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On Prime Numbers in Arithmetic Progressions

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Abstract

This thesis studies the distribution of prime numbers in arithmetic progressions. Two important results are stated and proven. The first is Dirichlet's theorem on arithmetic progressions which establishes a condition under which an arithmetic progression contains infinitely many prime numbers. The second is the Prime Number Theorem for arithmetic progressions, an analogue of the Prime Number Theorem, that gives an asymptotic formula for the number of primes in a given arithmetic progression. The framework underlying these results is developed through the study of Dirichlet characters and L-functions. The thesis concludes with a brief discussion on other results regarding primes in arithmetic progressions and current open problems in the area.

Sammanfattning

Denna uppsats studerar fördelningen av primtal i aritmetiska följder. Två viktiga resultat formuleras och bevisas. Det första är Dirichlets sats om aritmetiska följder som etablerar ett villkor under vilket en aritmetisk följd innehåller oändligt många primtal. Det andra är primtalssatsen för aritmetiska följder, en analog till primtalssatsen, som ger en asymptotisk formel för antalet primtal i en given aritmetisk följd. Ramverket som underbygger dessa result utvecklas genom studiet av Dirichletkaraktärer och L-funktioner. Uppsatsen avslutas med en kort diskussion om andra resultat angående primtal i aritmetiska följder och alltjämt olösta problem i området.

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Contents

1 Introduction

The prime numbers are among the most fundamental objects in number theory, arguably in all of mathematics. That there are infinitely many prime numbers has been established since antiquity, but what is less widely known, perhaps, is that there are also infinitely many prime numbers in any arithmetic progression $q\mathbb{Z} + a$ for which a and q are coprime positive integers. This result is known as Dirichlet's theorem on arithmetic progressions, and it will be proven in Section 4. A very celebrated result on the distribution of prime numbers is the Prime Number Theorem, which states that the number of primes smaller than or equal to some x is approximately equal to $x/\log x$, for large x. It turns out that an analogous result holds in any arithmetic progression $q\mathbb{Z} + a$ where $(a, q) = 1$. Indeed, we prove in Section 5 that in such a progression, the number of primes smaller than or equal to x is asymptotic to the expression

$$\frac{1}{\varphi(q)} \frac{x}{\log x},$$

where φ denotes Euler's totient function. The thesis begins, in Section 2, with a proof of Euclid's theorem that there are infinitely many prime numbers. This proof quite closely resembles some aspects of the proof of Dirichlet's theorem on arithmetic progressions given in Section 4, and the inclusion of this proof is therefore of instructive value. The proof comes from [3]. Section 3 is devoted to background theory needed for the proofs in Sections 4 and 5, in particular the important concepts of Dirichlet characters and L-series. It is worth mentioning here that it is customary in the study of L-series (and, more generally, Dirichlet series) to denote the complex argument s by $s = \sigma + it$. We will follow this custom, and so throughout this thesis σ will always denote the real part of s and t will always denote the complex part of s. The material in this section comes mostly from [1] (Subsections 3.1-3.2), [6] (Subsection 3.3), and [3] (Subsections 3.4-3.5). The proof of Dirichlet's theorem on arithmetic progressions that is presented in Section 4 is taken from [3], although we have somewhat altered the structure of the proof to make it easier to follow. The final major section, Section 5, is comprised of a proof of the Prime Number Theorem for arithmetic progressions. This proof is taken from [11], and it is an adaptation of a proof of the regular Prime Number Theorem found in [12], which in turn is inspired by a paper by Newman ([9]). The thesis concludes in Section 6 with a brief introduction to a fairly recent result regarding arithmetic progressions entirely composed of prime numbers. We also discuss the Generalized Riemann hypothesis, which, as the name suggests, is a more general form of the famous Riemann hypothesis. It is a conjecture on the zeros of L-functions and it turns out to have implications for the distribution of prime numbers in arithmetic progressions. The material in Section 6 is sourced from various sources, and those are referred to in the text. A number of other sources are used for individual proofs or small bits of material, these are clearly referenced. One of those sources is my own bachelor's thesis, which consists chiefly of an introduction to the Riemann zeta function and a proof of the Prime Number Theorem. Finally, a quick note on notation. There will be many instances of sums of the form

$$\sum_{p} \qquad \text{or} \qquad \sum_{p \equiv a\ (q)}$$

throughout this thesis. In the former case it is understood that the sum is taken over all prime numbers, while in the latter case the sum is taken over all prime numbers that are congruent to $a \pmod q$. There will also be numerous appearances of the logarithm function. These will invariably be denoted by log and it will be understood that this refers to the natural logarithm, with base e.

2 Euclid's theorem

That the prime numbers are infinitely many has been known since the days of Euclid in ancient Greece. Euclid himself gave a particularly simple proof of the theorem that now bears his name, and this proof may be found in Theorem 1.7 in [1]. To set the tone for this thesis we will present a somewhat more advanced, but nonetheless enlightening, proof. It is due to Leonhard Euler.

Theorem 2.1. There are infinitely many prime numbers.

Proof. Consider the series

$$\sum_p \frac{1}{p}. \tag{1}$$

We will show that this series is divergent. This implies the infinitude of the primes since any sum taken over finitely many (finite) terms is convergent. We start with the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

defined for real $s > 1$. Euler showed that this function may be written as an infinite product over the prime numbers, as such

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}. \tag{3}$$

See, for example, Sats 3.22 in [10] for a proof of this. We now take logarithms on both sides of (3). Since the logarithm function is continuous and the factors in the product on the right hand side are all positive (since $|p^{-s}| < p^{-1} < 1$), we may move the logarithm inside the product which turns the latter into a series. Hence, we get

$$\log \zeta(s) = \sum_p \big(-\log\big(1 - p^{-s}\big)\big) = \sum_p \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n}, \tag{4}$$

where the second equality comes from the series expansion of the natural logarithm, which is convergent because $|p^{-s}| < 1$. Now, it is clear from (2) that $\zeta(s)$ diverges to infinity as $s \to 1$ (it approaches the harmonic series, which has been known to be divergent since the 14th century), so the same holds true for $\log \zeta(s)$. The right hand side of (4), on the other hand, tends to

$$\sum_p \sum_{n=1}^{\infty} \frac{p^{-n}}{n} = \sum_p \frac{1}{p} + \sum_p \sum_{n=2}^{\infty} \frac{p^{-n}}{n}, \tag{5}$$

as $s \to 1$. Therefore we may prove that (1) diverges by showing that the double sum on the right hand side of (5) is bounded. That this is the case follows from

$$\sum_p \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \leq \sum_p \sum_{n=2}^{\infty} p^{-n} = \sum_p \frac{1}{p(p-1)} \leq \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1.$$

So the series (1) diverges and hence there are infinitely many prime numbers. □

It is worth mentioning that this proof actually proves something stronger that just that the prime numbers are infinitely many. Indeed it says something also about "how infinite" the primes are. For example, it is well known that the sum of the reciprocals of all square numbers, that is, the series $\sum_{n=1}^{\infty} n^{-2}$, is convergent, so in a sense the fact that the analogous series over the prime numbers diverges implies that there are more primes than squares. What is not known is whether there are infinitely many twin primes (pairs of prime numbers with a difference of precisely 2), but it is conjectured that so is the case. What has been proven however is that the sum

$$\sum_{p,p+2} \left(\frac{1}{p} + \frac{1}{p+2} \right),$$

taken over all pairs of twin primes, is convergent. This result, known as *Brun's theorem* after the Norwegian mathematician Viggo Brun who proved the convergence in 1915 (see [2] for an introduction to Brun's work) illustrates why the divergence of the series (1) is important. If the twin primes do turn out to be infinite as is believed, then the convergence of the above series implies that the twin primes become exceptionally few and far between as you go further out on the number line. Of course, if the series had diverged, then we would know with certainty that the set of twin primes is infinite, as is the case with the prime numbers.

3 Preliminaries

The main attractions of this thesis are proofs of Dirichlet's theorem on arithmetic progressions and of the Prime Number Theorem for arithmetic progressions. In order to prove these theorem we will first need to develop some underlying theory. We begin with the notion of Dirichlet series.

3.1 Dirichlet series

Definition 3.1. Let f be an arithmetic function, that is, a function $f : \mathbb{N} \to \mathbb{C}$. We define the *Dirichlet series* of f by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \tag{6}$$

for s a complex number. One can show that for every Dirichlet series there is an extended real number σ_c, called the *abscissa of convergence*, such that the series (6) converges for all s in the half-plane $\sigma > \sigma_c$, and diverges for all s in the half-plane $\sigma < \sigma_c$. There is also an extended real number σ_a, called the *abscissa of absolute convergence*, such that the series (6) converges absolutely for $\sigma > \sigma_a$, but not for $\sigma < \sigma_a$. Naturally, $\sigma_c \leq \sigma_a$ for any Dirichlet series. It is of course possible for the series (6) to converge everywhere, or nowhere, and in these cases we set $\sigma_c = -\infty$ and $\sigma_c = \infty$, respectively. The same goes for σ_a.

