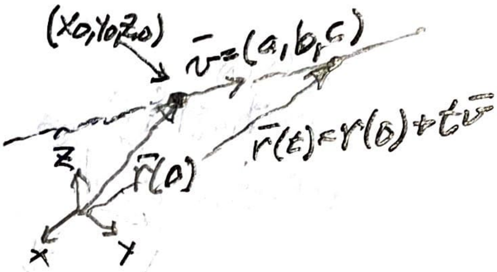
Extra räkneövning 30/11-'23, Analys B

Parametrisering av kurvor och ytor i R^3

Kurvor: $\bar{r}(t) = (x(t), y(t), z(t))$

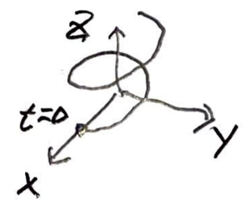
Ex: i) $(x_0 + at, y_0 + bt, z_0 + ct)$

rät linje

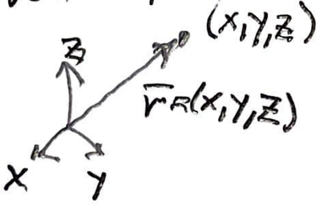


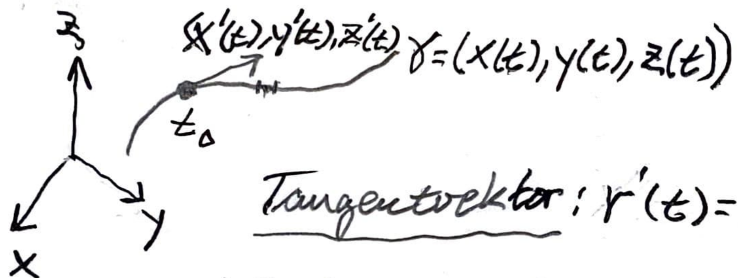
ii) $(\cos t, \sin t, t)$

"skruvlinje"



(x, y, z) en punkt i R^3 - eller en vektor $\bar{r}$





Tangentvektor: $r'(t) = (x'(t), y'(t), z'(t))$

bågelement: $ds = |r'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$

båglängd: $\int_\alpha^\beta |\bar{r}'(t)|\,dt = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t) + z'(t)^2}\,dt$

$\alpha \le t \le \beta$

Ytor: $\bar{r}(s, t) = (x(s, t), y(s, t), z(s, t))$

Ex: $\bar{r}(x, y) = (x, y, \overbrace{f(x, y)}^{z = f(x,y)})$

$\bar{r}(\theta, \varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$

$0 \le \theta \le \pi$

$0 \le \varphi \le 2\pi$

Sfär med radien R

$x^2 + y^2 + z^2 = R^2$

$(s, t) \in D \subset R^2$

Normalvektor: $\bar{N} = \bar{r}'_s \times \bar{r}'_t$ (sid 127 PB2)

Ytelement: $dS = |\bar{r}'_s \times \bar{r}'_t|\, ds\, dt$ (sid 305 i PB2)

Ex 1: $\bar{r}(x,y) = (x, y, f(x,y))$

$$\bar{r}'_x = (1, 0, f'_x)\ ; \quad \bar{r}'_y = (0, 1, f'_y)$$

$$\bar{N} = \bar{r}'_x \times \bar{r}'_y = (-f'_x, -f'_y, 1)$$ (sid 309 i PB2)

$$dS = |\bar{r}'_x \times \bar{r}'_y| = \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, dx\, dy$$

Ytan: $\iint_D dS = \iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, dx\, dy$

Ex2: sfär: $\bar{r} = (x, y, z) = R(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

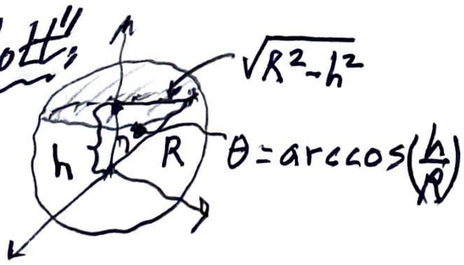
$x^2 + y^2 + z^2 = R^2$

$$\bar{N} = \bar{r}'_\theta \times r'_\varphi = \underbrace{R\sin\theta}_{\geq 0}\, \overset{(x,y,z)}{\bar{r}}$$ (sid. 127, 361 PB2)

$$dS = |\bar{r}'_\theta \times \bar{r}'_\varphi| = R\sin\theta\, \overset{\sqrt{x^2+y^2+z^2} = R}{|\bar{r}|}\, d\theta\, d\varphi = \underline{R^2 \sin\theta\, d\theta\, d\varphi}$$

