This shows that F_3 gives a map of A-linear hom-spectra. We get a map

$$M \otimes_A (X \otimes \mathbb{S}[G])^{hG} \to ((M \otimes_A X) \otimes \mathbb{S}[G])^{hG}.$$

(4) Let $F_4(X) := (X \otimes \mathbb{S}[G])_{hG}$ where G acts diagonally on $X \otimes \mathbb{S}[G]$. This preserves A-linear hom-spectra by the same argument as for F_3. We get a map

$$M \otimes_A (X \otimes \mathbb{S}[G])_{hG} \to ((M \otimes_A X) \otimes \mathbb{S}[G])_{hG}.$$

The functors F_1 and F_2 are naturally isomorphic, since the (internal) hom-functor preserves limits in the second variable. There is a natural transformation $F_3 \to F_4$ given by composing the naturality of the j map (see remark 4.9) by the functor $(-)^{hG}$. By [Rog08, Lemma 6.1.2], this is actually a natural isomorphism since $A \to B$ is G-Galois. There is also a natural transformation $F_4 \to F_3$ given by the norm natural isomorphism (see example 3.14). We set $X = B$ such that $X^{hG} \cong A$ and obtain a *commutative* diagram

$$\begin{array}{ccc}
M \otimes_A \mathrm{Hom}_A(B, A) & \xrightarrow{\nu} & \mathrm{Hom}_A(B, (M \otimes_A B)^{hG}) \\
\downarrow \cong & & \downarrow \cong \\
M \otimes_A \mathrm{Hom}_A(B, B)^{hG} & \longrightarrow & \mathrm{Hom}_A(B, M \otimes_A B)^{hG} \\
\uparrow \cong & & \uparrow \cong \\
M \otimes_A (X \otimes \mathbb{S}[G])^{hG} & \longrightarrow & ((M \otimes_A X) \otimes \mathbb{S}[G])^{hG} \\
\uparrow \cong & & \uparrow \cong \\
M \otimes_A (X \otimes \mathbb{S}[G])_{hG} & \longrightarrow & ((M \otimes_A X) \otimes \mathbb{S}[G])_{hG}
\end{array}$$

The bottom arrow is an isomorphism since the left adjoint $M \otimes_A -$ preserves colimits. Also notice that $(M \otimes_A B)^{hG} \cong M \otimes_A B^{hG}$ by lemma 4.15. Thus for $M = B$ the top row becomes $\mathrm{Hom}_A(B, B \otimes_A B^{hG}) = \mathrm{Hom}_A(B, B)$, which is precisely the ν we are after.

Since ν is now given as a zigzag of isomorphisms, it is also an isomorphism, and the proof is finished. □

4.3 Faithful Galois extensions

We define faithfulness in the context of higher algebra as in ordinary module theory.

Definition 4.17. Let R be an $\mathbb{E}_\infty$-ring. An R-module M is *faithful* if $- \otimes_R M \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is a conservative functor, that is, it reflects isomorphisms.

Remark 4.18. Since Mod_R is a stable ∞-category, a map $f \colon X \to Y$ is an isomorphism if and only if its cofiber $\mathrm{cofib}(f)$ is zero. Since $M \otimes_R -$ is a left adjoint, it preserves exact sequences. In particular,

$$M \otimes_R X \to M \otimes_R Y \to M \otimes_R \mathrm{cofib}(f)$$

is exact. Therefore, in order for M to be faithful, it suffices to show that if $M \otimes_R N \cong 0$, then $N \cong 0$.

Lemma 4.19. *A faithful G-Galois extension $A \to B$ is a weak finite cover of A.*

Proof. We will verify the axioms of a weak finite cover (see definition 4.1) for the choice $B' = B$. The assumption that $A \to B$ is faithful is equivalent to asking that the functor $- \otimes_A B$ is conservative, so that requirement is fulfilled. Next, we have that $B \otimes_A B \cong \prod_G B$, which is of the form required in the third axiom of definition 4.1.

Recall that B is a dualizable A-module. Then, its dual DB is also dualizable by the fact that we are in a symmetric monoidal ∞-category. This gives us a sequence of adjunctions $- \otimes_A DB \dashv - \otimes_A B \dashv - \otimes_A DB$. It follows that $- \otimes_A B$ preserves limits, so we are finished (by [Lur17, Corollary 3.2.2.5], limits in $\mathrm{CAlg}_A = \mathrm{CAlg}(\mathrm{Mod}_A)$ can be computed in Mod_A). ☐

As a result going in the opposite direction, we will shows that any weak finite cover is dualizable, using that we know that G-Galois extensions are dualizable. In order to state a very useful criterion to identify comonadic adjunctions, we briefly introduce some terminology regarding cosimplicial objects.

Definition 4.20. A cosimplicial object $X^\bullet$ in an ∞-category C is a functor $\Delta \to C$, where Δ denotes the simplex category. The *totalization of* $X^\bullet$ is $\lim_{[n]\in\Delta} X^n = \lim_{[n]\in\Delta} X([n])$. A cosimplicial object is *split* if we can extend $X^\bullet \colon \Delta \to C$ to a functor $\Delta_{-\infty} \to C$, where $\Delta_{-\infty}$ is defined by [RV22, Definition 2.3.9]:

(1) adding an initial object $[-1]$ to Δ, and

(2) adding extra degeneracy maps $s^{-1} \colon [n+1] \to [n]$ for all $n \geq -1$ with composition defined by the simplicial relations.

Definition 4.21 (Amitsur complex)**.** Let $A \to B$ be an A-algebra, and define $X^n := B^{\otimes_A n}$. This gives us a cosimplicial object $X_A(B)^\bullet$, with the face maps $d^i \colon X^{n-1} \to X^n$ given by maps of the form

$$b_1 \otimes \cdots \otimes b_{n-1} \mapsto b_1 \otimes \cdots \otimes b_i \otimes 1 \otimes b_{i+1} \cdots b_{n-1}$$

and the degeneracy maps s^i by multiplying components. This cosimplicial object can be coaugmented by adding the map $A \to B$. The *completion of B along A* is

$$A^{\wedge}_B := \operatorname{Tot} X_A(B)^\bullet = \lim_{[n]\in\Delta} X_A(B)^n \tag{9}$$

after [Rog08, Definition 8.2.1].

The completion $A^{\wedge}_B$ can be used to provide a criterion for when an algebra $A \to B$ is a G-Galois extension, by relating the completion to the homotopy fixpoints.

Lemma 4.22. *([Rog08, Proposition 8.2.8]) Let G be a finite group and B an A-algebra with an action of G (i.e. a functor $\operatorname{Fun}(BG, \mathrm{CAlg}_A)$). Suppose that $h\colon B \otimes_A B \to \prod_G B$ is an isomorphism. Then there is an isomorphism $h'\colon A^{\wedge}_B \to B^{hG}$.*

We will need to apply the Barr-Beck-Lurie criterion (remark 2.21) to a faithful G-Galois extension $A \to B$ inside the category $C = \mathrm{CAlg}_A^{\mathrm{w.cov}}$ of weak finite covers of A. We therefore need to show that $- \otimes_A B \dashv \underline{\mathrm{Hom}}_A(B, -)$ restricts to an adjunction $C \rightleftarrows C$.

Remark 4.23. Note that by [Lur17, Corollary 3.2.2.5], limits in $\mathrm{CAlg}_A = \mathrm{CAlg}(\mathrm{Mod}_A)$ can be computed in Mod_A, so $- \otimes_A B\colon \mathrm{CAlg}_A \to \mathrm{CAlg}_A$ also preserves all limits and is conservative if B is a dualizable A-module.

Lemma 4.24. *Let $A \to B$ be a G-Galois extension. The tensor hom adjunction $- \otimes_A$: $\mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_A : \underline{\mathrm{Hom}}(A, -)$ on A-algebras restricts to an adjunction $\mathrm{CAlg}_A^{\mathrm{w.cov}} \rightleftarrows \mathrm{CAlg}_A^{\mathrm{w.cov}}$.*

Proof. Given a weak finite cover X, we must verify that $X \otimes_A B$ is a weak finite cover. Take X' such that $- \otimes_A X'$ is conservative, commutes with limits and satisfies $X \otimes_A X' \cong \prod_{i=1}^d X'[e_i^{-1}]$. Then $- \otimes_A (B \otimes_A X')$ still satisfies the first two properties by composition, and we can compute that $X \otimes_A B \otimes_A (B \otimes_A X') \cong \prod_{g\in G} \prod_{i=1}^d (B \otimes X')[e_{g,i}^{-1}]$. To see that $\underline{\mathrm{Hom}}_A(B, -)$ also preserves weak finite covers, it suffices to note that by dualizability, $\underline{\mathrm{Hom}}_A(B, -) \cong DB \otimes_A -$, so we can apply the same argument. □

Theorem 4.25. *The faithful G-Galois extensions of an $\mathbb{E}_\infty$-ring R are precisely the G-torsors in the category $(\mathrm{CAlg}_R^{\mathrm{w.cov}})^{\mathrm{op}}$.*

Proof. Let a faithful G-Galois extension $A \to B$ be given. By lemma 4.19, B lives in $\mathrm{CAlg}_A^{\mathrm{w.cov}}$. We first show that in $(\mathrm{CAlg}_A^{\mathrm{w.cov}})^{\mathrm{op}}$, the map $(A \to B)^{\mathrm{op}}$ is an effective descent morphism. By the Barr-Beck-Lurie criterion (remark 2.21), lemma 4.24 and remark 4.23, it suffices to show that $- \otimes_A B$ is conservative and that it preserves limits in CAlg_A. That this functor is conservative follows from the assumption that B is faithful, and the dualizability of B implies that we have a chain of adjunctions $- \otimes_A DB \vdash - \otimes_A B \vdash - \otimes_A DB$ showing that $- \otimes_A B$ preserves limits. Finally, we have that $B \otimes_A B \cong \prod_G B$ in $(\mathrm{CAlg}_A)^{BG}$. This shows that B is a G-torsor once we pass to the opposite category $(\mathrm{CAlg}_A)^{\mathrm{op}}$. The entire preceding argument is compatible with the action of G, so we are done.

Conversely, let B be a G-torsor in $(\mathrm{CAlg}_A^{\mathrm{w.cov}})^{\mathrm{op}}$. By definition, B is a weak finite cover of A with an action of G such that there is another weak finite cover B' satisfying:

(1) $B' \otimes_A B \cong \prod_G B'$ as objects of $(\mathrm{CAlg}_A^{\mathrm{w.cov}})_{B'/}$ and respecting the action of G,

(2) $(A \to B')^{\mathrm{op}}$ is an effective descent morphism in $(\mathrm{CAlg}_A^{\mathrm{w.cov}})^{\mathrm{op}}$. By the Barr-Beck-Lurie criterion of remark 2.21, this implies in particular that the functor $-\otimes_A B'$ is conservative.