Example 3.2. Taking f to be the constant function 1 yields what is undoubtedly the most famous Dirichlet series, namely the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series is clearly absolutely convergent for $\sigma > 1$, so $\sigma_a \leq 1$. On the other hand, $s = 1$ yields the harmonic series, which we know to be divergent. Hence we must have $\sigma_c \geq 1$. Since $\sigma_c \leq \sigma_a$, we conclude that $\sigma_c = \sigma_a = 1$.

Theorem 3.3. Suppose that, for some arithmetic function f, the Dirichlet series (6) converges for $\sigma > \sigma_c$, and let $F(s)$ denote the convergent series. Then $F(s)$ is holomorphic in its half-plane of convergence $\sigma > \sigma_c$, and its derivative $F'(s)$ is represented in this half-plane by the Dirichlet series

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n) \log n}{n^s},$$

obtained by differentiating the series for $F(s)$ termwise. The same goes for all higher order derivatives.

Proof. See the proof to Theorem 11.12 in [1]. □

We recall the property of the Riemann zeta function $\zeta(s)$ that it can be represented as an infinite product taken over the prime numbers, as in (3). This is, in fact, not a property unique to the zeta function, as we show below.

Theorem 3.4. Let f be an arithmetic function and suppose that its Dirichlet series $F(s)$ converges absolutely for $\sigma > \sigma_a$. If f is a multiplicative function then its Dirichlet series can be written as

$$F(s) = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{(p^k)^s}, \tag{7}$$

for $\sigma > \sigma_a$. Furthermore, if f is completely multiplicative, then

$$F(s) = \prod_p \frac{1}{1 - f(p)p^{-s}}. \tag{8}$$

The formulas (7) and (8) are collectively known as the Euler products.

Proof. Fix a prime number p and consider the series

$$\sum_{k=0}^{\infty} \frac{f(p^k)}{(p^k)^s}. \tag{9}$$

We see that

$$\sum_{k=0}^{\infty} \left| \frac{f(p^k)}{(p^k)^s} \right| = \sum_{k=0}^{\infty} \frac{|f(p^k)|}{(p^k)^\sigma} \leq \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma},$$

and the right hand side converges for $\sigma > \sigma_a$ by assumption. Hence the series (9) is absolutely convergent, so that

$$P(x, s) = \prod_{p \leq x} \sum_{k=0}^{\infty} \frac{f(p^k)}{(p^k)^s}$$

defines a finite product of absolutely convergent series. Since the series are absolutely convergent they can be multiplied together and the terms rearranged at will. A typical term of $P(x, s)$ will therefore be of the form

$$\frac{f(p_1^{k_1})}{(p_1^{k_1})^s} \frac{f(p_2^{k_2})}{(p_2^{k_2})^s} \cdots \frac{f(p_n^{k_n})}{(p_n^{k_n})^s} = \frac{f(p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n})}{(p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n})^s},$$

using the multiplicativity of f, and where $p_1, \ldots, p_n$ are all primes smaller than or equal to x. By the fundamental theorem of arithmetic any number that only has prime factors smaller or equal to x can, uniquely, be written in the form

$$p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n},$$

and we can therefore express $P(x, s)$ as the series

$$P(x, s) = \sum_{n \in A} \frac{f(n)}{n^s},$$

where A denotes the set of all positive integers that only has prime factors smaller or equal to x. Thus,

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n^s} - P(x, s) \right| = \left| \sum_{n \notin A} \frac{f(n)}{n^s} \right|.$$

The positive integers n that are not in A are those that have at least one prime factor larger than x. These numbers are necessarily larger than x, so the right hand side above is

$$\leq \left| \sum_{n>x} \frac{f(n)}{n^s} \right| \leq \sum_{n>x} \left| \frac{f(n)}{n^s} \right| = \sum_{n>x} \frac{|f(n)|}{n^{\sigma}}.$$

The final series on the right tends to 0 as x tends to infinity, assuming that $\sigma > \sigma_a$, as $F(s)$ was assumed to converge absolutely there. Hence $P(x, s)$ tends to $\sum f(n)/n^s$ in the limit, so that

$$F(s) = \lim_{x \to \infty} \prod_{p \leq x} \sum_{k=0}^{\infty} \frac{f(p^k)}{(p^k)^s} = \prod_{p} \sum_{k=0}^{\infty} \frac{f(p^k)}{(p^k)^s}.$$

If f is completely multiplicative then $f(p^k) = f(p)^k$, and so the above becomes

$$F(s) = \prod_{p} \sum_{k=0}^{\infty} \left(\frac{f(p)}{p^s} \right)^k = \prod_{p} \frac{1}{1 - f(p)p^{-s}},$$

using the formula for the sum of a geometric series. □

3.2 Characters

In the proofs that are presented later in this thesis we will have immense use of a special kind of function called characters. We begin by introducing these in the general context of finite abelian groups.

Definition 3.5. Let G be a finite abelian group with identity element e, and with its operation written multiplicatively. A *character* on G is a function $f : G \to \mathbb{C}$, not identically equal to 0, with the multiplicative property that

$$f(ab) = f(a)f(b)$$

for all $a, b \in G$. In other words, f is a non-zero homomorphism from G to the group of complex numbers $\mathbb{C}$.

Theorem 3.6. *A group character $f : G \to \mathbb{C}$ only takes values that are roots of unity. Furthermore, if e denotes the identity element of G, then $f(e) = 1$.*

Proof. Let $c \in G$ be some element for which $f(c) \neq 0$, such an element exists since f is not the zero function. By the multiplicative property we have that

$$f(c)f(e) = f(ce) = f(c),$$

hence $f(e) = 1$ since we can cancel $f(c) \neq 0$ from both sides. Now let $d \in G$ be arbitrary. Since G is finite, there is an $n \in \mathbb{N}$ such that $d^n = e$. Hence

$$f(d)^n = f(d^n) = f(e) = 1,$$

so that $f(d)$ is some root of unity. $\square$

Definition 3.7. If one defines multiplication of characters by

$$(f_i f_j)(a) = f_i(a) f_j(a),$$

for all characters f_i, f_j on G and group elements $a \in G$, then it is easy to see that the set of characters of the finite abelian group G themselves form a group, which is also finite and abelian. This group is called the *character group* of G and is denoted by $\widehat{G}$. One sees that the identity element of $\widehat{G}$ must be the character that is identically equal to 1. It is called the *principal character* and is denoted by f_0.

Theorem 3.8. The inverse in $\widehat{G}$ of a character $f : G \to \mathbb{C}$ is the character $\overline{f}$ given by $\overline{f}(a) = \overline{f(a)}$.

Proof. Since the values of f are roots of unity by Theorem 3.6, we have, for any $a \in G$, that

$$(f\overline{f})(a) = f(a)\overline{f(a)} = |f(a)|^2 = 1.$$

so that $\overline{f}$ is the inverse of f. $\square$

The characters on a finite abelian group satisfy the following two so called *orthogonality conditions*.

Theorem 3.9. Let G be a finite abelian group with identity element e, and let $f_0 \in \widehat{G}$ be the principal character on G. Then the following hold:

(a) Fix a character $f \in \widehat{G}$. Then

$$\sum_{a \in G} f(a) = \begin{cases} |G| & \text{if } f = f_0, \\ 0 & \text{if } f \neq f_0. \end{cases}$$

(b) Fix a group element $a \in G$. Then

$$\sum_{f \in \widehat{G}} f(a) = \begin{cases} |\widehat{G}| & \text{if } a = e, \\ 0 & \text{if } a \neq e. \end{cases}$$

These results let us extract the principal character on, or the identity element of, a group.

Proof. (a) Of course, if f is the principal character, then $f(a) = 1$ for every $a \in G$, so that the sum equals $|G|$. If f is not the principal character, consider the following. Let $b \in G$ be some group element for which $f(b) \neq 1$, note that such a b must exist since the character is not the principal one. As a runs through the elements of G so does ab, thus

$$\sum_{a\in G} f(a) = \sum_{a\in G} f(ab) = \sum_{a\in G} f(a)f(b) = f(b)\sum_{a\in G} f(a).$$

Hence the sum must equal 0, since $f(b) \neq 1$.