Ytan: $\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} R^2 \sin\theta\, d\theta\, d\varphi = 2\pi R^2 \left[-\cos\theta\right]_0^{\pi} = \underline{4\pi R^2}$

"Kalott":



Ytan av "kalotten": $R^2 \int_{\theta=0}^{\arccos(h/R)} \int_{\varphi=0}^{2\pi} \sin\theta\, d\theta\, d\varphi =$

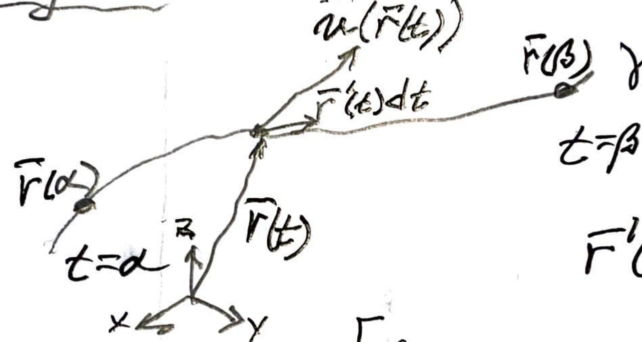
$$= 2\pi R^2 \left[-\cos\theta\right]_0^{\arccos(h/R)} = \underline{\underline{2\pi R^2\left(1 - \frac{h}{R}\right)}}$$

Vektorfält i R^3

$$\bar{u}(\underbrace{x,y,z}_{\bar{r}}) = \big(P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\big)$$

$$\left[\text{eller } \bar{u}(\bar{r}) = \big(P(\bar{r}), Q(\bar{r}), R(\bar{r})\big)\right]$$

Kurvintegral:



$$\gamma = \underbrace{(x(t), y(t), z(t))}_{\bar{r}(t)}$$

$$\bar{r}'(t) = (x'(t), y'(t), z'(t))$$

$$\int_{t=\alpha}^{\beta} \underbrace{\bar{u}(\bar{r}(t)) \cdot \bar{r}'(t)}_{\text{skalärprodukt}}\,dt = \left[\int_{\gamma} (P\,dx + Q\,dy + R\,dz)\right] =$$

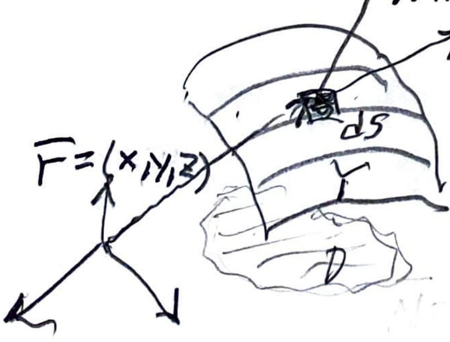
$$= \int_{\alpha}^{\beta} \Big(P(x(t),y(t),z(t))\,x'(t) + Q(x(t),y(t),z(t))\,y'(t) +$$

$$+ R(x(t),y(t),z(t))\,z'(t)\Big)\,dt$$

[Fysik: Arbete uträttat av ett kraftfält]

Flödesintegral:

$\bar{N}(\bar{r})$ = enhetsnormal $|N(\bar{r})| = 1$



$$\underbrace{\text{Flödet}}_{\Phi} \text{ genom } Y \text{ är } \iint_{Y} \overbrace{\bar{u}(\bar{r}) \cdot \bar{N}(\bar{r})}^{\text{skalärprodukt}}\,dS$$

$$\bar{r} = (x(s,t), y(s,t), z(s,t)) \Rightarrow \bar{N} = \frac{\bar{r}'_s \times \bar{r}'_t}{|\bar{r}'_s \times \bar{r}'_t|}$$

$$\phi = \iint_Y \bar{u}(\bar{r}) \cdot \bar{N}\, dS \; ; \text{ där } \bar{N} = \frac{\bar{r}'_s \times \bar{r}'_t}{|\bar{r}'_s \times \bar{r}'_t|} \text{ och}$$

$$dS = |\bar{r}'_s \times \bar{r}'_t|\, ds\, dt$$

$$\therefore \; \phi = \iint_{D} \bar{u}(\bar{r}) \cdot (\bar{r}'_s \times \bar{r}'_t)\, ds\, dt$$