Now apply $-\otimes_A B'$ to the map $h\colon B \otimes_A B \to \prod_G B$ to get

$$B \otimes_A B \otimes_A B' \to \prod_G B \otimes_A B'$$

which becomes an isomorphism $\prod_G \prod_G B' \to \prod_G \prod_G B'$ by the same reasoning as in the ordinary 1-categorical case (proposition 2.36). Since B' is faithful, we see that h is an isomorphism. We are now in a situation to apply lemma 4.22 to see that the completion $A^\wedge_B$ is isomorphic to B^{hG}. We therefore must show that the coaugmentation $A \to A^\wedge_B$ is an isomorphism. For this, note that tensoring the coaugmented Amitsur complex $X^\bullet$ by B' gives

$$B' \longrightarrow \textstyle\prod_G B' \rightrightarrows \prod_G \prod_G B' \,\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, \cdots$$

which we can describe by identifying elements of $\prod_G B'$ with functions $G \to B'$. Then, the cosimplicial structure is given by

$$(d^i\phi)(g_0, \ldots, g_n) := \phi(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) \qquad 0 \le i < n \tag{10}$$

$$(d^n\phi)(g_0, \ldots, g_n) := g_n \cdot \phi(g_0, \ldots, g_{n-1}) \tag{11}$$

$$(s^i\phi)(g_0, \ldots, g_n) := \phi(g_0, \ldots, g_i, 1, g_{i+1}, \ldots, g_n) \qquad 0 \le i \le n \tag{12}$$

where $\phi\colon G \to B'$. The coaugmenting map $d^{-1}\colon B' \to \prod_G B'$ is the diagonal. We let

$$(s^{-1}\phi)(g_0, \ldots, g_n) := \phi(1, g_0, \ldots, g_n)$$

be the "extra" codegeneracy splitting $X^\bullet$. This works since $s^{-1} \circ d^0 = \mathrm{id}$. Knowing that the complex $F(X^\bullet)$ splits, we can apply property (2) from the Barr-Beck-Lurie criterion to see that $A \to A^\wedge_B$ is an isomorphism after tensoring by the faithful algebra B', so it was an isomorphism to begin with. Therefore, $A \to A^\wedge_B \cong B^{hG}$ is an isomorphism. □

As a result, we see that weak finite covers can be trivialized by G-Galois extensions. Since these are rather well-behaved (e.g. they are dualizable), this can be useful tool (this result is used very often in [NP24]).

Corollary 4.26. *Let R be an $\mathbb{E}_\infty$, and S a weak finite cover of R. Then there is a faithful G-Galois extension $R \to T$ such that*

$$T \otimes_R S \cong \prod_{i=1}^k T[e_i^{-1}].$$

Proof. Combine theorem 4.25 with corollary 2.27. □

Remark 4.27. In contrast to Galois extensions of ordinary commutative rings, Galois extensions of $\mathbb{E}_\infty$-rings are not automatically faithful (that is, $- \otimes_A B$ need not be conservative). A consequence is that any Galois theory based on descent (such as the one we are discussing) is unable to see non-faithful Galois extensions, since faithfulness is critical to using base change to prove theorems.

5 K-theory

5.1 Grothendieck group of vector bundles

In this section, we will work over the complex numbers $\mathbb{C}$, but every construction works equally well over the reals $\mathbb{R}$, which we will need later. For now, let X be a paracompact Hausdorff space, such as a manifold. We let $\mathrm{Vect}_k(X)$ denote the isomorphism classes of complex vector bundles of rank $k \geq 0$ over X, and $\mathrm{Vect}(X) = \coprod_{k \geq 0} \mathrm{Vect}_k(X)$.

Definition 5.1 (Grothendieck group)**.** Given a paracompact Hausdorff space X, let $KU(X)$ be the group completion of the commutative monoid $\mathrm{Vect}(X)$, the set of isomorphism classes of complex vector bundles over X.

The first interesting fact is that $\mathrm{Vect}_k(X)$ is classified by Grassmanians, i.e. the map $X \mapsto \mathrm{Vect}_k(X)$ as a functor $\mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$ is representable. Indeed, isomorphism classes of rank k vector bundles are classified by the following space.

Definition 5.2. The *infinite Grassmanian* of k-planes is

$$\mathrm{Gr}_k(\mathbb{C}^\infty) = \{V \subseteq \mathbb{C}^\infty \mid V \text{ a rank } k \text{ linear subspace}\}$$

where $\mathbb{C}^\infty := \mathrm{colim}(\mathbb{C}^0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots)$.

The following is a classical fact [Cno26], [Hat17, Theorem 1.16].

Theorem 5.3. $\mathrm{Vect}_k(X)$ *is isomorphic to* $[X, \mathrm{Gr}_k(\mathbb{C}^\infty)]_u$*, the set of unbased homotopy classes of maps from* X *to* $\mathrm{Gr}_k(\mathbb{C}^\infty)$.

To get another characterization of $\mathrm{Gr}_k(\mathbb{C}^\infty)$, we show that it can be exhibited as a classifying space of the unitary group $U(k)$. Let

$$\begin{aligned} V_k(\mathbb{C}^\infty) &:= \{\text{orthonormal } k\text{-frames in } \mathbb{C}^\infty\} \\ &= \{(v_1, \ldots, v_k) \in (\mathbb{C}^\infty)^k \mid |v_i| = 1, v_i \perp v_j = 0 \text{ when } i \neq j\} \end{aligned}$$

be the *infinite Stiefel manifold of orthonormal k-frames.* There is an obvious map

$$\pi_k \colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$$

given by taking linear spans. Conveniently, π_k is a principal $U(k)$-bundle and the space $V_k(\mathbb{C}^\infty)$ is contractible. This puts us into the situation where π_k is actually the classifying space of $U(k)$, and we can write $\mathrm{Gr}_k(\mathbb{C}^\infty) \cong BU(k)$. Finally, taking the colimit as $k \to \infty$ gives us the space $\mathrm{Gr}(\mathbb{C}^\infty) = BU := \mathrm{colim}_{k\to\infty} BU(k)$ where the maps $BU(n) \to BU(m)$ are given for $m \geq n$ by the inclusion $U(n) \to U(m)$ into the first n coordinates.

Using this, we can show that $KU(X)$ is representable as a functor on nice spaces X. In fact, we have $[X, \mathbb{Z} \times BU]_u \cong KU(X)$ where the $\mathbb{Z}$ component accounts for the fact that if X is not connected, different components may have different ranks. If we take based homotopy classes, we have $[X, \mathbb{Z} \times BU] \cong \widetilde{KU}(X)$, where $\widetilde{KU}$ is the *reduced K-theory group* defined as the kernel of the map $KU(X) \to \mathbb{Z}$ assigning to a virtual vector bundle over X the rank at the basepoint of X. In other words, $\widetilde{KU}$ sees only the rank zero virtual vector bundles.

5.2 The Segal construction

It turns out that complex K-theory can be turned into an extraordinary cohomology theory, and in particular, it can be represented by a spectrum KU. From the spectrum KU, we want for finite CW-complexes X that $KU^0(X) \cong \widetilde{KU}(X)$ i.e. the cohomology theory represented by KU should have as 0-th cohomology group the K-group we defined. We now aim to construct the spectra KU (for complex K-theory) and KO (for real K-theory) which have the full $\mathbb{E}_\infty$-ring structure, that is, the structure of a commutative algebra object in $\mathrm{Sp}^{\otimes}$). For this, we will begin by defining related connective spectra ku and ko using the recognition theorem (theorem 3.39). The following theorem is a very general ∞-categorical version of Segal's original construction.

Theorem 5.4. *([GGN15, Theorem 8.6]) The functor*

$$\mathrm{SymMonCat}_\infty \xrightarrow{(1)} \mathrm{CMon}(\mathcal{S}) \xrightarrow{(2)} \mathrm{CGroup}(\mathcal{S}) \xrightarrow{(3)} \mathrm{Sp}^{\geq 0}$$

defined as the composition of the functors

(1) Given a symmetric monoidal ∞-category C, take its groupoid core $C^{\simeq}$, the subcategory of C consisting of all isomorphisms. This will be an ∞-groupoid, an element of $\mathcal{S}$. Moreover, the symmetric monoidal structure on C gives rise to a monoid structure on $C^{\simeq}$.

(2) Apply the group completion functor $M \mapsto M^{\mathrm{grp}} = \mathbf{\Omega B}M$ from remark 3.40.

(3) Apply $\mathbf{B}^\infty$ to get a connective spectrum from a grouplike $\mathbb{E}_\infty$-space.

Then, this functor is lax symmetric monoidal. In particular, it sends commutative algebra objects in $\mathrm{SymMonCat}_\infty$ to commutative algebra objects in $\mathrm{Sp}^{\geq 0}$, which are connective $\mathbb{E}_\infty$-rings.

Whenever we apply this theorem, we will be starting with an ordinary 1-category which is potentially enriched over topological spaces. Therefore, it will suffice to consider a very strict ring structure on our categories, which is the following [BGH22, Definition 6.5.4].

Definition 5.5 (Bipermutative category)**.** A *bipermutative category* C is an ordinary 1-category with two binary operations $\oplus, \otimes \colon C \times C \to C$, two objects $0, 1$ and natural transformations $\tau_\oplus, \tau_\otimes$ such that for each $\star \in \{\oplus, \otimes\}$ (we write $0_\oplus = 0$ and $0_\otimes = 1$):

(1) $(X \star Y) \star Z = X \star (Y \star Z)$,

(2) $0_\star \star X = X \star 0_\star = X$,

(3) the natural transformation $\tau_\star \colon \star \to \star$ satisfies $\tau_\star^2 = \mathrm{id}_\star$,

(4) the diagrams

$$\begin{array}{ccc} X \star 0_\star & \xrightarrow{\tau_\star} & 0_\star \star X \\ & \searrow\!\!\!= \qquad =\!\!\!\swarrow & \\ & X & \end{array} \qquad \begin{array}{ccc} X \star Y \star Z & \xrightarrow{\tau_\star} & Z \star X \star Y \\ {\scriptstyle \mathrm{id}_X \star \tau_\star} \searrow & & \nearrow {\scriptstyle \tau_\star \star \mathrm{id}_Y} \\ & X \star Z \star Y & \end{array}$$

commute,

(5) $0 \otimes X = X \otimes 0 = 0$,

(6) $(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z)$,

(7) the diagram

$$\begin{array}{ccc} (X \oplus Y) \otimes Z & = & (X \otimes Z) \oplus (Y \otimes Z) \\ {\scriptstyle \tau \otimes \mathrm{id}_Z}\downarrow & & \downarrow{\scriptstyle \tau} \\ (Y \oplus X) \otimes Z & = & (Y \otimes Z) \oplus (X \otimes Z) \end{array}$$

commutes,

(8) for the isomorphism $d \colon X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z)$ defined as the composition

$$X \otimes (Y \oplus Z) \to (Y \oplus Z) \otimes X = (Y \otimes X) \oplus (Z \otimes X) \to (X \otimes Y) \oplus (X \otimes Z),$$

we should have that

$$\begin{array}{ccc} (X \oplus Y) \otimes (Z \oplus W) & \longrightarrow & (X \oplus Y) \otimes Z \oplus (X \oplus Y) \otimes W \\ \| & & \| \\ X \otimes (Z \oplus W) \oplus Y \otimes (Z \oplus W) & & X \otimes Z \oplus Y \otimes Z \oplus X \otimes W \oplus Y \otimes W \\ {\scriptstyle d \oplus d}\downarrow & \nearrow & \\ X \otimes Z \oplus X \otimes W \oplus Y \otimes Z \oplus Y \otimes W & & \end{array}$$

commutes.