(b) We know from Theorem 3.6 that $f(e) = 1$ for every character on G, so if $a = e$ then the sum consists of a 1 for every $f \in \widehat{G}$, and so equals $|\widehat{G}|$. Now suppose $a \neq e$. Let f_1 be a character for which $f_1(a) \neq 1$. Since the characters form a finite group under multiplication we see that $f_1 f$ runs through the entire set of characters as f does. Therefore,

$$\sum_{f\in\widehat{G}} f(a) = \sum_{f\in\widehat{G}} (f_1 f)(a) = \sum_{f\in\widehat{G}} f_1(a)f(a) = f_1(a)\sum_{f\in\widehat{G}} f(a),$$

which implies that the sum equals 0, since $f_1(a) \neq 1$.

□

Theorem 3.10. For any finite abelian group G there are as many characters on G as there are elements of G. That is, $|\widehat{G}| = |G|$.

Proof. See for example Theorem 6.8 in [1]. □

3.3 Dirichlet characters

We now consider a specific finite abelian group, namely the group $(\mathbb{Z}/q\mathbb{Z})^\times$ of reduced residue classes modulo some positive integer q. Recall that the elements of this group are those integers n in $\{1, \dots, q\}$ for which $(n, q) = 1$, and that the group operation is multiplication (mod q). The order of $(\mathbb{Z}/q\mathbb{Z})^\times$ is given by Euler's totient function $\varphi(q)$.

Definition 3.11. Let q be a positive integer and suppose that $f(n)$ is a group character on $(\mathbb{Z}/q\mathbb{Z})^\times$. We define a function $\chi : \mathbb{Z} \to \mathbb{C}$ by

$$\chi(n) = \begin{cases} f([n]) & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1, \end{cases} \tag{10}$$

where $[n]$ is the unique representative of n in $(\mathbb{Z}/q\mathbb{Z})^\times$. The function $\chi(n)$ thus defined is a special kind of arithmetic function called a *Dirichlet character of modulus q*.

The letter χ will henceforth be reserved for Dirichlet characters. For any positive integer q there is a Dirichlet character (mod q) that corresponds to the principal character. It is called *the principal Dirichlet character* (mod q), is denoted by χ_0, and is given by

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1. \end{cases} \tag{11}$$

Theorem 3.12. Let q be a positive integer. Any Dirichlet character (mod q) is completely multiplicative, and also q-periodic.

Proof. Let χ be a Dirichlet character (mod q) and let f be the group character on $(\mathbb{Z}/q\mathbb{Z})^\times$ that induces χ according to Definition 3.11. For integers a and b we know that their product ab is coprime with q if and only if both a and b are. Indeed, if $(ab, q) = 1$, then $(a, q) = (b, q) = 1$, and so

$$\chi(ab) = f(ab) = f(a)f(b) = \chi(a)\chi(b),$$

by the multiplicative property of group characters. Likewise, if $(ab, q) > 1$, then $\chi(ab)$ is 0 and the same is true of at least one of $\chi(a)$ or $\chi(b)$, hence $\chi(ab) = \chi(a)\chi(b)$. This shows that χ is completely multiplicative.

Furthermore, it is clear that $(a + q, q) = (a, q)$, so if $(a + q, q) = (a, q) = 1$, then

$$\chi(a + q) = f(a + q) = f(a) = \chi(a),$$

since f is defined on the residue classes modulo q. Also, if $(a+q, q) = (a, q) > 1$, then $\chi(a + q) = 0 = \chi(a)$. This shows that χ is q-periodic. □

Corollary 3.12.1. Any Dirichlet character χ satisfies that $\chi(-1) = \pm 1$.

Proof. Since χ is completely multiplicative, we have that

$$\chi(-1)^2 = \chi((-1)^2) = \chi(1) = 1,$$

since all characters satisfy that $\chi(1) = 1$ by Theorem 3.6. This clearly implies that $\chi(-1) = -1$ or 1. □

As was the case with the characters of a general finite abelian group, the Dirichlet characters (mod q) form a finite abelian character group for any positive integer q. This group is denoted by

$$\widehat{(\mathbb{Z}/q\mathbb{Z})^\times}.$$

The group operation is again pointwise multiplication and the neutral element is the principal Dirichlet character (mod q), defined by (11). Dirichlet characters too adhere to certain orthogonality relations. These are given below.

Theorem 3.13. Let q be a positive integer.

(a) Let χ be a Dirichlet character (mod q). Then

$$\sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0, \end{cases}$$

(b) Let $a \in (\mathbb{Z}/q\mathbb{Z})^\times$. Then

$$\sum_{\chi \in \widehat{(\mathbb{Z}/q\mathbb{Z})^\times}} \chi(a) = \begin{cases} \varphi(q) & \text{if } a = 1, \\ 0 & \text{if } a \neq 1. \end{cases}$$

Proof. This follows immediately from Theorem 3.9 once we note that $(\mathbb{Z}/q\mathbb{Z})^\times$ has order $\varphi(q)$ and that, by Theorem 3.10, this is true also of $\widehat{(\mathbb{Z}/q\mathbb{Z})^\times}$. □

3.4 L-series

Recall that for an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ we define its Dirichlet series by

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

whenever the series converges.

Definition 3.14. We noted in Definition 3.11 that a Dirichlet character χ is a kind of arithmetic function. The corresponding Dirichlet series is called an *L-series*, and is denoted by $L(s, \chi)$.

Theorem 3.15. The function $L(s, \chi)$, for any Dirichlet character χ, is holomorphic for $\sigma > 1$. Furthermore, the derivative $L'(s, \chi)$ is obtained by term-wise differentiation,

$$L'(s, \chi) = -\sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n^s}.$$

Proof. Note that

$$\sum_{n=1}^{\infty} \left| \frac{\chi(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|\chi(n)|}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

because the values of $\chi(n)$ are always roots of unity or 0. Thus $L(s, \chi)$ is absolutely convergent for any $\sigma > 1$, and so $\sigma_c \leq \sigma_a \leq 1$. The result then follows from Theorem 3.3. □

Just like for general Dirichlet series, there is a notion of an Euler product for L-series.

Theorem 3.16. For any Dirichlet character χ, its L-series can be written

$$L(s, \chi) = \prod_p \frac{1}{1 - \chi(p) p^{-s}},$$

for $\sigma > 1$.

Proof. A Dirichlet character is completely multiplicative by definition, so the result follows immediately from (8). □

Notice how this implies that $L(s,\chi) \neq 0$ whenever $\sigma > 1$, since every factor in the product for $L(s,\chi)$ is non-zero. For the principal Dirichlet character modulo q we have the following corollary.

Corollary 3.16.1. For the L-series of the principal character χ_0 we have

$$L(s,\chi_0) = \zeta(s)\prod_{p\,|\,q}(1-p^{-s}). \tag{12}$$

Proof. From the previous theorem we know that

$$L(s,\chi_0) = \prod_p \frac{1}{1-\chi_0(p)p^{-s}}.$$

By definition, $\chi_0(p)$ equals 0 whenever $(p,q) > 1$, that is, whenever p divides q, and equals 1 otherwise. Hence

$$\prod_p \frac{1}{1-\chi_0(p)p^{-s}} = \prod_{p\,\nmid\,q} \frac{1}{1-p^{-s}} = \prod_p \frac{1}{1-p^{-s}} \prod_{p\,|\,q}(1-p^{-s}) = \zeta(s)\prod_{p\,|\,q}(1-p^{-s})$$

invoking the Euler product for $\zeta(s)$. This proves the result. □

We thus see that the L-series of the principal character differs from the Riemann zeta function by multiplication of only finitely many factors. In particular, this implies that $L(s,\chi_0)$ diverges to infinity as $s \to 1$, since this is true of $\zeta(s)$ and the factors $1-p^{-1}$ are clearly non-zero for all prime divisors p of q.

3.5 Analytic continuation of L-series

So far we have defined the functions $L(s,\chi)$ only for $\sigma > 1$. In the proof of Dirichlet's theorem on arithmetic progressions we will be interested in the behaviour of $L(s,\chi)$ in a larger half-plane, namely for $\sigma > 0$. We will therefore look briefly at how one may analytically continue the function $L(s,\chi)$, for principal and non-principal χ.

Theorem 3.17. The function $L(s,\chi_0) - \frac{\varphi(q)}{q}\frac{1}{s-1}$ is holomorphic for $\sigma > 0$.