$(s,t) \in D$

Konservativa vektorfält och potentialer

Ett **konservativt** vektorfält är ett fält $\bar{u}(\bar{r})$ som kan skrivas $\bar{u} = \text{grad}\, U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$ i hela sin definitionsmängd för någon funktion $U \in C^1$.

Om $\bar{u}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ är ett **nödvändigt** villkor för att $\bar{u}(\bar{r})$ skall vara konservativt att $\text{rot}(\bar{u}) = \bar{0}$, dvs. att

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \; ; \; \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \; ; \; \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Kurvintegraler av konservativa fält beror bara på start- och slutpunkt

Antag att $U(x,y,z)$ är en potential till vektorfältet $\bar{u}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

Låt γ vara en kurva i $\mathbb{R}^3$, parametriserad av $\bar{r}(t) = (x(t), y(t), z(t))$; $\alpha \le t \le \beta$

Då gäller enligt kedjeregeln:

$$\frac{d}{dt} U(\bar{r}(t)) = \underbrace{\frac{\partial U}{\partial x}(\bar{r}(t))}_{P(\bar{r}(t))} x'(t) + \underbrace{\frac{\partial U}{\partial y}(\bar{r}(t))}_{Q(\bar{r}(t))} y'(t) + \underbrace{\frac{\partial U}{\partial z}(\bar{r}(t))}_{R(\bar{r}(t))} z'(t)$$

$= \bar{u}(\bar{r}(t)) \cdot \bar{r}'(t)$ eftersom $U(\bar{r})$ är en potential till $\bar{u}$.

$$\int_\alpha^\beta \bar{u}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt = \int_\alpha^\beta \frac{d}{dt} U(\bar{r}(t))\,dt = \underline{U(\bar{r}(\beta)) - U(\bar{r}(\alpha))}$$

$\bar{r}(\beta) = (x(\beta), y(\beta), z(\beta))$ är <u>slutpunkt</u> och

$\bar{r}(\alpha)$ är <u>startpunkt</u>

10.2) Beräkna $\int_\gamma \underbrace{(3x^2+6y)}_{P}dx - \underbrace{14yz}_{Q}dy + \underbrace{20xz^2}_{R}dz$

där $\gamma = (t, t^2, t^3)$ $\quad 0 \le t \le 1$

Lösning: $\gamma = (x(t), y(t), z(t))$

$$\int_\gamma Pdx + Qdy + Rdz =$$

$$= \int_0^1 \Big((3t^2+6t^2)\cdot \overbrace{1}^{x'(t)} - 14t^2\cdot t^3 \cdot \overbrace{2t}^{y'(t)} + 20\cdot t \cdot (t^3)^2 \cdot 3t^2 \Big) dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt =$$

$$= \Big[3t^3 - 4t^7 + 6\cdot t^{10} \Big]_0^1 = \underline{\underline{5}}$$

ÖPB2 10.11) Beräkna flödet Φ av vektorfältet $\bar{u} = (x, y, z+1)$

upp genom ytan $z = 1 - x^2 - y^2$ $\quad z \geq 0$

$$\Phi = \iint_Y \bar{u}(\bar{r}(x,y)) \cdot \bar{N}(\bar{r}(x,y)) \cdot dS$$

Ytan parametriseras som $\underbrace{(x, y, 1-x^2-y^2)}_{\bar{r}(x,y)}$

där $x^2 + y^2 \le 1$

$$\bar{u}(r(x,y)) = (x, y, 2 - x^2 - y^2)$$

10.11 forts

$\bar{r}_x' = (1, 0, -2x)$

$\bar{r}_y' = (0, 1, -2y) \Rightarrow \bar{r}_x' \times \bar{r}_y' = (2x, 2y, 1)$ *)