Bipermutative categories are a source of commutative algebra objects in $\mathrm{SymMonCat}_\infty$ (we implicitly embed 1-categories into ∞-categories) [GGN15]. In fact, all of our applications of theorem 5.4 will begin with such bipermutative categories.

Example 5.6. Let $\Sigma^{\simeq}$ denote the category with objects $0, 1, 2, \ldots$ and morphisms

$$\Sigma^{\simeq}(n,m) = \begin{cases} \Sigma_n & \text{if } n = m \\ \emptyset & \text{otherwise} \end{cases}.$$

This has an additive structure given by $n \oplus m := n + m$ with permutations acting on the blocks of size n and m. Similarly, there is a multiplicative structure given by $n \otimes m := nm$. By definition, $\oplus, \otimes$ satisfy the strict unitality, associativity and distributivity axioms, making $\Sigma^{\simeq}$ into a bipermutative category. Then, applying the Segal construction yields the sphere spectrum $\mathbb{S}$ [BGH22, Section 6.5]. This is also sometimes called the *Barratt-Priddy-Segal-Quillen* theorem, or a variant thereof.

Theorem 5.7. *Let $\mathcal{V}_{\mathbb{C}}$ denote the category with objects $0, 1, 2, \ldots$ and morphisms*

$$\mathcal{V}_{\mathbb{C}}(n,m) = \begin{cases} U(n) & \textit{if } n = m \\ \emptyset & \textit{otherwise} \end{cases}.$$

This has additive and mulitplicative structure given by $\oplus$ and $\otimes$, as well as the corresponding operations on matrices. Applying the above construction to this category yields a connective ring spectrum ku with $\Omega^\infty ku = BU \times \mathbb{Z}$ [BGH22, Section 6.5].

In particular, we get that

$$\pi_n(ku) = [\mathbb{S}^n, ku] \cong [S^n, \Omega^\infty ku] \cong \pi_n(\mathbb{Z} \times BU) \cong \widetilde{KU}(S^n).$$

Of course, this is not the periodic spectrum KU we really want. We will obtain KU on the algebraic level as $KU \cong ku[u^{-1}]$ where $u \in \pi_2(ku)$ is the *Bott element* [EKMM97, Theorem VIII.4.3]. The approach via the Segal construction has the advantage that maps on the level of vector spaces directly give us maps of ring spectra, e.g. *complexification*

$$c\colon \mathcal{V}_{\mathbb{R}} \to \mathcal{V}_{\mathbb{C}}, \qquad O(n) \hookrightarrow U(n) \tag{13}$$

immediately gives a map $ko \to ku$ of $\mathbb{E}_\infty$-rings.

5.3 Bott periodicity

One of the most important results about K-theory is that the homotopy groups repeat in a period of 2 for complex K-theory, and period 8 for real K-theory. This is called Bott periodicity. We will use very strong variants of Bott periodicity without proof to derive a few results that we need. The following theorems are all used in [Rog08].

Theorem 5.8. *There are equivalences*

$$\Omega(\mathbb{Z} \times BU) \cong U, \qquad \Omega U \cong \mathbb{Z} \times BU$$

for complex K-theory and similar for real K-theory, which is 8-periodic instead of 2-periodic. On homotopy groups, we have

$$\pi_*(ku) \cong \mathbb{Z}[u], \qquad \pi_*(ko) \cong \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 - 4B)$$

where $|u| = 2$, $|\eta| = 1$, $|A| = 4$ and $|B| = 8$. Then, u and B are called the complex and real Bott elements, respectively. In particular the stable homotopy groups of ko are shown in table 1.

n	0	1	2	3	4	5	6	7
$\pi_n(\mathbb{Z} \times BO)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

Table 1: Stable homotopy groups of ko, equivalently the homotopy groups of $\mathbb{Z} \times BO$.

Definition 5.9. Let $KU := ku[u^{-1}]$ and $KO := ko[B^{-1}]$ be the periodic K-theory spectra.

Another consequence of Bott periodicity is the following, which will be quite helpful for us.

Theorem 5.10. *There is a cofiber sequence of spectra*

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU.$$

where η is induced by the stable Hopf map $\mathbb{S}^1 \to \mathbb{S}^0$ and c is complexification.

5.4 Module structure of K-theory spectra

The complexification $ko \to ku$ also gives us a map $KO \to KU$. To see this, we need to show that the image of $B \in \pi_8(KO)$ is invertible in $\pi_8(KU)$. Taking Ω^∞ of the (co)fiber sequence of theorem 5.10 yields

$$U/O \to \mathbb{Z} \times BO \to \mathbb{Z} \times BU$$

where $\Omega^\infty(\Sigma KO) \cong U/O$ follows from Bott periodicity. Furthermore, Bott periodicity gives us $\Omega(U/O) \cong \mathbb{Z} \times BO$, so $\pi_8(U/O) \cong \pi_7(\Omega(U/O)) \cong \pi_7(\mathbb{Z} \times BO)$. We have a long exact sequence

$$\ldots \to \pi_7(\mathbb{Z} \times BO) \to \pi_8(\mathbb{Z} \times BO) \to \pi_8(\mathbb{Z} \times BU) \to \pi_6(\mathbb{Z} \times BO) \to \ldots$$

and we have $\pi_n(\mathbb{Z}\times BO) \cong \widetilde{KO}(S^n)$ which is as in table 1 and we see that $\pi_7(ko) = \pi_6(ko) = 0$. Therefore, $\pi_8(\mathbb{Z} \times BO) \cong \pi_8(\mathbb{Z} \times BU)$ is an isomorphism. Thus, $c_*\colon \pi_*ko \to \pi_*ku$ sends the generator of degree 8, B necessarily to $\pm u^4$, which shows that we can localize to get $KO \to KU$. The result is:

Lemma 5.11. *Complexification $ko \to ku$ descends to a map $c\colon KO \to KU$.*

We can use the same fiber sequence to obtain a short exact sequence

$$0 \to \pi_4(\mathbb{Z} \times BO) \to \pi_4(\mathbb{Z} \times BU) \to \pi_2(\mathbb{Z} \times BO) \to 0$$

where we know by assumption that this is of the form $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Therefore, the only possibility is that the first map is $x \mapsto \pm 2x$, i.e. that $A \mapsto \pm 2u^2$. By flipping the signs of the generators A, B if necessary, we see that on homotopy groups,

$$\begin{aligned} c_*\colon \pi_* ko &\to \pi_* ku \\ \eta &\mapsto 0 \\ A &\mapsto 2u^2 \\ B &\mapsto u^4\ . \end{aligned} \tag{14}$$

Construction 5.12. We have a functor $\tau\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$ which is the identity on objects and sends $X \in U(n)$ to its complex conjugate $\overline{X} \in U(n)$. We now show that τ induces a nontrivial operation on $\widetilde{KU}(S^2)$. Let H denote the canonical line bundle $\mathbb{C}^2 \to \mathbb{C}P^1$. We will use the elementary method of *clutching functions* $S^1 \to \mathrm{GL}_1(\mathbb{C})$ to describe this vector bundle, by dividing $\mathbb{C}P^1$ into two discs $\{[1:z]\}$ and $\{[z:1]\}$ and considering the circle $S^1 = \{[1:e^{ix}] \mid x \in \mathbb{R}\}$ lying in the intersection (see e.g. [Hat17, Chapter 1]). Then we have the transition map

$$H^{-1}(\{[1:e^{ix}]\}) \times \mathbb{C} \to H^{-1}(\{[e^{-ix}:1]\}) \times \mathbb{C}, \qquad ([1:e^{ix}], z) \mapsto ([1:e^{ix}], e^{ix}z)$$

by following the trivializations and their inverses $([1:e^{ix}], z) \mapsto (z, ze^{ix}) \mapsto ([1:e^{ix}], e^{ix}z)$. Therefore, the clutching function $S^1 \to \mathrm{GL}_1(\mathbb{C})$ is just the identity $e^{ix} \mapsto e^{ix}$ on S^1. On the other hand, if we consider the conjugate line bundle $\overline{H}$, then the transition map becomes $([1:e^{ix}], z) \mapsto (\overline{z}, \overline{z}e^{ix}) \mapsto ([1:e^{ix}], e^{-ix}z)$, so the clutching function is $e^{ix} \mapsto e^{-ix}$. By [Hat17, Proposition 1.11], this shows that the vector bundles H and $\overline{H}$ are indeed non-isomorphic, so $\tau\colon ku \to ku$ is not homotopic to the identity. Since $\tau^2 = \mathrm{id}_{ku}$, it follows that on homotopy groups, we must have

$$\tau_*\colon \mathbb{Z}[u] \to \mathbb{Z}[u], \qquad u \mapsto -u$$

since it is not $u \mapsto u$ (otherwise it would be homotopic to id_{ku}). Note that the image of u determines the map since we know τ to be a map of $\mathbb{E}_\infty$-rings by the Segal construction. Finally, we see that we can descend to periodic complex K-theory $KU = ku[u^{-1}]$ and obtain $\tau\colon KU \to KU$ as a KO-algebra map.