Proof. In light of the simple expression of $L(s,\chi_0)$ involving $\zeta(s)$ given by Corollary 3.16.1, we write $\zeta(s)$ in the following form:

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$

(see [10], Sats 3.27, for a rigorous derivation of this identity). The integral above converges absolutely for any $\sigma > 0$, so this defines $\zeta(s)$ as a meromorphic function in the half-plane $\sigma > 0$, with the only pole being a simple one at $s = 1$

(and which, as one readily verifies, has residue 1). By Corollary 3.16.1, this is true also of $L(s, \chi_0)$, except that the residue at $s = 1$ in this case becomes

$$\prod_{p \mid q} (1 - p^{-1}) = \frac{\varphi(q)}{q},$$

invoking the product form of the Euler totient function $\varphi(q)$ (see Theorem 2.4 in [1]). $\square$

Theorem 3.18. For any non-principal Dirichlet character χ, the function $L(s, \chi)$ can be extended to a function that is holomorphic for all $\sigma > 0$. This function shall also be denoted by $L(s, \chi)$.

Proof. Denote by $S(n)$ the partial sums

$$S(n) = \sum_{k \leq n} \chi(k).$$

Then $\chi(n) = S(n) - S(n-1)$, and thus, for $\sigma > 1$,

$$\begin{aligned} L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{S(n) - S(n-1)}{n^s} \\ &= \sum_{n=1}^{\infty} S(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\ &= \sum_{n=1}^{\infty} S(n) \int_n^{n+1} \frac{s}{x^{s+1}} dx \\ &= s \int_1^{\infty} \frac{S(x)}{x^{s+1}} dx. \end{aligned}$$

Now, since any q consecutive integers constitute a complete set of the residue classes modulo q, the sum $\sum \chi(n)$ over any q consecutive integers is 0 by Theorem 3.13 a). This implies that $S(x)$ is a bounded function of x, and hence the final integral above converges absolutely for all $\sigma > 0$. The expression for $L(s, \chi)$ above furthermore does not have any poles anywhere. We have thus extended $L(s, \chi)$ to a holomorphic function on the half-plane $\sigma > 0$. $\square$

As a note on terminology, we shall henceforth refer to the analytically continued and, for $\sigma > 0$, holomorphic (meromorphic in the case of χ_0) function $L(s, \chi)$ as the *L-function* of χ, rather than the *L-series*, terminology which shall be reserved for the representation of the function as a series, valid for $\sigma > 1$.

4 Dirichlet's theorem on arithmetic progressions

An obvious consequence of Euclid's theorem on the infinitude of the primes is that the sequence of odd numbers

$$1, 3, 5, \dots, 1 + 2n, \dots$$

contains infinitely many prime numbers, since all primes but one are odd. A question one may ask is whether there is some condition one may impose in order for this to hold for an arbitrary arithmetic progression

$$a, a + q, a + 2q, \dots, a + nq, \dots, \tag{13}$$

with $a, q \geq 1$. Of course, if $d = (a, q)$ is the greatest common divisor of a and q then every number in the sequence is divisible by d, so if $d > 1$ there cannot be any primes in the sequence other than, perhaps, a itself. In other words, a necessary condition for the arithmetic progression to contain infinitely many primes is that $d = 1$. What Dirichlet showed is the quite remarkable fact that this very condition is also sufficient, so that any arithmetic progression (13) contains infinitely many prime numbers as long as a and q are coprime. This result, now known as *Dirichlet's theorem on arithmetic progressions*, is what we will prove in this section. Specifically, we will prove that the series

$$\sum_{p \equiv a \ (q)} \frac{1}{p}, \tag{14}$$

where we sum over all primes congruent to $a \pmod q$, diverges. This is analogous to how we in Theorem 2.1 proved that there are infinitely many prime numbers by showing that the series $\sum_p p^{-1}$, taken over all primes, diverges. To make it easier to follow along in the proof, it is divided into a few smaller proofs. We begin with a lemma.

Lemma 4.1. Let a and q be coprime positive integers. Then

$$\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod q, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

where the sum is taken over all Dirichlet characters $\pmod q$.

Proof. By Theorems 3.8 and 3.12 we have that

$$\overline{\chi}(a)\chi(n) = \chi(a)^{-1}\chi(n) = \chi(a^{-1})\chi(n) = \chi(a^{-1}n).$$

The result then follows from Theorem 3.13 b) since $a^{-1}n \equiv 1 \pmod q$ if and only if $n \equiv a \pmod q$. □

Theorem 4.2. Let a and q be positive coprime integers. Then the following holds,

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\log L(s,\chi)=\sum_{p\equiv a\ (q)}\frac{1}{p^s}+\sum_{\substack{p\\ p^n\equiv a\ (q)}}\sum_{n=2}^{\infty}\frac{p^{-ns}}{n},$$

where the sum in the left hand side is taken over all Dirichlet characters of modulus q.

Proof. To begin with, consider some Dirichlet character χ of modulus q. By Theorem 3.16, if $\sigma>1$, the L-series of χ can be written as the infinite product

$$L(s,\chi)=\prod_{p}\frac{1}{1-\chi(p)p^{-s}}. \tag{16}$$

Since every factor in the product is positive and the logarithm function is continuous, taking the logarithm of both sides of (16) turns the product into a series, and so

$$\log L(s,\chi)=\sum_{p}-\log\big(1-\chi(p)p^{-s}\big)=\sum_{p}\sum_{n=1}^{\infty}\frac{\chi(p)^n p^{-ns}}{n}, \tag{17}$$

where, like in the proof of Theorem 2.1, we have used the series expansion of the logarithm which converges because $|\chi(p)p^{-s}|\le|p^{-s}|=p^{-\sigma}<p^{-1}<1$. Using the complete multiplicity of the Dirichlet characters we have that $\chi(p)^n=\chi(p^n)$, and so (17) becomes

$$\log L(s,\chi)=\sum_{p}\sum_{n=1}^{\infty}\frac{\chi(p^n)p^{-ns}}{n} \tag{18}$$

Consider now the following linear combination of logarithms of L-series,

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\log L(s,\chi).$$

Using (18) and doing a little reshuffling of the terms yields

$$\begin{aligned}\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\log L(s,\chi)&=\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\sum_{p}\sum_{n=1}^{\infty}\frac{\chi(p^n)p^{-ns}}{n}\\&=\sum_{p}\sum_{n=1}^{\infty}\frac{p^{-ns}}{n}\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\chi(p^n).\end{aligned}$$

Now using Lemma 4.1 (with p^n in place of n) we see that the previous expression can be written as

$$\sum_{\substack{p\\ p^n\equiv a\ (q)}}\sum_{n=1}^{\infty}\frac{p^{-ns}}{n}=\sum_{p\equiv a\ (q)}\frac{1}{p^s}+\sum_{\substack{p\\ p^n\equiv a\ (q)}}\sum_{n=2}^{\infty}\frac{p^{-ns}}{n},$$

so that, all in all,

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\log L(s,\chi)=\sum_{p\equiv a\,(q)}\frac{1}{p^s}+\sum_{\substack{p\\ p^n\equiv a\,(q)}}\sum_{n=2}^{\infty}\frac{p^{-ns}}{n}, \tag{19}$$

which we were to show. □

To conclude the proof of Dirichlet's theorem on arithmetic progressions we will first need to prove that $L(1,\chi)\neq 0$ for an arbitrary non-principal Dirichlet character. The proof transpires rather differently according to whether the character is real-valued or complex-valued, the proof is thus divided into two. We begin with the easier one.

Theorem 4.3. Let χ_1 be a complex Dirichlet character, in the sense that not all of its values are real. Then $L(1,\chi_1)\neq 0$.

Proof. Note that the following equality,

$$\frac{1}{\varphi(q)}\log\prod_{\chi}L(s,\chi)=\frac{1}{\varphi(q)}\sum_{\chi}\log L(s,\chi)=\sum_{\substack{p\\ p^n\equiv 1\,(q)}}\sum_{n=1}^{\infty}\frac{p^{-ns}}{n},$$

follows from Theorem 4.2 by taking $a=1$. This implies that

$$\frac{1}{\varphi(q)}\log\prod_{\chi}|L(s,\chi)|=\sum_{\substack{p\\ p^n\equiv 1\,(q)}}\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}.$$

Clearly every term in the double sum is positive. Hence $\log\prod_{\chi}|L(s,\chi)|\geq 0$, so that

$$\prod_{\chi}|L(s,\chi)|\geq 1. \tag{20}$$

Now suppose that $L(1,\chi_1)=0$. We have $\chi_1\neq\overline{\chi_1}$, and by continuity it holds that

$$L(1,\overline{\chi_1})=\overline{L(1,\chi_1)}=0$$

as well. We know that $L(s,\chi_0)$ has a simple pole at $s=1$, but for every other χ the function $L(s,\chi)$ is definitely bounded as $s\to 1$ since it is continuous there. Hence the product $\prod_{\chi}|L(s,\chi)|$ has a zero of multiplicity at least two (coming from χ_1 and $\overline{\chi_1}$) and only a simple pole at $s=1$. These zeroes cancel out the pole and the product is therefore holomorphic for $\sigma>0$ and also satisfies

$$\prod_{\chi}|L(1,\chi)|=0.$$

Naturally, this contradicts (20). Hence we conclude that $L(1,\chi_1)\neq 0$. □

Theorem 4.4. Let χ_1 be a real-valued (and non-principal) Dirichlet character. Then $L(1, \chi_1) \neq 0$.