$$\Phi = \iint_Y \bar{u}(\bar{r}(x,y)) \cdot \bar{N}\, dS = \iint_{x^2+y^2\le 1} (x, y, \underbrace{2 - x^2 - y^2}_{1+z}) \cdot (2x, 2y, 1)\, dx\, dy$$

$$= \iint_{x^2+y^2\le 1} (2x^2 + 2y^2 + 2 - x^2 - y^2)\, dx\, dy = \iint_{x^2+y^2\le 1} (x^2 + y^2 + 2)\, dx\, dy$$

$$= \{\text{Polära koordinater}\} = \int_{r=0}^{1} \int_{\varphi=0}^{2\pi} (r^2 + 2)\, r\, dr\, d\varphi =$$

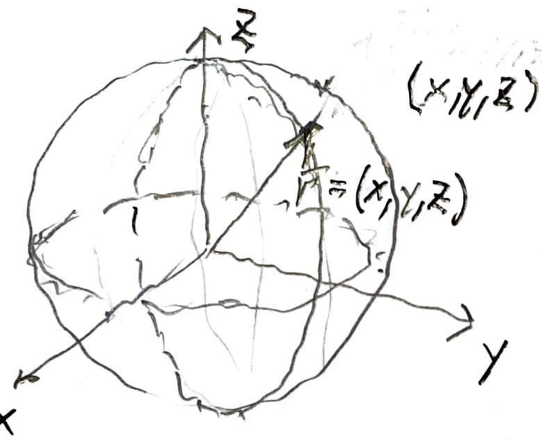
$$= 2\pi \left[\frac{r^4}{4} + r^2 \right]_0^1 = 2\pi \cdot \frac{5}{4} = \underline{\underline{\frac{5\pi}{2}}}$$

$$*) \quad \bar{r}_x' \times \bar{r}_y' = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \bar{e}_x(2x) - \bar{e}_y(-2y) + 1 \cdot \bar{e}_z =$$

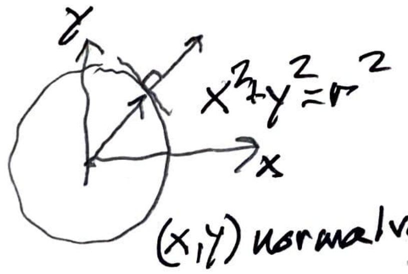
$$= \underline{(2x, 2y, 1)}$$

10.12) Beräkna flödet ϕ av fältet $\bar{F}(\bar{r}) = \frac{\bar{r}}{|\bar{r}|^3}$ ut genom sfären $x^2+y^2+z^2=R^2$

Lösning: $\bar{r}=(x,y,z)$ och $|\bar{r}|=\sqrt{x^2+y^2+z^2}=R$



jämför R^2



$\bar{r}=(x,y,z)$ är normalvektor i punkten (x,y,z) på sfären. Enhetsnormalen är $N=\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}=\frac{(x,y,z)}{R}$ (sid 364 PB2)

$$\bar{F}=\frac{\bar{r}}{|\bar{r}|^3}=\frac{(x,y,z)}{R^3} \Rightarrow \phi=\iint_{x^2+y^2+z^2=R^2} \frac{(x,y,z)\cdot(x,y,z)}{R^4}\,dS$$

$$\phi=\frac{1}{R^4}\iint_{\text{sfären}} \underbrace{(x^2+y^2+z^2)}_{R^2}\,dS=\frac{1}{R^2}\overbrace{\iint_{\text{sfären}} dS}^{4\pi R^2}=\underline{\underline{4\pi}}$$

10.23) En lampskärm har formen av en "stympad" sfär med ekvationen $x^2+y^2+z^2=2$, $z \le 1$

a) Gör en parameterframställning av L med sfäriska koordinater

b) Bestäm flödet ut genom L från fältet

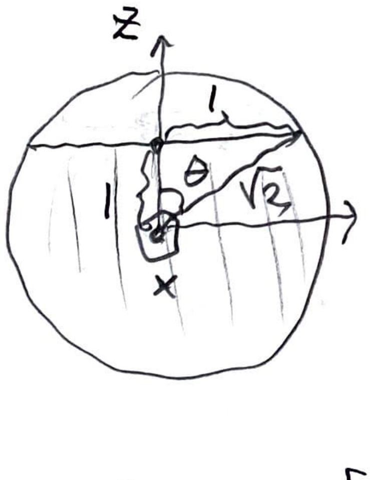
$$\bar{u}(\bar{r}) = C\frac{\bar{r}}{|\bar{r}|^3} = \frac{C(x,y,z)}{2\sqrt{2}}$$