Construction 5.13. There is a functor $r\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{R}}$ given by sending $n \mapsto 2n$ and by choosing compatible homomorphisms $U(n) \to O(2n)$. For example, we could use the isomorphism of $\mathbb{R}$-vector spaces $\mathbb{C} \to \mathbb{R}^2$ given by $z = x + iy \mapsto (x, y)$, using this to define isomorphisms $\mathbb{C}^n \to \mathbb{R}^{2n}$ and finally the inclusion $U(n) \to O(2n)$. We call this operation *realification*. Since this functor respects the bipermutative structures present, we get a ring map $ku \to ko$. In fact, the cofiber sequence of spectra of theorem 5.10 has differential

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\partial} \Sigma^2 KO$$

with $\partial = \Sigma^2 r \circ \beta^{-1}$ where $\beta\colon \Sigma^2 KU \to KU$ is the Bott isomorphism (see [Rog08, equation (5.3.2)]. We can then use the long exact sequence on homotopy to compute that $\partial_d = 0$ in degrees $d \in \{0, 1, 3, 5, 7\}$ modulo 8. The remaining degrees are involved in exact sequences:

$$\cdots \longrightarrow \mathbb{Z}/2 \xrightarrow{c_2} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{c_4} \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z}/2 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_6} \mathbb{Z} \longrightarrow 0.$$

Thus, by shifting according to $\Sigma^2 r = \partial \circ \beta$, we see that r is given on homotopy by

$$\begin{aligned} r\colon \pi_* ku &\to \pi_* ko \\ 1 &\mapsto 2 \\ u &\mapsto \eta^2 \\ u^2 &\mapsto A \\ u^3 &\mapsto 0. \end{aligned}$$

5.5 Complexification is a Galois extension

For the remainder of this section, we will be showing that complexification $c\colon KO \to KU$ turns KU into a C_2-Galois extension of KO, giving us a nontrivial, and also non-algebraic example of a Galois extension. This is maybe the simplest example of a truly new type of Galois extension, and it also shows that the Galois category $\mathrm{CAlg}_{KO}^{\mathrm{w.cov}}$ is nontrivial.

Lemma 5.14. *The map $h\colon KU \otimes_{KO} KU \to \prod_{\mathbb{Z}/2} KU$ is an isomorphism.*

Proof. We follow Rognes' proof [Rog08, Proposition 5.3.1], filling in a few details but referring to his proof whenever results about Bott periodicity are concerned. Since the KO-module structure of KU is given by $c\colon KO \to KU$, we write $c_! := KU \otimes_{KO} -$. This admits a right adjoint $c^*\colon \mathrm{Mod}_{KU} \to \mathrm{Mod}_{KO}$ given by restriction of scalars[14]. In particular, $KU \otimes_{KO} -$ is a left adjoint, so it is exact and preserves cofiber sequences in stable the ∞-category Mod_{KO}.

Hence we have a cofiber sequence

$$KU \otimes_{KO} \Sigma KO \to KU \otimes_{KO} KO \to KU \otimes_{KO} KU \to KU \otimes_{KO} \Sigma^2 KO$$

[14] to define this in the ∞-categorical context requires some work, see e.g. [Lur17, Section 4.5.3] and [Cno26].

of KU-modules, where the KU-module structure is given by multiplication onto the first components. To see that h is an isomorphism, define a second cofiber sequence

$$KU \xrightarrow{\Delta} \prod_{\mathbb{Z}/2} KU \xrightarrow{\delta} KU$$

where Δ is the diagonal map and δ the difference map. We have already constructed exact sequences of this form in eq. (4). We consider the diagram

$$\begin{array}{ccccc} KU \otimes_{KO} KO & \xrightarrow{1\otimes c} & KU \otimes_{KO} KU & \xrightarrow{1\otimes\partial} & KU \otimes_{KO} \Sigma^2 KO \\ \mu\downarrow & & \downarrow h & & \downarrow \\ KU & \xrightarrow{\Delta} & \prod_{\mathbb{Z}/2} KU & \xrightarrow{\delta} & KU \end{array}$$

We will show that this diagram commutes in Mod_{KO}, i.e. that it commutes up to stable homotopy equivalence. In the left square, notice that since the KO-module structure on KU is given by $c\colon KO \to KU$, μ factors as the composition

$$KU \otimes_{KO} KO \xrightarrow{1\otimes c} KU \otimes_{KO} KU \xrightarrow{\mu_{KU}} KU$$

where μ_{KU} can be seen as a map $KU \otimes_{KO} KU \to KU$ by lemma 3.28. We need to show that $h \circ (1 \otimes c) \simeq \Delta \circ \mu_{KU} \circ (1 \otimes c)$. For this, we use that $\tau \circ c = c$ (a complexification is invariant under conjugation, this can be seen already on the level of the categories $\mathcal{V}_{\mathbb{C}}$ and $\mathcal{V}_{\mathbb{R}}$). Into the first component of $\prod_{\mathbb{Z}/2} KU$, both maps are $\mu_{KU} \circ (1 \otimes c)$. Into the second component, they are $\mu_{KU} \circ (1 \otimes c)$ and $\mu_{KU} \circ (1 \otimes \tau \circ c)$ which are equivalent by $\tau \circ c = c$. Therefore, the left square commutes.

We can precompose the right hand square with the map $1 \otimes \beta\colon KU \otimes_{KO} \Sigma^2 KU \to KU \otimes_{KO} KU$ to obtain a diagram of KU-module maps

$$\begin{array}{ccc} KU \otimes_{KO} \Sigma^2 KU & \xrightarrow{1\otimes\partial\circ\beta} & KO \otimes_{KO} \Sigma^2 KO \\ h\circ(1\otimes\beta)\downarrow & & \cong\downarrow\beta' \\ \prod_{\mathbb{Z}/2} KU & \xrightarrow{\delta} & KU \end{array} \tag{15}$$

where the KU-module structure on the upper row is given by the extension of scalars functor $- \otimes_{KO} KU\colon \mathrm{Mod}_{KO} \to \mathrm{Mod}_{KU}$. The isomorphism β' is given by

$$KU \otimes_{KO} \Sigma^2 KO \xrightarrow{\cong} \Sigma^2 KU \otimes_{KO} KO \xrightarrow{\cong} \Sigma^2 KU \xrightarrow{\beta} KU. \tag{16}$$

Given a KO-module X, a KU-module Y and a KU-module morphism $f\colon c_! X \to Y$, we obtain a KO-module map via the composition

$$X \xrightarrow{i} KO \otimes_{KO} X \xrightarrow{c\otimes 1} KU \otimes_{KO} X \xrightarrow{f} c^* Y.$$

Using this adjunction, commutativity of the square (15) is now equivalent to showing that

$$\delta \circ h \circ (c \otimes \beta) \circ i \cong \beta' \circ (c \otimes \partial \circ \beta) \circ i \tag{17}$$

as KO-module maps $\Sigma^2 KU \to KU$. It is a fact (which we do not prove) that $\partial \circ \beta \cong \Sigma^2 r$. By definition, we have $\delta \circ h \cong \mu \circ (1 \otimes (1 - \tau))$ where $\mu\colon KU \otimes_{KO} KU \to KU$ is the ring multiplication of KU and $\tau\colon KU \to KU$ is complex conjugation. To understand β', observe that the map $\Sigma^2 KU \otimes_{KO} KO \to \Sigma^2 KU$ in equation (16) is given by $\Sigma^2(\mu \circ (1 \otimes c))$, since the KO-module structure on KU is given by c. Denoting the swap map by $s\colon KU \otimes_{KO} \Sigma^2 KO \to \Sigma^2 KU \otimes_{KO} KO$, we have

$$\beta' \circ (c \otimes \partial \circ \beta) \circ i \cong \beta \circ \Sigma^2(\mu \circ (1 \otimes c)) \circ s \circ (c \otimes \Sigma^2 r) \circ i \cong \beta \circ \Sigma^2(c \circ r)$$

where the last isomorphism follows from considering where the $\mathbb{S}^2$ and KU components of $\Sigma^2 KU$ go into the multiplication μ. We now consider $c \circ r\colon KU \to KU$ on the level of homotopy rings $\pi_* KU \cong \mathbb{Z}[u^{\pm 1}]$. Using what we know about c (equation (14)) and r (construction 5.13), this is the map

$$1 \mapsto 2, \qquad u \mapsto 0, \qquad u^2 \mapsto 2u^2, \qquad u^3 \mapsto 0, \qquad \dots$$

and we see that $c \circ r \cong 1 + \tau$. Moreover, there is also an equivalence $\beta \circ \Sigma^2(1 + \tau) \cong (1 - \tau) \circ \beta$ which involves the structure of the Bott periodicity map β, so we do not prove it here. Combining these results yields

$$\beta' \circ (c \otimes \partial \circ \beta) \circ i \cong \beta \circ \Sigma^2(1 + \tau) \cong (1 - \tau) \circ \beta.$$

Finally,

$$\delta \circ h \circ (c \otimes \beta) \circ i \cong \mu \circ (c \otimes (1 - \tau) \circ \beta) \circ i \cong (1 - \tau) \circ \beta$$

so we see that (17) holds, finishing the proof. □

Lemma 5.15. *The map $c\colon KO \to KU$ identifies KU as a faithfully flat KO-module.*

Proof. By remark 4.18 we need to show that if N is a KO-module such that $N \otimes_{KO} KU \cong 0$, then $N \cong 0$. We can tensor the Bott periodicity cofiber sequence by N to obtain a cofiber sequence

$$N \otimes_{KO} \Sigma KO \xrightarrow{1 \otimes \eta} N \otimes_{KO} KO \xrightarrow{1 \otimes c} N \otimes_{KO} KU.$$

By assumption, $N \otimes_{KO} KU \cong 0$, so the map $1 \otimes \eta$ must have been an isomorphism. Moreover, the first map is equivalent to $\Sigma N \to N$ classified by the stable Hopf map $\eta\colon \mathbb{S}^1 \to \mathbb{S}^0$. By iterating this isomorphism, we see that $\Sigma^k N \cong N$ for all $k > 0$. By known computations, $\eta^4 = 0$ in $\pi_4(\mathbb{S})$, i.e. the map $\Sigma^4 N \to N$ given by applying η to each of the four suspension coordinates is zero. This implies that $N \cong 0$, and we are finished. □

Lemma 5.16. *The map $KO \to KU^{hC_2}$ given by the structure of the limit KU^{hC_2} taken in $\mathrm{Mod}_{KO}^{BC_2}$ is an isomorphism in* CAlg.

Proof. Consider the homotopy fixed point spectral sequence for the action of C_2 on $KO \to KU$ by complex conjugation.

$$E^2_{s,t} = H^{-s}(C_2; \pi_t(KU)) \quad \Longrightarrow \quad \pi_{t+s}(KU^{hC_2}).$$

We now compute the E^2 page. Notice that $\pi_* KU = \mathbb{Z}[u]$, with C_2 acting on u by flipping the sign. Therefore, we have to calculate $H^s(C_2; \mathbb{Z})$ with C_2 acting trivially on $\mathbb{Z}$ for $t = 4k$, and $H^s(C_2; \mathbb{Z}\{x\})$ with C_2 acting by sign flipping on $\mathbb{Z} = \mathbb{Z}\{x\}$ for $t = 4k + 2$.