Proof. Suppose that $L(1, \chi_1) = 0$. This zero cancels out the simple pole of $L(s, \chi_0)$ at $s = 1$ (see Corollary 3.16.1) to make the product

$$L(s, \chi_1)L(s, \chi_0)$$

holomorphic at $s = 1$, and indeed for all $\sigma > 0$. Furthermore, the function $L(2s, \chi_0)$ is holomorphic, and non-zero, for $\sigma > 1/2$. The function

$$\psi(s) = \frac{L(s, \chi_1)L(s, \chi_0)}{L(2s, \chi_0)} \tag{21}$$

is therefore holomorphic for $\sigma > 1/2$. For $\sigma > 1$ the three different L-series in the definition of ψ all have convergent Euler products. Multiplying these respective Euler products we see that

$$\psi(s) = \prod_p \frac{(1 - \chi_1(p)p^{-s})^{-1}(1 - \chi_0(p)p^{-s})^{-1}}{(1 - \chi_0(p)p^{-2s})^{-1}}.$$

For primes p that divide q we naturally have $\chi_1(p) = \chi_0(p) = 0$ so that the factor corresponding to p is 1. Those primes can therefore be disregarded. But note also that if p is such that $\chi_1(p) = -1$ then the factor corresponding to it becomes

$$\frac{1 - p^{-2s}}{(1 + p^{-s})(1 - p^{-s})} = 1,$$

hence the only primes that contribute to the product are those for which $\chi_1(p) = 1$. Thus we get that the Euler product for ψ is

$$\psi(s) = \prod_{\chi_1(p)=1} \frac{1 - p^{-2s}}{(1 - p^{-s})(1 - p^{-s})} = \prod_{\chi_1(p)=1} \frac{1 + p^{-s}}{1 - p^{-s}}.$$

We see from this Euler product that $\psi(s) \geq 0$ and that $\psi(s) \to 1$ as $\sigma \to \infty$. Hence if we express ψ as a Dirichlet series,

$$\psi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{22}$$

then $a_n \geq 0$ for all n and, in particular, $a_1 = 1$. Since ψ is holomorphic for $\sigma > 1/2$, it may be expanded in a power series about the point $s = 2$,

$$\psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!}\psi^{(m)}(2)(s - 2)^m, \tag{23}$$

with a radius of convergence of at least $\frac{3}{2}$. The coefficients $\psi^{(m)}(2)$ can be attained by formally differentiating the Dirichlet series (22) m times, by Theorem 3.3. Doing so yields

$$\psi^{(m)}(2) = (-1)^m \sum_{n=1}^{\infty} \frac{a_n(\log n)^m}{n^2}.$$

Note that, since $a_n \geq 0$, the terms

$$b_m = \sum_{n=1}^{\infty} \frac{a_n (\log n)^m}{n^2}$$

are non-negative for all m. Now (23) becomes

$$\psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!}(-1)^m b_m (s-2)^m = \sum_{m=0}^{\infty} \frac{1}{m!} b_m (2-s)^m.$$

If s is real and $\frac{1}{2} < s < 2$ then every term in the series above is non-negative, and so

$$\psi(s) \geq \psi(2) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} \geq a_1 = 1.$$

However, going back to (21), we see clearly that $\psi(s) \to 0$ as $s \to 1/2$, since $L(2s, \chi_0)$ has a pole at $s = 1/2$. Thus we have a contradiction, and so we conclude that $L(1, \chi_1) \neq 0$. □

We are finally ready to prove the main result of this section.

Theorem 4.5. Let a and q be coprime positive integers. The series

$$\sum_{p \equiv a\ (q)} \frac{1}{p}.$$

is divergent.

Proof. In Theorem 4.2 we saw that

$$\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \log L(s, \chi) = \sum_{p \equiv a\ (q)} \frac{1}{p^s} + \sum_{\substack{p \\ p^n \equiv a\ (q)}} \sum_{n=2}^{\infty} \frac{p^{-ns}}{n}. \tag{24}$$

As $s \to 1$ the double sum in above is bounded. Indeed, it is clearly smaller than

$$\sum_{p} \sum_{n=2}^{\infty} \frac{p^{-n}}{n},$$

which we showed to be bounded in the proof to Theorem 2.1. Therefore, if we can show that the left hand side of (24) diverges as $s \to 1$, then so does

$$\sum_{p \equiv a\ (q)} \frac{1}{p^s}.$$

The term in the finite sum on the left hand side of (24) that corresponds to the principal Dirichlet character is

$$\frac{1}{\varphi(q)} \overline{\chi_0}(a) \log L(s, \chi_0) = \frac{1}{\varphi(q)} \log L(s, \chi_0).$$

In Corollary 3.16.1 we saw that the following formula holds for the L-series of the principal character,

$$L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s}),$$

and therefore that $L(s, \chi_0) \to \infty$ as $s \to 1$, since this is true of $\zeta(s)$ and the other factors are obviously bounded. Hence $\log L(s, \chi_0)$ too tends to infinity as s tends to 1. So to show that the left hand side of (24) diverges as $s \to 1$ it is enough to show that every term $\log L(s, \chi)$, where $\chi \neq \chi_0$, remains bounded as $s \to 1$. Since $L(s, \chi)$ is holomorphic for $\sigma > 0$, in particular continuous at $s = 1$, this is equivalent to showing that $L(1, \chi) \neq 0$. But this was shown in Theorems 4.3 and 4.4. Hence we are done. □

We have successfully proven that the series (14) diverges for any coprime pair of positive integers a and q. Naturally, this implies that there are infinitely many prime numbers congruent to a (mod q). Hence Dirichlet's theorem in arithmetic progressions has been proven.

5 The Prime Number Theorem for arithmetic progressions

Now that we know that there are infinitely many primes in each arithmetic progression

$$a, a+q, a+2q, \dots,$$

subject to the condition $(a,q)=1$, a very natural follow-up question is to ask how many primes there are in each congruence class modulo q? A fundamental and celebrated result in analytic number theory is the *Prime Number Theorem* (PNT), which states that the number of prime numbers up to some x is asymptotic to the quotient $x/\log x$. If it was assumed that the primes, modulo q, were equidistributed, in some sense, between the $\varphi(q)$ different reduced residue classes, it would follow from the PNT that

$$\pi(x;q,a) \sim \frac{1}{\varphi(q)} \frac{x}{\log x}.$$

It is this formula, dubbed the *Prime Number Theorem for arithmetic progressions*, that we seek to prove in this section. The proof of the Prime Number Theorem for arithmetic progressions is divided into a number of smaller proofs, and we start with the hardest one.

Theorem 5.1. Let χ be any Dirichlet character. Then $L(s,\chi) \neq 0$ for $\sigma \geq 1$.

Before we proceed with this claim, we prove a simple lemma.

Lemma 5.2. Suppose that $f : \Omega \to \mathbb{C}$ has a zero of order k at $s=\alpha$ and a pole of order l at $s=\beta$. Then the logarithmic derivative f'/f has simple poles at $s=\alpha$ and $s=\beta$, with residues k and $-l$ respectively.