Lösning

a) $r(\theta,\varphi) = \sqrt{2}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

b) $\bar{u} = \frac{c(x,y,z)}{2\sqrt{2}}$ och $\bar{N} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{\sqrt{2}}$

$$\Phi = \iint_L \bar{u}\cdot\bar{N}\,dS = \frac{c}{4}\iint_L \underbrace{(x^2+y^2+z^2)}_{2}\,dS = \frac{c}{2}\iint_L dS$$



$\cos\theta = \sin\theta = 1/\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore\ \Phi = \frac{c}{2}\int_{\theta=\pi/4}^{\pi}\int_{\varphi=0}^{2\pi} \overbrace{R^2\sin\theta\,d\theta\,d\varphi}^{dS} = \{R^2=2\}$$

$$\Rightarrow \Phi = c\cdot 2\pi\left[-\cos\theta\right]_{\pi/4}^{\pi} = \underline{\underline{2\pi c\left(1+\frac{1}{\sqrt{2}}\right)}}$$

10.28) a) Bestäm konstanterna a och b så att fältet

$\bar{F} = (ax^2 + xy, xy + y^2, byz + b)$ blir källfritt

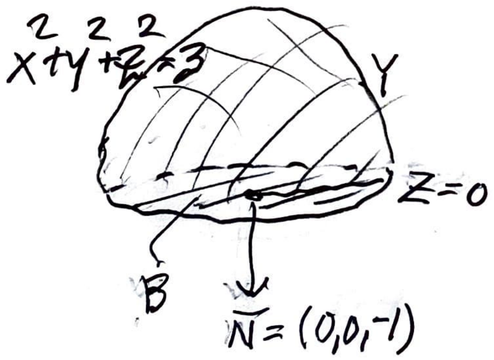
b) För dessa värden på a och b beräkna flödet av $\bar{F}$ ut genom sfären $x^2+y^2+z^2=3$, $z \geq 0$

Lösning:

a) $\operatorname{div} \bar{F} = (2ax + y + x + 2y + by) = 0$

$x(2a+1) + y(3+b) = 0$ om $\underline{a = -\frac{1}{2} \text{ och } b = -3}$

b) Gauss sats:



Enligt Gauss sats:

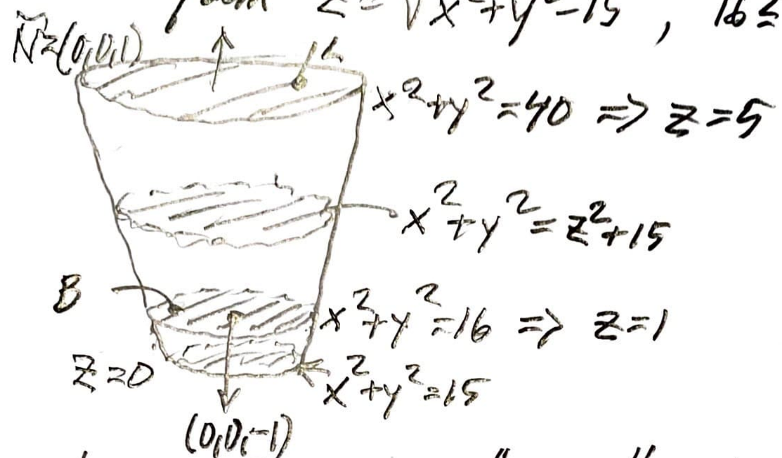
$$\iint_Y \bar{F} \cdot \bar{N}\, dS + \iint_B \bar{F} \cdot \bar{N} = 0$$

eftersom $\operatorname{div} \bar{F} = 0$

$$\iint_B \bar{F} \cdot \bar{N} = \{z=0\} = \iint_B \left(-\frac{x^2}{2} + xy, xy + y^2, -3\right) \cdot (0,0,-1)\, dS =$$

$$= 3 \cdot \pi (\sqrt{3})^2 = 9\pi \Rightarrow \iint_Y \bar{F} \cdot \bar{N}\, dS = \underline{\underline{-9\pi}}$$