For the former case, this is the computation of the group cohomology $H^n(C_2; \mathbb{Z}) = \mathrm{Ext}^n_{\mathbb{Z}[C_2]}(\mathbb{Z}, \mathbb{Z})$ so we find a projective resolution of $\mathbb{Z}$. A natural choice is

$$\cdots \xrightarrow{1-\tau} \mathbb{Z}[C_2] \xrightarrow{1+\tau} \mathbb{Z}[C_2] \xrightarrow{1-\tau} \mathbb{Z}[C_2] \xrightarrow{\tau \mapsto 1} \mathbb{Z} \longrightarrow 0$$

where $C_2 = \{e, \tau\}$ and the map $\mathbb{Z}[C_2] \to \mathbb{Z}$ is given by the projection $\tau \mapsto 1$. Every summand is free, hence projective, and the sequence is exact since:

1. The kernel of the projection $\tau \mapsto 1$ is precisely elements of the form $(1-\tau)x$ for $x \in \mathbb{Z}$.

2. The kernel of $(x + \tau y) \mapsto (1 \pm \tau)(x + \tau y) = (1+\tau)(x \mp y)$ is generated by $1 - \tau$.

Then apply $\mathrm{Hom}_{\mathbb{Z}[C_2]}(-, \mathbb{Z})$ to get the cochain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots.$$

Thus,

$$H^s(C_2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & s = 0 \\ \mathbb{Z}/2 & s > 0 \text{ and } s \text{ even} \\ 0 & s \text{ odd.} \end{cases}$$

For the case of $\mathbb{Z}\{x\}$, we can do a similar computation. Since in this case the action of C_2 on $\mathbb{Z}\{x\}$ is $\tau x = -x$, the free resolution and resulting cochain complex are

$$\cdots \xrightarrow{1-\tau} \mathbb{Z}[C_2] \xrightarrow{1+\tau} \mathbb{Z}[C_2] \xrightarrow{\tau \mapsto -1} \mathbb{Z} \longrightarrow 0 \qquad \rightsquigarrow \qquad \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots$$

and we get

$$H^s(C_2; \mathbb{Z}\{x\}) = \begin{cases} \mathbb{Z}/2 & p \text{ odd} \\ 0 & p \text{ even.} \end{cases}$$

Using these two computations, we have computed the E^2 page of the homotopy fixed point spectral sequence to be

$$E^2_{s,t} = \mathbb{Z}[a, t^{\pm 1}]/(2a)$$

where a is the generator of bidegree $(-1, 2)$ and t the generator of bidegree $(4, 0)$. The E^2 page is shown in fig. 1. To proceed requires some nontrivial input, namely that $d^3(t) = a^3$.

There are various approaches to show this, such as the computational approach in [HS14, Section 5], using equivariant homotopy theory and Atiyah's $K\mathbb{R}$-theory [Hau24, Section 3] or by using a map of spectral sequences from the Adams-Novikov spectral sequences [Hed18]. We will assume this input and follow the presentation of [Hed18] quite closely. Then, it follows that $d^3(a^3) = d^3 \circ d^3(t) = 0$. From this, we can work backwards using the differential graded structure and get

$$d^3(a^3) = d^3(a)a^2 + (-1)^{|a|}ad^3(a^2) = d^3(a)a^2$$

where we use that $d^3(a^2) = d^3(a)a - ad^3(a) = 0$ since $|d^3(a)| = 4 - 4 = 0$. Since $a^2 \neq 0$, it must be that $d^3(a) = 0$ and the entire differential structure now follows from the multiplicative structure. The result is depicted in fig. 2a, where we do not draw all classes, but all of the depicted differentials are nonzero. Then, since most of the differentials are isomorphisms $\mathbb{Z}/2 \to \mathbb{Z}/2$, all classes to with x-coordinate less than -2 vanish on the E^4 page. We write b for the class of $2t$ in $E^4_{0,4}$ which remains after $d^3 \colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$. Then, the resulting bigraded ring can be written as

$$E^4_{*,*} = \mathbb{Z}[a, b, t^{\pm 2}]/(2a, a^3, ab, b^2 - 4t^2).$$

The E^4 page is depicted in fig. 2b, and we can see that since all remaining classes are in the columns $0, -1, -2$, there are no further nonzero differentials. This implies $E^4_{*,*} = E^\infty_{*,*}$ and we see that $\pi_*(KU^{hC_2}) \cong \pi_*(KO)$, finishing the proof.

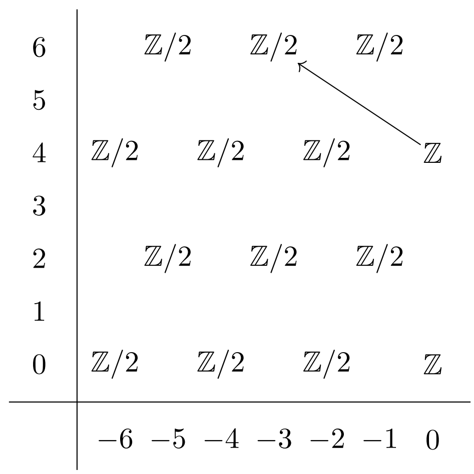
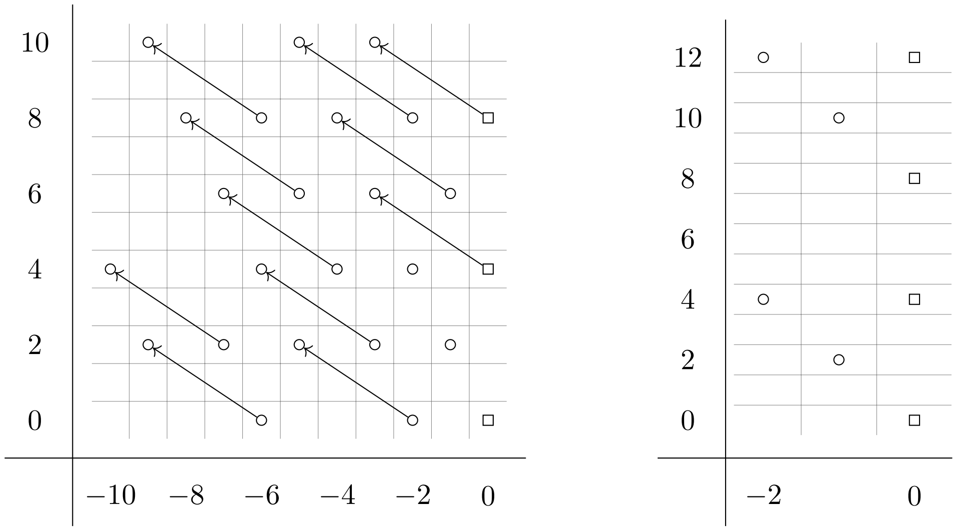


Figure 1: The E^3 page of the HFPSS for KU^{hC_2}, with a highlighted differential.

□

Theorem 5.17. *The absolute Galois group* $\mathrm{Gal}(KO)$ *of* KO*, obtained by choosing a basepoint for the fundamental group* $\pi_{\leq 1}(\mathrm{CAlg}^{\mathrm{w.cov}}_{KO})$ *of* $\mathrm{CAlg}^{\mathrm{w.cov}}_{KO}$*, is* $\mathbb{Z}/2$*.*

Proof. We will need some additional results for this proof. Most critically, since KU is even periodic and $\pi_0 KU = \mathbb{Z}$, [Mat16, Proposition 6.37] tells us that $\pi_{\leq 1}(\mathrm{CAlg}^{\mathrm{w.cov}}_{KU}) \cong$



(a) The E^3 page. (b) The E^4 page. No classes to the left remain.

Figure 2: The E^3 and E^4 pages of the HFPSS for KU^{hC_2}. Squares are copies of $\mathbb{Z}$ and circles are copies of $\mathbb{Z}/2$.

$\pi_{\leq 1}(\mathrm{CAlg}^{\text{ét}}_{\operatorname{Spec}\mathbb{Z}})$. After choosing a basepoint, this latter groupoid becomes $\pi_1^{\text{ét}}(\operatorname{Spec}\mathbb{Z})$, which is well-studied in algebraic geometry and number theory. Since there are no nontrivial finite étale $\mathbb{Z}$-algebras due to *Minkowski's theorem* (see [Sza09, Remark 5.2.5] and the citations within), one has $\pi_1^{\text{ét}}(\operatorname{Spec}\mathbb{Z}) = 1$. In particular, the only faithful Galois extension of KU is KU itself. Moreover $\pi_1(KO) \cong \mathbb{Z}/2$ [Mat16, Corollary 10.5]. It follows that $\mathrm{CAlg}^{\text{w.cov}}_{KO} \simeq \mathrm{FinSet}_{\mathbb{Z}/2}$ and example 2.26 tells us how to find all Galois extensions of KO.

□

Concluding remarks

We have explored a mechanism to take an algebraic ∞-category such as Mod_R to a special kind of an ordinary 1-category: a Galois category. This allows us to define a π_1 of such ∞-categories. The "1-categorization" is built into the theory from the beginning. There are other approaches such as in the paper of Hoyois [Hoy18] which assign to sufficiently nice ∞-categories a *fundamental pro-∞-groupoid*, avoiding the truncation present in the theory presented here. It is possible that the profinite fundamental groupoids we we compute here can be seen as a truncation of such a profinite-∞-groupoid, but it is not clear to this author how to approach this question (e.g. what is a candidate ∞-topos to use as input into the theory in [Hoy18]?).

Appendix A ∞-categories

A.1 Properties

We follow the approach of the ongoing book project of Cnossen [Cno26] and treat ∞-categories informally by listing axioms that they satisfy. This approach should streamline the use of ∞-categories in our context. Every axiom we list is satisfied by the *quasicategories* introduced by Joyal and developed extensively by Lurie [Lur09]. We will define quasicategories in section A.7. Moreover, we ignore any issues about set-theoretic size, trusting that the cited sources adequately consider these issues.

Axiom A.1. There are structures called ∞-categories, such that:

(1) we can speak of functors $f\colon C \to D$ between ∞-categories C, D,

(2) we can compose functors $f\colon C \to D$, $g\colon D \to E$, denoted $g \circ f$, and every ∞-category C has an identity functor $\mathrm{id}_C\colon C \to C$,

(3) given functors $f, f'\colon C \to D$, we can speak of natural isomorphisms $\eta\colon f \cong f'$ between functors, and there are identity natural isomorphisms $\mathrm{id}_f\colon f \cong f$ as well as inverse natural transformations $\eta^{-1}\colon f' \cong f$,

(4) given natural isomorphisms $\eta, \eta'\colon f \cong f'$, we may speak of 3-isomorphisms between η and η', and so on...