Proof. Write f as

$$f(s) = (s-\alpha)^k g(s)$$

where g is holomorphic and non-zero at $s=\alpha$. Then in a vicinity of α we have

$$\begin{aligned} f'(s) &= k(s-\alpha)^{k-1} g(s) + (s-\alpha)^k g'(s) \\ &= f(s)\left(\frac{k}{s-\alpha} + \frac{g'(s)}{g(s)}\right). \end{aligned}$$

Hence,

$$\frac{f'}{f}(s) = \frac{k}{s-\alpha} + \frac{g'}{g}(s),$$

whereby the first result follows since g'/g is holomorphic at $s=\alpha$. The second result is proven analogously, see Lemma 4.14 of [10] for the details. □

Proof of Theorem 5.1. For $\sigma > 1$ this follows immediately, as we have previously noted, from the fact that $L(s,\chi)$ has a convergent Euler product in that region. We also proved in Theorems 4.3 and 4.4 that $L(1,\chi) \neq 0$. It remains therefore to show that $L(1+it,\chi) \neq 0$ for $t \neq 0$. As in the proof of Theorem 4.3, we consider the function

$$\mathcal{L}(s) = \prod_{\chi} L(s,\chi),$$

where the product is taken over all Dirichlet characters modulo q. We know from Corollary 3.16.1 that $L(s,\chi_0)$ has a simple pole at $s=1$, hence $\mathcal{L}(s)$ does too. Now let $\alpha \neq 0$ and suppose that $\mathcal{L}(s)$ has a zero at $s = 1+i\alpha$ of order $\mu \geq 0$. Let also ν denote the order of the zero at $s = 1+2i\alpha$ (ν may of course be 0). Note that the function $\mathcal{L}(s)$ is real whenever the input s is real. Indeed,

$$\overline{\mathcal{L}(s)} = \prod_{\chi} \overline{L(s,\chi)} = \prod_{\chi} L(s,\overline{\chi}) = \mathcal{L}(s),$$

because as χ runs over all Dirichlet characters modulo q, so does $\overline{\chi}$. Hence the zeros of $\mathcal{L}(s)$ are symmetric with respect to the real axis, so $\mathcal{L}(s)$ has zeros also at $s = 1-i\alpha$ and $s = 1-2i\alpha$ with multiplicities μ and ν, respectively.

Now consider the Euler product for $L(s,\chi)$ given in Theorem 3.16. Taking logarithms of both sides yields,

$$\log L(s,\chi) = \sum_p -\log\big(1-\chi(p)p^{-s}\big).$$

The function $L(s,\chi)$, for any χ, is holomorphic for $\sigma > 1$, so we may differentiate the above expression to get

$$\begin{aligned} -\frac{L'}{L}(s,\chi) = -\frac{d}{ds}\log L(s,\chi) &= \sum_p \frac{d}{ds}\log\big(1-\chi(p)p^{-s}\big) \\ &= \sum_p \frac{\chi(p)p^{-s}\log p}{1-\chi(p)p^{-s}} \\ &= \sum_p \frac{\chi(p)p^{s}\log p}{p^s(p^s-\chi(p))} \\ &= \sum_p \frac{\chi(p)(p^s-\chi(p))\log p + \chi(p)^2\log p}{p^s(p^s-\chi(p))} \\ &= \sum_p \frac{\chi(p)\log p}{p^s} + \sum_p \frac{\chi(p)^2\log p}{p^s(p^s-\chi(p))} \\ &=: \phi(s,\chi) + h(s,\chi). \end{aligned} \tag{25}$$

The term-wise differentiation of the series (that is, the interchanging of the derivative and the limit in the second equality) is justified by the locally uniform convergence of the series expression for $\log L(s,\chi)$ (see Lemma 3, Section 11, of [1]). If $\chi \neq \chi_0$ then, by Theorem 3.18, $L(s,\chi)$ and by extension $L'(s,\chi)$ are holomorphic for $\sigma > 0$. So the function $(L'/L)(s,\chi)$ is meromorphic for $\sigma > 0$ with the possible poles being the zeros of $L(s,\chi)$. Furthermore, the function

$$h(s,\chi) = \sum_p \frac{\chi(p)^2\log p}{p^s(p^s-\chi(p))}$$

is holomorphic for $\sigma > 1/2$, irrespective of the character χ. Hence the function $\phi(s,\chi)$, defined by

$$\phi(s,\chi) = \sum_p \frac{\chi(p)\log p}{p^s} \tag{26}$$

for $\sigma > 1$, is meromorphically extended to $\sigma > 1/2$ by (25), with poles at the zeros of $L(s,\chi)$. The function $(L'/L)(s,\chi_0)$ (note the principal character) has a simple pole at $s = 1$ owing to the pole of $L(s,\chi_0)$ at that point, by Lemma 5.2. Hence $\phi(s,\chi_0)$ is meromorphic for $\sigma > 1/2$ with simple poles at $s = 1$ and at those zeros of $L(s,\chi_0)$ that lie in $\sigma > 1/2$. Summing (25) over all Dirichlet characters modulo q we get

$$\begin{aligned} -\sum_\chi \frac{L'}{L}(s,\chi) &= \sum_\chi \phi(s,\chi) + \sum_\chi h(s,\chi) \\ &=: \Phi(s;q) + h(s), \end{aligned}$$

where $h(s)$ is some function, holomorphic for $\sigma > 1/2$. On the other hand,

$$\begin{aligned} \sum_\chi \frac{L'}{L}(s,\chi) &= \sum_\chi \frac{d}{ds}\log L(s,\chi) \\ &= \frac{d}{ds}\log \prod_\chi L(s,\chi) \\ &= \frac{d}{ds}\log \mathcal{L}(s,\chi) \\ &= \frac{\mathcal{L}'}{\mathcal{L}}(s), \end{aligned}$$

so that

$$-\frac{\mathcal{L}'}{\mathcal{L}}(s) = \Phi(s;q) + h(s).$$

Since $h(s)$ is holomorphic for $\sigma > 1/2$ the poles of $\Phi(s;q)$ in this region and its residues at those poles are equal to those of $-\mathcal{L}'/\mathcal{L}$. The poles of $\mathcal{L}'/\mathcal{L}$ are precisely the poles and zeros of $\mathcal{L}$, and the poles are all simple. Recall that $\mathcal{L}(s)$ has a simple pole at $s = 1$ and, by assumption, zeros at $1 \pm i\alpha$ and $1 \pm 2i\alpha$ with multiplicities μ and ν, respectively. Therefore, by Lemma 5.2, we find that these residues are

$$\begin{aligned} &\mathrm{Res}_{s=1} \frac{\mathcal{L}'}{\mathcal{L}}(s) = -1, \\ &\mathrm{Res}_{s=1\pm i\alpha} \frac{\mathcal{L}'}{\mathcal{L}}(s) = \mu, \\ &\mathrm{Res}_{s=1\pm 2i\alpha} \frac{\mathcal{L}'}{\mathcal{L}}(s) = \nu. \end{aligned}$$

Hence,

$$\begin{aligned} 1 &= \operatorname{Res}_{s=1} \quad \Phi(s;q) = \lim_{\varepsilon\to 0} \varepsilon\Phi(1+\varepsilon;q), \\ -\mu &= \operatorname{Res}_{s=1\pm i\alpha} \Phi(s;q) = \lim_{\varepsilon\to 0} \varepsilon\Phi(1+\varepsilon\pm i\alpha;q), \\ -\nu &= \operatorname{Res}_{s=1\pm 2i\alpha} \Phi(s;q) = \lim_{\varepsilon\to 0} \varepsilon\Phi(1+\varepsilon\pm 2i\alpha;q). \end{aligned} \tag{27}$$

Note that, by the definitions of $\Phi(s;q)$ and $\phi(s,\chi)$, and by Lemma 4.1, we have

$$\begin{aligned} \Phi(s;q) = \sum_{\chi} \phi(s,\chi) &= \sum_{\chi}\sum_{p} \frac{\chi(p)\log p}{p^s} \\ &= \sum_{p} \frac{\log p}{p^s} \sum_{\chi} \chi(p) \\ &= \sum_{p\equiv 1\,(q)} \frac{\varphi(q)\log p}{p^s}, \end{aligned} \tag{28}$$

for $\sigma > 1$. Now suppose that $\varepsilon > 0$ is small and let us sum the values of $\Phi(s;q)$ at points close to its five poles with binomial coefficients $1, 4, 6, 4, 1$. Then we see that

$$\begin{aligned} \sum_{r=0}^{4} \binom{r}{4} \Phi(1+\varepsilon+(r-2)i\alpha;q) &= \sum_{r=0}^{4} \binom{r}{4} \sum_{p\equiv 1\,(q)} \frac{\varphi(q)\log p}{p^{1+\varepsilon+(r-2)i\alpha}} \\ &= \sum_{p\equiv 1\,(q)} \frac{\varphi(q)\log p}{p^{1+\varepsilon}} \sum_{r=0}^{4} \binom{r}{4} p^{(4-r)i\alpha/2} p^{-ri\alpha/2} \\ &= \sum_{p\equiv 1\,(q)} \frac{\varphi(q)\log p}{p^{1+\varepsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4 \\ &\geq 0 \end{aligned}$$

because the value inside the brackets is real. On the other hand, multiplying the left hand side by ε and letting $\varepsilon \to 0^+$, (27) implies that

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \sum_{r=0}^{4} \binom{r}{4} \varepsilon\Phi(1+\varepsilon+(r-2)i\alpha;q) &= -\nu - 4\mu + 6 - 4\mu - \nu \\ &= 6 - 8\mu - 2\nu. \end{aligned}$$

Since this must be greater or equal to 0, and $\mu, \nu \geq 0$, we must have $\mu = 0$. That is to say, the zero of $\mathcal{L}(s)$ at $s = 1 + i\alpha$ has multiplicity 0, i.e., there is no zero there at all. Since α was an arbitrary non-zero real number, this proves that $\mathcal{L}(s)$, and hence $L(s,\chi)$, cannot have any zeroes for $\sigma \geq 1$. □

Theorem 5.3. Let a and q be positive coprime integers, and define, for $\sigma > 1$,

$$\Phi(s;q,a) = \sum_{\chi} \overline{\chi(a)}\phi(s,\chi),$$

where the sum is taken over all Dirichlet characters $\pmod q$, and where $\phi(s,\chi)$ is defined by (26). Then the function

$$\Phi(s;q,a) - \frac{1}{s-1}$$

is holomorphic for $\sigma \geq 1$.