10.31) Beräkna flödet av fältet $\bar{F}=(x,y,z)$ genom ytan $z=\sqrt{x^2+y^2-15}$, $16\le x^2+y^2\le 40$



$x^2+y^2=z^2+15$

$1\le z\le 5$

Slutområdet med "locket" L ($z=5$) och botten B ($z=1$)

Om Flödet genom Ytan kallas Φ gäller enligt Gauss sats:

$$\Phi + \iint_L \overset{(x,y,5)}{\bar{F}}\cdot \underbrace{\bar{N}}_{(0,0,1)}\, dS + \iint_B \overset{(x,y,1)}{\bar{F}}\cdot \underbrace{\bar{N}}_{(0,0,-1)}\, dS = \iiint_\Omega \underbrace{\operatorname{div}\bar{F}}_{3}\, dx\,dy\,dz$$

$$\Phi + 5\underbrace{\iint_L dS}_{r=\sqrt{40}\Rightarrow \pi\cdot 40} - \underbrace{\iint_B dS}_{r=4\Rightarrow \pi\cdot 16} = 3\iiint_\Omega dx\,dy\,dz = 3\int_{z=1}^{5}\left[\iint_{x^2+y^2\le\sqrt{z^2+15}} dx\,dy\right]dz =$$

$$\Phi + 200\pi - 16\pi = 3\int_{z=1}^{5}\pi(z^2+15)\,dz = 3\pi\left[\frac{z^3}{3}+15z\right]_1^5 = 304\pi$$

$$\Phi = 304\pi - 200\pi + 16\pi = \underline{\underline{120\pi}}$$

10.31 (Utan Gauss)

$$\bar{F}(\bar{r}) = (x,y,z) \quad ; \quad r(x,y) = (x,y,\sqrt{x^2+y^2-15}) \quad ; \quad 16 \le x^2+y^2 \le 40$$

$$\bar{r}_x' \times \bar{r}_y' = \left(\frac{x}{\sqrt{x^2+y^2-15}}, \frac{y}{\sqrt{x^2+y^2-15}}, -1 \right)$$

$$\text{Flödet } \Phi = \iint_Y \bar{F}(\bar{r}) \circ (\bar{r}_x' \times \bar{r}_y')\, dx\,dy$$

$$= \iint_{16 \le x^2+y^2 \le 40} \left(\frac{x^2+y^2}{\sqrt{x^2+y^2-15}} - z \right) dx\,dy = \left\{ z = \sqrt{x^2+y^2-15} \right\} =$$

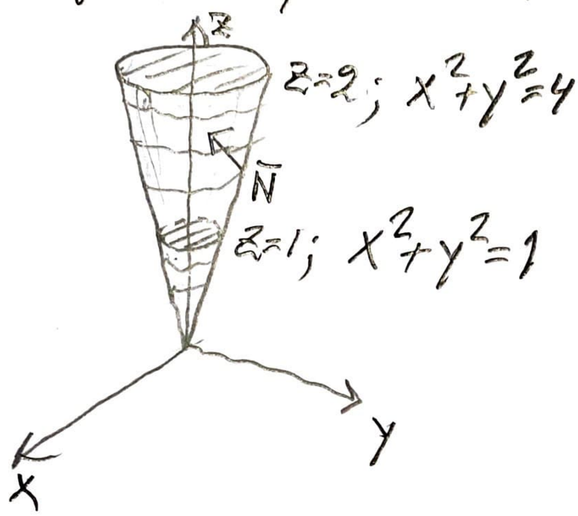
$$= \iint_{16 \le x^2+y^2 \le 40} \frac{x^2+y^2-(x^2+y^2-15)}{\sqrt{x^2+y^2-15}}\, dx\,dy = \left\{ \begin{array}{l} \text{Polära} \\ \text{koordinater} \end{array} \right\} =$$

$$= 2\pi \int_4^{\sqrt{40}} \frac{15}{\sqrt{r^2-15}}\, r\,dr = 30\pi \left[\sqrt{r^2-15} \right]_4^{\sqrt{40}} =$$