Remark A.1. Since we have the structure of categories and functors with a defined composition, we can speak of commutative diagrams, such as squares

$$\begin{array}{ccc} C & \xrightarrow{g} & C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ D & \xrightarrow{h} & D'. \end{array}$$

In contrast to ordinary 1-category theory, what we imply by the commutativity of such a square is that we have the *data* of a particular natural isomorphism $\eta\colon h \circ f \cong f' \circ g$.

Axiom A.2. Given any functor $f\colon C \to D$, there are natural isomorphisms $\mathrm{id}_D \circ f \cong f$ and $f \circ \mathrm{id}_C \cong f$. We also have natural isomorphisms $(f \circ g) \circ h \cong f \circ (g \circ h)$. Similarly, this structure goes for natural isomorphisms and onwards, where we also have isomorphisms $\alpha \circ \alpha^{-1} \cong \mathrm{id}_f$ and $\alpha^{-1} \circ \alpha \cong \mathrm{id}_{f'}$.

There are also various other axioms concerning horizontal and vertical composition of natural transformations, but we do not have any need for these, and the rules for composition of n-isomorphisms are somewhat involved. For the complete axiomatic approach, one can consult the in-progress work [CCNW26]. To populate the concept of ∞-categories, we introduce the next axiom.

Axiom A.3. Any ordinary 1-category gives rise to an ∞-category. Any functor of 1-categories, is a functor of ∞-categories. Any ordinary natural isomorphism of functors between ordinary 1-categories is a natural isomorphism of the corresponding functors between ∞-categories. Composition works for this assignment as expected.

Axiom A.4. For any two ∞-categories C, D, there is an ∞-category $\mathrm{Fun}(C, D)$, called the ∞-category of functors. This comes with a functor $\mathrm{ev}\colon \mathrm{Fun}(C, D) \times C \to D$.

The simplex category Δ (see definition A.30) is key to the definition of simplicial sets and quasicategories. In the approach presented here, we also need the notions involved.

Definition A.2. Let $[n]$ for $n \geq 0$ denote the ordinary 1-category defined by the linearly ordered set $\{0 \leq 1 \leq \cdots \leq n\}$. We define the following functors:

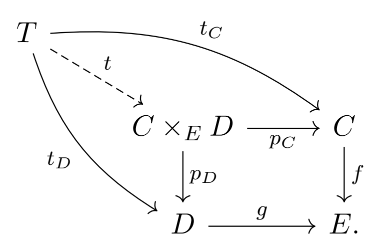
(1) For any $n \geq 1$ and $0 \leq i \leq n$, the *face map* $d_i^n\colon [n-1] \to [n]$ is defined by the unique injective order-preserving map of posets which omits i from its image. It is common to abuse notation by writing d_i instead of d_i^n.

(2) For any $n \geq 0$ and $0 \leq i \leq n$, the *degeneracy map* $s_i^n\colon [n+1] \to [n]$ is defined by the unique surjective order-preserving map of posets for which the preimage of $\{i\}$ has two elements. It is common to abuse notation by writing s_i instead of s_i^n.

Definition A.3. For an ∞-category C, we denote by $\mathrm{Ob}(C)$ the collection of functors $[0] \to C$; these are the *objects* of C. The *morphisms* of C are functors $[1] \to C$. To a morphism $f\colon [1] \to C$, we can associate two objects $[0] \to C$ by applying the face maps to $[1]$. We write $f\colon x \to y$ if x is the object $[0] \xrightarrow{d_1} [1] \to C$ and y is the object $[0] \xrightarrow{d_0} [1] \to C$.

To create new ∞-categories from old, categorical constructions can be very useful. The next axiom introduces pullbacks of ∞-categories.

Axiom A.5. Let two functors $f\colon C \to E$ and $g\colon D \to E$ between ∞-categories C, D, E be given. Then there is an ∞-category $C \times_E D$ together with functors $p_C\colon C \times_E D \to C$, $p_D\colon C \times_E D \to D$ such that $f \circ p_C \cong g \circ p_D$. Moreover, for any ∞-category T and functors $t_C\colon T \to C$, $t_D\colon T \to D$, the following diagram commutes



We note that the fact that the above diagram commutes means that there are natural isomorphisms witnessing commutativity of all faces. The fact that the dotted line exists therefore implies the existence of new natural isomorphisms $\eta_D \colon p_D \circ t \cong t_D$ and $\eta_C \colon p_C \circ t \cong t_C$. Moreover, there are 3-isomorphisms showing compatibility of the new natural isomorphisms with the old structure. Since it is not our goal to be precise, we refer the reader to [Cno26] and [CCNW26] for a precise formulation of the higher data involved in this axiom.

Similarly, we define products, coproducts and pushouts of ∞-categories.

Axiom A.6. There are ∞-categories $C \times D$ and $C \sqcup D$, $C \sqcup_E D$ defined by the usual universal property. The formulation of the axioms are analogous to axiom A.5.

Definition A.4. A *commutative square* in C is a functor $[1] \times [1] \to C$. A *commutative triangle* in C is a functor $[2] \to C$.

To properly work with commutative diagrams, we want to be able to build them using pushouts of the ∞-categories $[0], [1], \ldots$. We omit these axioms, since they are intuitively clear and are easily proven in the quasicategorical model. To define composition, we need the *Segal axiom*, which can be expressed as an equivalence of ∞-categories.

Definition A.5. A functor $f \colon C \to D$ is an *equivalence* if there exists another functor $g \colon C \to D$ and natural isomorphisms $\mathrm{id}_C \cong g \circ f$, $\mathrm{id}_D \cong f \circ g$.

Axiom A.7 (Segal axiom)**.** The restriction functor

$$\mathrm{Fun}([2], C) \longrightarrow \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C)$$

given by restricting to the edges $0 \to 1$ and $1 \to 2$ of

$$[2] = \begin{array}{ccc} 0 & \longrightarrow & 2 \\ & \searrow \quad \nearrow & \\ & 1 & \end{array},$$

is an equivalence.

This axiom is a way of encoding the fact that inner horns can be filled in, which is the defining property of a quasicategory. It allows us to define the composition of morphisms in an ∞-category.

Definition A.6. Given the Segal axiom, we can define the *composition functor* as

$$\mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \to \mathrm{Fun}([2], C) \xrightarrow{d_1^*} \mathrm{Fun}([1], C).$$

Remark A.7. The composition *functor* is well-defined, but this does not mean that given two composible morphisms f, g, the composite $g \circ f$ is uniquely defined. In fact, the language we are using only allows us to speak of $\mathrm{Fun}([1], C)$ and $f \colon x \to y$ should be seen more as notational convenience. Using quasicategories, morphisms are elements of an actual set, and composition is only well-defined up to a contractible space of choices for a composite.

Definition A.8 (Isomorphism)**.** A morphism $f\colon x \to y$ is *invertible* or is an *isomorphism* if there is another morphism $g\colon y \to x$ such that there are commutative triangles

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & \searrow^{\mathrm{id}_x} & \downarrow g \\ & & x \end{array} \qquad \begin{array}{ccc} y & \xrightarrow{g} & x \\ & \searrow^{\mathrm{id}_y} & \downarrow f \\ & & y \end{array} .$$

Definition A.9 (∞-groupoid)**.** An ∞-category C is called an ∞*-groupoid*, *space* or *homotopy type* if all of its morphisms are invertible.

Axiom A.8 (Groupoid core)**.** For any ∞-category C, there is an ∞-groupoid $C^{\simeq}$ called the *groupoid core* of C, as well as a functor $\gamma_C\colon C^{\simeq} \to C$. These satisfy that every functor $F\colon X \to C$ from an ∞-groupoid X to C factors through C, and also that for every pair of functors $F, G\colon X \to C^{\simeq}$, every natural isomorphism $\gamma_C \circ F \cong \gamma_C \circ G$ can be lifted to a natural isomorphism $F \cong G$.

Definition A.10 (Hom spaces)**.** Given objects x, y of an ∞-category C, we can form the pullback

$$\begin{array}{ccc} \mathrm{Hom}_C(x,y) & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & & \downarrow (\mathrm{ev}_0, \mathrm{ev}_1) \\ * & \xrightarrow{(x,y)} & C \times C \end{array}$$

and get an ∞-category $\mathrm{Hom}_C(x,y)$ (sometimes also denoted $\mathrm{Map}_C(x,y)$), called the *hom space* or *mapping space*. This terminology is justified by the fact that $\mathrm{Hom}_C(x,y)$ is an ∞-groupoid.

Remark A.11. What these axioms essentially do is describe the ∞-category Cat_∞ of small ∞-categories. In the quasicategories model, the hom-spaces in this category are given by $\mathrm{Hom}_{\mathrm{Cat}_\infty}(C, D) = \mathrm{Fun}(C, D)^{\simeq}$. That is, we only care about *invertible functors* between functors, i.e. natural isomorphisms and higher.

So far, we can only talk about natural isomorphisms, but of course, we want to talk about general natural transformations as well.

Definition A.12 (Natural transformation)**.** If C, D are ∞-categories, a *natural transformation* is a morphism in the functor category $\mathrm{Fun}(C, D)$. Equivalently, this is a functor $[1] \times C \to D$.

Remark A.13. By definition, a natural transformation $F \to G$ is a morphism in $\mathrm{Fun}(C, D)$. The evaluation functor $\mathrm{ev}\colon \mathrm{Fun}(C, D) \times C \to D$ provides for an object x in C and a functor F and object $F(x)$ in D, and for a morphism $x \to y$ in C and a natural transformation $F \to G$ a morphism $F(x) \to G(x)$ in D. This recovers the usual way we think about natural transformations in ordinary 1-category theory, which is as a compatible family of morphisms $\{G(x) \to F(x)\}$. It is common when working informally with ∞-categories to adopt this viewpoint.

Remark A.14. Maybe due to the fact that isomorphisms in the ∞-categories of spaces and spectra correspond to what are classically called *(weak) equivalences*, some authors call isomorphisms "equivalences". In this work, we mostly use the term isomorphism.

Definition A.15 (∞-groupoid)**.** An ∞-category in which every morphism is an isomorphism is called an ∞-groupoid, or *space*.

Definition A.16. The *homotopy category* of an ∞-category C is the ordinary 1-category with objects $\mathrm{Ob}(C)$ and morphisms $\mathrm{Hom}_{\mathrm{Ho}(C)}(X, Y) := \pi_0 \mathrm{Map}_C(X, Y)$, the collection of path components of the mapping space $\mathrm{Map}_C(X, Y)$.