Proof. By the very definition of $\Phi(s;q,a)$ we have

$$\Phi(s;q,a) = \sum_{\chi \neq \chi_0} \overline{\chi(a)}\phi(s,\chi) + \phi(s,\chi_0).$$

Equation (25) shows that $\phi(s,\chi)$ is holomorphic for $\sigma \geq 1$ since $L(s,\chi)$ is holomorphic and non-zero in that region by Theorems 3.18 and 5.1, and because $h(s,\chi)$ is holomorphic for $\sigma > 1/2$. Furthermore, $\phi(s,\chi_0)$ is meromorphic for $\sigma \geq 1$ with only a simple pole at $s = 1$. By Lemma 5.2 the residue at the pole is 1 since it is a simple pole of $L(s,\chi_0)$. Hence $\phi(s,\chi_0) - \frac{1}{s-1}$ is holomorphic for $\sigma \geq 1$, and the result follows. □

Theorem 5.4. Let a and q be coprime positive integers and define the function

$$\theta(x;q,a) = \varphi(q) \sum_{\substack{p \leq x \\ p \equiv a\ (q)}} \log p.$$

Then $\theta(x;q,a) = O(x)$, where the implied constant depends on q.

Proof. Clearly,

$$\theta(x;q,a) \leq \varphi(q) \sum_{p \leq x} \log p =: \varphi(q)\theta(x),$$

thus it suffices to show that $\theta(x) = O(x)$. For any $n \in \mathbb{N}$, we have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \geq \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \geq \prod_{n<p\leq 2n} p,$$

where the last inequality comes from the fact that every prime between n and $2n$ appears in the prime factorization of the numerator but not in that of the denominator, hence they all divide $\binom{2n}{n}$. Taking logarithms and setting $n = 2^{k-1}$ for some k, we get

$$\sum_{2^{k-1}<p\leq 2^k} \log p \leq 2^k \log 2,$$

which implies that

$$\sum_{p \leq 2^k} \log p \leq (2^k + 2^{k-1} + \ldots + 1)\log 2 < 2^{k+1} \log 2.$$

Hence,

$$\sum_{p \leq x} \log p \leq Cx$$

for any $C \geq 2\log 2$, which shows that $\theta(x) = O(x)$. □

The following theorem will be used in the proof of Theorem 5.6. See [12] for the proof.

Theorem 5.5 (Analytic Theorem)**.** Let $f(t)$, defined for $t \geq 0$, be a bounded and locally integrable function and suppose that the function

$$g(s) = \int_0^\infty f(t)e^{-st}dt, \qquad \mathrm{Re}(s) > 0,$$

extends to a holomorphic function on an open neighbourhood of the closed half-plane $\mathrm{Re}(s) \geq 0$. Then

$$\int_0^\infty f(t)dt$$

converges, and equals $g(0)$.

Theorem 5.6. The integral

$$\int_1^\infty \frac{\theta(x;q,a) - x}{x^2}dx$$

is convergent.

Proof. Let $\overline{a}$ be a multiplicative inverse of $a \pmod q$. Then $\overline{\chi(a)} = \chi(\overline{a})$ and by the definition of $\Phi(s;q,a)$ and a calculation analogous to (28) we see that

$$\Phi(s;q,a) = \sum_{\chi} \overline{\chi(a)}\phi(s,\chi) = \sum_{p \equiv a\ (q)} \frac{\varphi(q)\log p}{p^s}.$$

From the definition we see that $\theta(x;q,a)$ has jumps of size $\varphi(q)\log p$ at every $x = p$ where $p \equiv a \pmod q$. Hence the sum above can be written as the Riemann-Stieltjes integral

$$\Phi(s;q,a) = \int_1^\infty \frac{d\theta(x;q,a)}{x^s}.$$

Integrations by parts yields

$$\begin{aligned}\int_1^\infty \frac{d\theta(x;q,a)}{x^s} &= \frac{\theta(x;q,a)}{x^s}\bigg|_1^\infty - \int_1^\infty \theta(x;q,a)d\left(\frac{1}{x^s}\right) \\ &= \frac{\theta(x;q,a)}{x^s}\bigg|_1^\infty + s\int_1^\infty \frac{\theta(x;q,a)}{x^{s+1}}dx.\end{aligned}$$

Since $\theta(x;q,a) = O(x)$ by Theorem 5.4 and $\theta(1;q,a) = 0$ by definition, the first term in the last expression is zero as long as $\sigma > 1$. Therefore, with the substitution $x = e^t$, we finally get

$$\Phi(s;q,a) = s\int_0^\infty \frac{\theta(e^t;q,a)}{e^{t(s+1)}}e^t dt = s\int_0^\infty e^{-st}\theta(e^t;q,a)dt. \tag{29}$$

Now consider the function $f(t) = \theta(e^t; q, a)e^{-t} - 1$. From Theorem 5.4 we know that $\theta(e^t; q, a) = O(e^t)$ and so f is bounded. It is also locally integrable since it is discontinuous only at a discrete set of points. Furthermore, by (29),

$$g(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-(s+1)t}\theta(e^t; q, a)dt - \int_0^\infty e^{-st}dt$$
$$= \frac{\Phi(s+1; q, a)}{s+1} - \frac{1}{s}.$$

Using the result from Theorem 5.3, this means that $g(s)$ is holomorphic for $\sigma \geq 0$. Hence we can use the Analytic Theorem (Theorem 5.5) to conclude that

$$g(0) = \int_0^\infty f(t)dt = \int_0^\infty (\theta(e^t; q, a)e^{-t} - 1)dt$$
$$= \int_0^\infty \frac{\theta(e^t; q, a) - e^t}{e^t}dt$$
$$= \int_1^\infty \frac{\theta(x; q, a) - x}{x^2}dx$$

converges. □

Theorem 5.7. *$\theta(x; q, a) \sim x$.*

Proof. Let $\lambda > 1$ and assume that there are arbitrarily large x for which $\theta(x; q, a)/x \geq \lambda$. For such x we see, since $\theta(\lambda x; q, a) \geq \theta(x; q, a) \geq \lambda x$, that

$$\int_x^{\lambda x} \frac{\theta(t; q, a) - t}{t^2}dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt = \int_1^\lambda \frac{\lambda - s}{s^2}ds > 0,$$

where the equality follows from the substitution $t = sx$. But this contradicts Theorem 5.6, because the convergence of that integral necessitates that the contribution above tend to zero as x tends to infinity. Conversely, let $\omega < 1$ and assume that there are arbitrarily large x such that $\theta(x; q, a)/x \leq \omega$. Then

$$\int_{\omega x}^{x} \frac{\theta(t; q, a) - t}{t^2}dx \leq \int_{\omega x}^{x} \frac{\omega x - t}{t^2}dt = \int_\omega^1 \frac{\omega - s}{s^2}ds < 0$$

for such x. Again, this contradicts Theorem 5.6. This proves that $\theta(x; q, a)/x \to 1$ as $x \to \infty$, that is, $\theta(x; q, a) \sim x$. □

We are now ready to deduce the main result.