$$= 30\pi(\sqrt{25}-1) = \underline{\underline{120\pi}}$$

10.32) Beräkna flödet av $\bar{u}(x,y,z)=\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, z\right)$

genom ytan $z=\sqrt{x^2+y^2}$ $\quad 1\le z\le 2$



$\bar{N}$ har positiv z-koordinat

Parametrisera ytan $\bar{r}(x,y)=(x,y,\sqrt{x^2+y^2})$

$$\bar{N}=\bar{r}_x'\times\bar{r}_y'=\left(1,0,\frac{x}{\sqrt{x^2+y^2}}\right)\times\left(0,1,\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$=\left(-\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$$

Flödet $\Phi=\iint_Y \bar{u}\cdot\bar{N}\,dS=\iint_Y \bar{u}\cdot(\bar{r}_x'\times\bar{r}_y')\,dxdy=$

$$=\iint_{1\le x^2+y^2\le 4}\left(\frac{-x^2}{(x^2+y^2)^{3/2}}-\frac{y^2}{(x^2+y^2)^{3/2}}+z\right)dxdy=\left\{z=\sqrt{x^2+y^2}\right\}=$$

$$=\iint_{1\le x^2+y^2\le 4}\left(-\frac{1}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}\right)dxdy=\{\text{Polärakoordinater}\}=$$

$$=2\pi\int_1^2\left(-\frac{1}{r}+r\right)r\,dr=2\pi\left[-r+\frac{r^3}{3}\right]_1^2=\underline{\underline{\frac{8\pi}{3}}}$$

Ex. Är vektorfältet $\bar{F}(x,y,z) = (y^2 + (2x-y)z, x(2y-z), x(x-y))$ konservativt? Bestäm i så fall en potential till $\bar{F}$. Beräkna $\int_\gamma \bar{F} \cdot d\bar{r}$ där $\gamma = (\cos t, \sin t, t)$ där $0 \le t \le \pi/2$.

Lösning $\bar{F} \in C^\infty$, R^3 är enkelt sammanhängande $\Rightarrow$ $\bar{F}$ har en potential om $\operatorname{rot} \bar{F} = \bar{0}$.

$$\operatorname{rot} \bar{F} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+(2x-y)z & x(2y-z) & x(x-y) \end{vmatrix} =$$

$$= \bar{e}_x(-x+x) - \bar{e}_y(2x-y-(2x-y)) + \bar{e}_z(2y-z-(2y-z)) =$$

$= (0,0,0) \Rightarrow \bar{F}$ är virvelfritt $\Rightarrow$ F är konservativt

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Potential U om $\nabla U = \bar{F} \Rightarrow$

$$\Rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = y^2 + (2x-y)z & (1) \\ \dfrac{\partial U}{\partial y} = x(2y-z) & (2) \\ \dfrac{\partial U}{\partial z} = x(x-y) & (3) \end{cases}$$

$$(3) \Rightarrow U(x,y,z) = x(x-y)z + \varphi(x,y)$$

$$(2) \Rightarrow \frac{\partial U}{\partial y} = -xz + \frac{\partial \varphi}{\partial y}(x,y) = x(2y-z) \Rightarrow$$

$$\Rightarrow \frac{\partial \varphi}{\partial y}(x,y) = 2xy \Rightarrow \varphi(x,y) = xy^2 + \theta(x)$$

$$\therefore U(x,y,z) = x(x-y)z + xy^2 + \theta(x)$$

$$(1) \Rightarrow \frac{\partial U}{\partial x} = \cancel{2xz - yz} + \cancel{y^2} + \theta'(x) = \cancel{y^2} + \cancel{(2x-y)z}$$

$$\theta'(x) = 0 \Rightarrow \theta(x) = C$$

$$\therefore U(x,y,z) = x(x-y)z + xy^2 + C = \underline{x^2 z - xyz + xy^2} \; (C=0)$$

$$\gamma = (\cos t, \sin t, t) \; ; \; 0 \le t \le \pi/2$$

$$\int_\gamma \bar{F} \cdot d\bar{r} = U(0,1,\tfrac{\pi}{2}) - U(1,0,0) = 0 - 0 = \underline{\underline{0}}$$

(oberoende av vägen)