Lemma A.17. *For a morphism $f\colon X \to Y$ in C, the following are equivalent:*

(1) f is an isomorphism,

(2) $[f]$ is an isomorphism in $\mathrm{Ho}(C)$,

(3) there is a morphism $g\colon Y \to X$ such that $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$,

(4) there are morphisms $g_1, g_2\colon Y \to X$ such that $[f \circ g_1] = [\mathrm{id}_Y]$ and $[g_2 \circ f] = [\mathrm{id}_X]$.

There is an important analog of the Yoneda lemma. In the ∞-categorical world, a presheaf on C is an object of $\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$.

Theorem A.18. *Let F be a presheaf on C, and x an object of C. Then evaluation at $\mathrm{id}_x \in \mathrm{Hom}_C(x, x)$ induces an isomorphism*

$$\mathrm{Fun}_{\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})}(\mathrm{Hom}_C(-, x), F) \xrightarrow{\sim} F(x)$$

and in particular, the Yoneda embedding $C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ is fully faithful.

A.2 The ∞-category of spaces

Axiom A.9. There exist ∞-categories $\mathcal{S}$ and Cat_∞, the ∞-category of (small) spaces (or ∞-groupoids) and the ∞-category of (small) ∞-categories, respectively. There is an inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$.

Remark A.19. In the quasicategorical model, one constructs $\mathcal{S}$ by taking the homotopy coherent nerve of the category of Kan complexes enriched in Kan complexes, e.g. [Lur26, Tag 00TZ]. We will give an idea of what this construction achieves in construction A.36.

Viewing the ordinary 1-category of Kan complexes Kan as a discrete ∞-category allows us to consider the *localization functor* $\mathrm{Kan} \to \mathcal{S}$ which is the identity on objects, but has the property that it sends homotopy equivalences to isomorphisms in $\mathcal{S}$. The homotopy category $\mathrm{Ho}(\mathcal{S})$ is then isomorphic to the category $\mathrm{Kan}[w^{-1}]$ given by inverting all weak homotopy equivalences.

Note that at the same time, it is true that by using the construction alluded to in remark A.19, a morphism $f\colon X \to Y$ in $\mathcal{S}$ is an isomorphism if and only if it is a homotopy equivalence of the Kan complexes X and Y. Thus, the category $\mathrm{Kan}[w^{-1}]$ localized at weak homotopy equivalences is equivalent as ∞-categories to $\mathrm{Kan}[h^{-1}]$ localized at homotopy equivalences.

A.3 Limits

Definition A.20. Let $F\colon I \to C$ be a functor of ∞-categories I, C. Let $\mathrm{const}_X\colon I \to C$ denote the functor sending all objects to X and all (higher) morphisms to the identity. Then we define:

(1) A *cone on* F is an object X of C with a natural transformation $\varepsilon\colon \mathrm{const}_X \to F$.

(2) A cone on F is a called a *limit cone* if for every other object Y of C the map

$$\mathrm{Hom}_C(Y, X) \to \mathrm{Hom}_{\mathrm{Fun}(C,D)}(\mathrm{const}_Y, \mathrm{const}_X) \to \mathrm{Hom}_{\mathrm{Fun}(C,D)}(\mathrm{const}_Y, F)$$

is an equivalence in the category of spaces $\mathcal{S}$.

This definition is a direct translation of how one may define limits in ordinary 1-categories. Limits are unique up to isomorphism (in the ∞-category C, so for example unique up to homotopy equivalence if $C = \mathcal{S}$).

A.4 Adjoint functors

As in ordinary category theory, adjoint functors play a huge role in the use of ∞-categories, so we define them and state some properties that they enjoy, which are ∞-categorical versions of familiar statements about ordinary adjoint functors.

Definition A.21. Let $F\colon C \to D$ and $G\colon D \to C$ be two functors between ∞-categories C, D. An *adjunction* $F \dashv G$ is a natural isomorphism

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-))$$

of functors $C^{\mathrm{op}} \times C \to \mathcal{S}$. We may also denote this situation $F : C \rightleftarrows D : G$.

As in ordinary 1-category theory, an adjunction is equivalently described by the data of two natural transformations $\eta\colon \mathrm{id}_C \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_D$ satisfying triangle identities

One of our most common uses of adjunctions is the following theorem, which asserts that left adjoints preserve colimits and right adjoints preserve limits.

Theorem A.22. *Let $F\colon C \rightleftarrows D\colon G$ be an adjunction. Then F preserves all colimits which exist in C and G preserves all limits which exist in D.*

A.5 Presentable ∞-categories

There is a particularly nice class of ∞-categories called *presentable* ∞-categories. One reason why they are useful in ∞-category theory is that the ∞-categorical analog of the adjoint functor theorem holds for them. Moreover, the category of spectra is initial in a subcategory of the category of all stable, symmetric monoidal and presentable ∞-categories (this is a consequence of [Lur17, Proposition 3.2.1.8] and is mentioned in the introduction to [Lur17, Section 4.2.8]), which is a further reason why they come up very often in higher algebra.

An ∞-category C is presentable if it admits small colimits and is accessible, which means that it arises as the category of ind-objects (cf. appendix B) of some small ∞-category. The precise definition is rather technical, but since all ∞-categories we encounter are stable, the following proposition captures what we need and further clarifies the importance of presentability for higher algebra.

Proposition A.23. *([Lur17, Corollary 1.4.4.2]) A stable ∞-category C is* presentable *if and only if:*

(1) C admits small coproducts,

(2) $\mathrm{Ho}(C)$ is locally small,

(3) C admits a κ-compact generator X for some regular cardinal κ. This means that $\pi_0 \operatorname{Hom}_C(X, Y) \simeq *$ implies that $Y \cong 0$, and that the functor $\operatorname{Hom}_C(X, -)$ commutes with κ-filtered colimits[15]*.*

Definition A.24. A symmetric monoidal category $p\colon C^{\otimes} \to N(\mathrm{Fin}_*)$ is presentable if its underlying category $C = C^{\otimes}_{\langle 1\rangle}$ is.

Example A.25. The stable ∞-category of spectra Sp is presentable by [Lur17, Proposition 1.4.4.4] since Sp is the stabilization of $\mathcal{S}$, which is presentable. The compact generator of Sp is the sphere spectrum $\mathbb{S}$.

Example A.26. Mod_R for an $\mathbb{E}_\infty$-ring R is presentable by [Lur17, Corollary 4.2.3.7].

The ∞-categories we encounter will be presentable (and stable):

Theorem A.27. *Since $\mathrm{Sp}, \mathrm{Mod}_R$ are presentable, so are Sp^{BG} and Mod_R^{BG}*

Proof. That applying $\mathrm{Fun}(BG, -)$ to a presentable ∞-category yields another presentable ∞-category is the content of [Lur09, Proposition 5.5.3.6]. □

[15] κ-filteredness is a generalization of being a filtered category (see definition B.1) where instead of requiring that every *finite* diagram has a cocone (this is equivalent to our definition), we require that every diagram over a category with $< \kappa$ objects has a cocone

A.6 Monads

In section 2.4, we speak of *effective descent morphisms* in ∞-categories. In this short section, we illustrate that the definition for ordinary 1-categories has a reasonable generalization, following definitions in [Gep20, Section 4.1]. We skip defining the non-commutative versions of algebras, modules, etc. since they are not otherwise relevant to our purposes. A nice introduction is [Gep20], with all details in [Lur17].

Definition A.28. Let C be an ∞-category. A *monad* on C is an algebra object[16] of $\mathrm{End}(C, C)$. Using the map $\mathrm{End}(C, C) \times C \to C$, we can consider (left) modules in C over a monad T.

As in the ordinary case, an adjoint pair $(L, R) : C \rightleftarrows D$ gives rise to a monad $T = R \circ L$. We can construct a functor $D \to \mathrm{LMod}_T(C)$ as in [Gep20, Remark 4.1.5].

Definition A.29. An adjunction (L, R) is *monadic* if $D \to \mathrm{LMod}_T(C)$ is an equivalence.

This now allows us speak of effective descent morphisms in ∞-categories, which is necessary to define Galois contexts.

A.7 Quasicategories

Due to the close relation between topological spaces and ∞-categories, it is not surprising that certain simplicial sets provide a good model for ∞-categories.

Definition A.30 (Simplex category)**.** The *simplex category* Δ is the ordinary 1-category with objects $[0], [1], [2], \ldots$ and morphisms the order-preserving maps between the posets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$.

Definition A.31 (Simplicial set)**.** A *simplicial set* is a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. The category of simplicial sets sSet is the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. The *standard* n-simplex is the functor represented by $[n]$, i.e. $\Delta^n := \mathrm{Hom}_\Delta(-, [n])$ The set of n-simplices of X is the set $X_n := X([n])$. By the Yoneda lemma, this is the same as $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X)$.

We consider two main sources of simplicial sets:

(1) Given an ordinary 1-category C, we can consider its *nerve* $N(C)$, which is a simplicial set with n-simplices given by sequences of composable arrows

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

in C, face maps given by composing two arrows or omitting them at the edges and degeneracy maps given by inserting identities.

[16] this is defined very similarly to definition 3.24, although the category $N(\mathrm{Fin}_*)$ is replaced by Δ^{op}, see [Gep20, Definition 2.3.14]

(2) Given a topological space X, we can consider its *singular simplicial set* $\mathrm{Sing}(X)$, with n-simplices given by $\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$, where $|\Delta^n|$ denotes the standard geometric n-simplex. The face and degeneracy maps

These two constructions admit certain lifting properties. To state these, we first need to define horns.

Definition A.32. For any $n \geq 0$ and $0 \leq i \leq n$, define the *horn* $\Lambda^n_i \in \mathrm{sSet}$ as a sub-simplicial set of Δ^n with m-simplices given by those maps $\alpha\colon [m] \to [n]$ for which $[n] \not\subseteq \alpha([m]) \cup \{i\}$ [Lur26, Tag 000U]. Roughly speaking, the horns arise by omitting from Δ^n the unique non-degenerate n-simplex as well as the i-th face. There is a natural inclusion $\Lambda^n_i \to \Delta^n$.

Lemma A.33. *Given any ordinary 1-category C, the nerve admits* unique *lifts*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & N(C) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for every $0 < i < n$.

Lemma A.34. *Given any topological space X, the singular simplicial set admits (not necessarily unique) lifts*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Sing}\, X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for every $0 \leq i \leq n$.

From this point of view, a quasicategory is a common generalization of the two notions:

Definition A.35 (Quasicategory)**.** A *quasicategory* is a simplicial set X which admits (not-necessarily unique) lifts

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for every $0 < i < n$.