Theorem 5.8.

$$\pi(x, q, a) \sim \frac{1}{\varphi(q)} \frac{x}{\log x}.$$

Proof. By the definition of $\theta(x; q, a)$ we have

$$\theta(x; q, a) = \varphi(q) \sum_{\substack{p \leq x \\ p \equiv a\ (q)}} \log p \leq \varphi(q) \sum_{\substack{p \leq x \\ p \equiv a\ (q)}} \log x = \varphi(q)\pi(x; q, a) \log x.$$

Hence

$$\frac{\varphi(q)\pi(x;q,a)\log x}{x} \geq \frac{\theta(x;q,a)}{x} \to 1 \tag{30}$$

as $x \to \infty$, by Theorem 5.7. On the other hand, for any $0 < \varepsilon < 1$ we have

$$\begin{aligned}
\theta(x;q,a) &\geq \varphi(q) \sum_{\substack{x^{1-\varepsilon} \leq p \leq x \\ p \equiv a\ (q)}} \log p \\
&\geq \varphi(q) \sum_{\substack{x^{1-\varepsilon} \leq p \leq x \\ p \equiv a\ (q)}} (1-\varepsilon)\log x \\
&= \varphi(q)(1-\varepsilon)\log x\big(\pi(x;q,a) + O(x^{1-\varepsilon})\big)
\end{aligned}$$

since clearly $\pi(x^{1-\varepsilon};q,a) = O(x^{1-\varepsilon})$. Then

$$\frac{\varphi(q)\pi(x;q,a)\log x}{x} \leq \frac{\theta(x;q,a)}{(1-\varepsilon)x} + O_q(x^{-\varepsilon}\log x) \to \frac{1}{1-\varepsilon}$$

as $x \to \infty$, since $x^{-\varepsilon}\log x \to 0$. Letting $\varepsilon \to 0^+$, this implies that

$$\frac{\varphi(q)\pi(x;q,a)\log x}{x} \leq 1 \tag{31}$$

for large enough x. Combining (30) and (31), we find that

$$1 \leq \frac{\varphi(q)\pi(x;q,a)\log x}{x} \leq 1$$

for sufficiently large x, hence the squeeze theorem implies that

$$\lim_{x\to\infty} \frac{\varphi(q)\pi(x;q,a)\log x}{x} = 1.$$

In other words,

$$\pi(x;q,a) \sim \frac{1}{\varphi(q)}\frac{x}{\log x}.$$

$\square$

6 Recent and potential future developments

6.1 Green-Tao theorem

In this thesis we have presented a proof of Dirichlet's theorem on arithmetic progressions, which states that any arithmetic progression where the initial term and the common difference are coprime contains infinitely many prime numbers. Of course, this is not to say that every number in such a progression is prime. In fact, this could not possibly be the case, because one can show that there exist arbitrarily long sequences of composite numbers, so any arithmetic progression will eventually contain a composite number. However, what can be shown is that, for any positive integer k, there exist arithmetic progressions of length k that only contain prime numbers. This is known as the *Green-Tao theorem*, and it was proven in 2004 by Ben Green and Terence Tao (see [5] for the original paper which proves the theorem). Actually, the paper proves a slightly stronger claim. We say that a subset A of the prime numbers is of *positive relative upper density* if

$$\limsup_{N\to\infty} \frac{|A \cap [1, N]|}{\pi(N)} > 0,$$

where $\pi(N)$ is the prime counting function. Then the stronger claim (which is referred to as *Szemerédi's theorem in the primes* after a theorem by Endre Szemerédi) is that any subset A of the prime numbers of positive relative upper density contains contains infinitely many arithmetic progressions of length k, for any k. Taking A to be the set of every prime number produces the special case that is the Green-Tao theorem (indeed, in that case $|A \cap [1, N]| = \pi(N)$, so the limit equals 1 and A is thus of positive relative upper density).

6.2 The Generalized Riemann hypothesis (GRH)

In Section 3.5 we showed how the L-series $L(s, \chi)$, initially defined for $\sigma > 1$, can be analytically continued to a function that is meromorphic for all $\sigma > 0$, with a pole at $s = 1$ only if χ is the principal character. This extension was perfectly sufficient for our purposes, but it turns out that the L-function $L(s, \chi)$ can be analytically continued much further than that. Indeed, one can show that $L(s, \chi)$ extends to an entire function, that is, one that is holomorphic in the entire plane, if χ is non-principal. The function $L(s, \chi_0)$ on the other hand can be analytically continued to a function that is meromorphic in the entire plane, still with a pole only at $s = 1$ (see Chapter VII in [8] for details). This is analogous to how the Riemann zeta function $\zeta(s)$ is extended to a function meromorphic in the entire plane with only a single pole at $s = 1$.

Speaking of the Riemann zeta function, one can show that it satisfies a certain functional equation. From this equation it is immediately obvious that it has zeros at every even negative integer, these are the so called trivial zeros of the Riemann zeta function. It follows also from the functional equation that the non-trivial zeros of $\zeta(s)$ lie in the critical strip $0 < \sigma < 1$ and that they are symmetric about the line $\sigma = 1/2$. The Riemann hypothesis is the conjecture that every non-trivial zero lies on the line $\sigma = 1/2$. Were this conjecture to be true, one can show that the error term in the Prime Number Theorem is the best possible, in a certain sense (see Section 5 of [10] for a more detailed discussion). As it turns out, all of this applies more generally to L-functions. We say that a Dirichlet character χ of modulus q is *imprimitive* if it is induced by a Dirichlet character χ' of smaller modulus, in the sense that

$$\chi(n) = \begin{cases} \chi'(n) & \text{if} \quad (n,q) = 1, \\ 0 & \text{if} \quad (n,q) \neq 1. \end{cases}$$

If a Dirichlet character cannot be induced by a character of smaller modulus it is *primitive*. The L-functions of primitive Dirichlet characters, like the Riemann zeta function, too satisfy a functional equation. Namely, for χ a primitive Dirichlet character of modulus q, the following one,

$$\Gamma\left(\frac{s+\kappa}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+\kappa}{2}} L(s,\chi) = \varepsilon(\chi)\Gamma\left(\frac{1-s+\kappa}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{1-s+\kappa}{2}} L(1-s,\overline{\chi}),$$

where Γ is the Gamma function, $\varepsilon(\chi)$ is a (non-zero) number easily computed from the properties of χ, and κ is 0 or 1 according to

$$\kappa = \kappa(\chi) = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1 \end{cases}$$

(recall that $\chi(-1) = \pm 1$ by Corollary 3.12.1). From this functional equation one can conclude that $L(s,\chi)$ must have zeroes at the points $-\kappa, -\kappa-2, -\kappa-4, \ldots$ and so on (these are the poles of $\Gamma(((s+\kappa)/2)$. These zeroes are analogous of the zeroes of $\zeta(s)$ at the negative even integers, and are referred to as the *trivial zeroes* of $L(s,\chi)$. Keeping with the analogy, the *non-trivial zeroes* of $L(s,\chi)$ are those in the strip $0 < \sigma < 1$ (recall that $L(s,\chi) \neq 0$ for $\sigma \geq 1$ by Theorem 5.1). It is believed that the non-trivial zeroes of $L(s,\chi)$ for any primitive Dirichlet character χ all have real part $1/2$, and this is known as the *Generalized Riemann hypothesis* (GRH). See Corollary 10.8 and the subsequent discussion in [7] for a more detailed account. The GRH is important (in the context of this thesis at least, it should be stressed that the GRH has many potential implications) because of what it tells us about the error term in the Prime Number Theorem for arithmetic progressions. The best result that has been achieved unconditionally, that is, without assuming any unproven conjectures, is given by the *Siegel-Walfisz Theorem*. It states that, given $A, B > 0$ fixed,

$$\psi(x;q,a) = \frac{1}{\varphi(q)}\frac{x}{\log x}\left(1 + O\left(\frac{1}{(\log x)^B}\right)\right)$$

if $(a,q) = 1$, where $q \leq (\log x)^A$. If one assumes that the GRH is true, however, we can substantially improve this estimate. Indeed, one can show that the GRH implies that, for any $\varepsilon > 0$,

$$\psi(x;q,a) = \frac{1}{\varphi(q)}\frac{x}{\log x} + O(x^{\frac{1}{2}+\varepsilon}),$$

for $(a,q) = 1$. Note that these formulas are given in terms of the function $\psi(x;q,a)$ rather than $\pi(x;q,a)$ with which we have dealt thus far. It is called the *second Chebyshev function* in arithmetic progressions (the *first Chebyshev function* in arithmetic progressions is the function $\theta(x;q,a)$ defined in Theorem 5.4), and is defined by

$$\psi(x;q,a) = \sum_{\substack{p^n \leq x \\ p^n \equiv a\ (q)}} \log p.$$

The second Chebyshev function shows up a lot in different proofs of the Prime Number Theorem for arithmetic progression (though, not the one we presented), because it is typically easier to work with than $\pi(x;q,a)$ (see Chapter 30 in [4] for a comprehensive exposition of the error term in the Prime Number Theorem for arithmetic progressions).

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