To construct some of the most important ∞-categories, such as the ∞-category of spaces $\mathcal{S}$ or the ∞-category of small ∞-categories Cat_∞, one uses a construction similar to the nerve called the *homotopy coherent nerve* [Lur26, Tag 00KM]. The intuition behind this construction is that the input is an ordinary 1-category C enriched in Kan complexes, and the resulting ∞-category $N^{\mathrm{hc}}(C)$ should have as 2-simplices diagrams

$$\begin{array}{ccc} x & \xrightarrow{\;h\;} & z \\ & {}_{f}\searrow \quad \nearrow_{g} & \\ & y & \end{array}$$

together with the *data* of a homotopy equivalence $g \circ f \simeq h$. The higher simplices similarly come with the data of homotopies between homotopies.

Construction A.36. Let Kan be the category of Kan complexes enriched over Kan complexes. Then $\mathcal{S} := N^{\mathrm{hc}}(\mathrm{Kan})$ is the quasicategory representing the ∞-category of spaces.

Construction A.37. Let QCat be the category of quasicategories enriched over Kan complexes. The hom-spaces are given by taking the groupoid core $\mathrm{Fun}(C, D)^{\simeq}$ whenever C, D are quasicategories. Then $\mathrm{Cat}_\infty := N^{\mathrm{hc}}(\mathrm{QCat})$ is the quasicategory representing the ∞-category of small ∞-categories.

The substantial work of Lurie on quasicategories [Lur09; Lur26] shows that quasicategories and Cat_∞ satisfy all axioms stated in the first part of this appendix.

Appendix B Pro- and ind-objects

In this appendix, we give the definition of Pro- and Ind-objects, which we make use of to define profinite groups and especially profinite groupoids.

Definition B.1. (Filtered category [KS06, Definition 3.1.1]) A category D is *filtered* if:

(1) D is nonempty,

(2) for any two object x, y in D, there is z such that there are morphisms $x \to z$ and $y \to z$, and

(3) for any pair of parallel morphisms $f, g \colon x \rightrightarrows y$, there is a morphism $h \colon y \to z$ such that $hf = hg$.

A category D is *cofiltered* if D^{op} is filtered.

Proposition B.2. *If $C \cong \mathrm{colim}_{d \in D} F(d)$ for a filtered category D and a functor $F \colon D \to \mathrm{Cat}$ with only small categories in its image, then for all objects x of C, we can find some d such that x is in the image of $F(d) \to C$.*

Proof. Since any category in the image of F is small, the colimit $\mathrm{colim}_{d \in D} F(d)$ can be built as follows:

(1) objects are equivalence classes (d, x) where $d \in D$ and $x \in F(d)$, with $(d, x) \sim (e, y)$ if there are arrows $d \to f$ and $e \to f$ such that the corresponding functors $F(d) \to F(f)$ and $F(e) \to F(f)$ map x and y to the same object in $F(f)$.

(2) morphisms $[d, x] \to [e, y]$ are equivalence classes (f, ψ) where there are arrows $\alpha\colon d \to f$ and $\beta\colon e \to f$, and $\psi\colon F(\alpha)(x) \to F(\beta)(y)$ is an arrow in $F(f)$. The equivalence relation we impose is that $\psi \sim \phi$ whenever they become equal in some further stage.

This construction works since every class of objects and morphisms is actually a set (by smallness) so we are essentially computing the colimit in Set, which we understand. The claim follows immediately. □

Definition B.3. ([KS06, Definition 6.1.1]) Let C be any category.

(1) An *ind-object* of C is an object of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ which is isomorphic to a colimit $\mathrm{colim}_{d \in D} \mathrm{Hom}_C(-, \alpha(d))$, where D is a small filtered category and $\alpha\colon D \to C$ is a functor.

(2) The full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ consisting of ind-objects is denoted $\mathrm{Ind}(C)$. There is a natural embedding $C \to \mathrm{Ind}(C)$ given by sending an object c to $\mathrm{Hom}_C(-, c)$.

(3) Dually, a *pro-object* is an object of $\mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$ of the form $\lim_{d \in D} \mathrm{Hom}_C(\alpha(d), -)$ where D is a small cofiltered category and $\alpha\colon D \to C$.

(4) The full subcategory of $\mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$ consisting of pro-objects is denoted $\mathrm{Pro}(C)$. There is a natural inclusion $C \to \mathrm{Pro}(C)$ given by sending c to $\mathrm{Hom}_C(c, -)$.

Remark B.4. Informally, we can think of ind-objects as being given by formal diagrams $D \to C$ for some filtered category D. Formally, given such a diagram $F\colon D \to C$, the limit $\mathrm{colim}_{d \in D} \mathrm{Hom}_C(-, F(d))$ is the ind-object in question.

Dually, pro-objects may be represented by diagrams over cofiltered categories.

Example B.5. Profinite groups are precisely the pro-objects of the category of finite groups.

References

[71] *Revêtements étales et groupe fondamental*. Vol. Vol. 224. Lecture Notes in Mathematics. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. Springer-Verlag, Berlin-New York, 1971, pp. xxii+447.

[BGH22] Andrew J. Blumberg, Teena Gerhardt, and Michael A. Hill, eds. *Stable categories and structured ring spectra*. Vol. 69. Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, 2022, pp. xi+426. ISBN: 978-1-009-12329-7.

[BJ01] Francis Borceux and George Janelidze. *Galois theories*. Vol. 72. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001, pp. xiv+341. ISBN: 0-521-80309-8. DOI: 10.1017/CBO9780511619939. URL: https://doi.org/10.1017/CBO9780511619939.

[CCNW26] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. *Synthetic category theory*. 2026. URL: https://sites.google.com/view/bastiaan-cnossen/home.

[Cno26] Bastiaan Cnossen. *Stable homotopy theory and higher algebra*. 2026. URL: https://sites.google.com/view/bastiaan-cnossen/home.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*. Vol. 47. Mathematical Surveys and Monographs. With an appendix by M. Cole. American Mathematical Society, Providence, RI, 1997, pp. xii+249. ISBN: 0-8218-0638-6. DOI: 10.1090/surv/047. URL: https://doi.org/10.1090/surv/047.

[For17] Timothy J. Ford. *Separable algebras*. Vol. 183. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2017, pp. xxi+637. ISBN: 978-1-4704-3770-1. DOI: 10.1090/gsm/183. URL: https://doi.org/10.1090/gsm/183.

[Gep20] David Gepner. "An introduction to higher categorical algebra". In: *Handbook of homotopy theory*. CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, [2020] ©2020, pp. 487–548. ISBN: 978-0-815-36970-7.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of multiplicative infinite loop space machines". In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2015.15.3107. URL: https://doi.org/10.2140/agt.2015.15.3107.

[Gre07] J. P. C. Greenlees. "First steps in brave new commutative algebra". In: *Interactions between homotopy theory and algebra*. Vol. 436. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 239–275. ISBN: 978-0-8218-3814-3. DOI: 10.1090/conm/436/08412. URL: https://doi.org/10.1090/conm/436/08412.

[Gre92] Cornelius Greither. *Cyclic Galois extensions of commutative rings*. Vol. 1534. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1992, pp. x+145. ISBN: 3-540-56350-4. DOI: 10.1007/BFb0089165. URL: https://doi.org/10.1007/BFb0089165.

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Hat17] Allen Hatcher. *Vector bundles and K-theory (Version 2.2)*. 2017. URL: https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html.

[Hau24] Markus Hausmann. "Extended lecture notes: equivariant homotopy theory at EAST 2024". In: (2024). URL: https://www.math.uni-bonn.de/people/hausmann/EAST.pdf.

[Hed18] Alice Hedenlund. "The Tate construction and spectral sequences". In: (2018). URL: https://www.math.ru.nl/~sagave/east2018/tate_construction_talk_EAST2018.pdf.

[Hoy18] Marc Hoyois. "Higher Galois theory". In: *J. Pure Appl. Algebra* 222.7 (2018), pp. 1859–1877. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2017.08.010. URL: https://doi.org/10.1016/j.jpaa.2017.08.010.

[HS14] Drew Heard and Vesna Stojanoska. "K-theory, reality, and duality". In: *J. K-Theory* 14.3 (2014), pp. 526–555. ISSN: 1865-2433,1865-5394. DOI: 10.1017/is014007001jkt275. URL: https://doi.org/10.1017/is014007001jkt275.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*. Vol. 332. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. x+497. ISBN: 978-3-540-27949-5; 3-540-27949-0. DOI: 10.1007/3-540-27950-4. URL: https://doi.org/10.1007/3-540-27950-4.

[Len08] Hendrik W. Lenstra Jr. *Galois theory of schemes*. 2008. URL: https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur17] Jacob Lurie. *Higher algebra*. 2017. URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[Lur26] Jacob Lurie. *Kerodon*. https://kerodon.net. 2026.

[Mat16] Akhil Mathew. "The Galois group of a stable homotopy theory". In: *Adv. Math.* 291 (2016), pp. 403–541. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2015.12.017. URL: https://doi.org/10.1016/j.aim.2015.12.017.

[Mat80] Hideyuki Matsumura. *Commutative algebra*. Second. Vol. 56. Mathematics Lecture Note Series. Benjamin/Cummings Publishing Co., Inc., Reading, MA, 1980, pp. xv+313. ISBN: 0-8053-7026-9.

[NP24] Niko Naumann and Luca Pol. "Separable commutative algebras and Galois theory in stable homotopy theories". In: *Adv. Math.* 449 (2024), Paper No. 109736, 67. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2024.109736. URL: https://doi.org/10.1016/j.aim.2024.109736.

[Rog08] John Rognes. "Galois extensions of structured ring spectra. Stably dualizable groups". In: *Mem. Amer. Math. Soc.* 192.898 (2008), pp. viii+137. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/0898. URL: https://doi.org/10.1090/memo/0898.

[Rot09] Joseph J. Rotman. *An introduction to homological algebra*. Second. Universitext. Springer, New York, 2009, pp. xiv+709. ISBN: 978-0-387-24527-0. DOI: 10.1007/b98977. URL: https://doi.org/10.1007/b98977.

[RV22] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*. Vol. 194. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022, pp. xix+759. ISBN: 978-1-108-83798-9. DOI: 10.1017/9781108936880. URL: https://doi.org/10.1017/9781108936880.

[RZ10] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Second. Vol. 40. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2010, pp. xvi+464. ISBN: 978-3-642-01641-7. DOI: 10.1007/978-3-642-01642-4. URL: https://doi.org/10.1007/978-3-642-01642-4.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2009, pp. x+270. ISBN: 978-0-521-88850-9. DOI: 10.1017/CBO9780511627064. URL: https://doi.org/10.1017/CBO9780511627064.