Extra räkneövning Analys B – 4 januari 2024

Tentamen 3 januari 2023

1) Beräkna kurvintegralen av vektorfältet

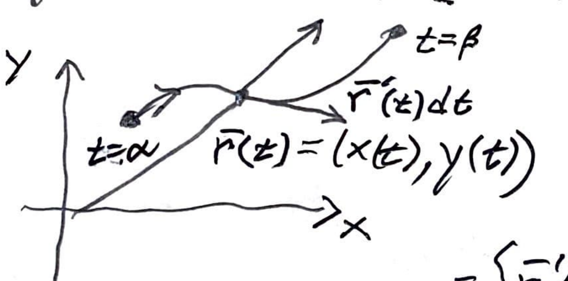
$\bar{F} = (y^2, x^2+2)$

a) från $(-1,0)$ till $(0,2)$ längs med en axelparallell ellips vars storaxel är två gånger längre än dess lillaxel.

b) längs med en cirkelbåge från $(-1,0)$ till $(0,1)$

Lösning

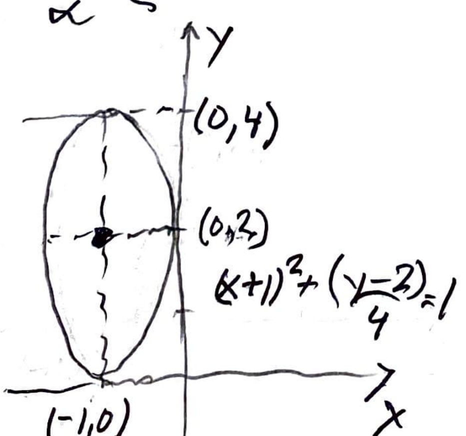
$\bar{F}(\bar{r}(t)) = (P(\bar{r}(t)), Q(\bar{r}(t)))$

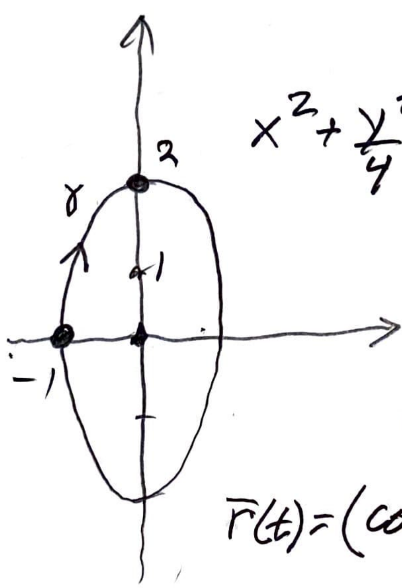


$$\int_{t=\alpha} \bar{F}(\bar{r}(t))\cdot\bar{r}'(t)\,dt =$$

$$= \{\bar{r}'(t) = (x'(t), y'(t))\} =$$

$$= \int_{\alpha}^{\beta} \Big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\Big)\,dt$$





$x^2 + \frac{y^2}{4} = 1$

$\begin{cases} x = \cos t \\ y = 2\sin t \end{cases}$

$t: \pi \to \pi/2$

$\bar{r}(t) = (\cos t, 2\sin t)$

$$I=\int_\gamma \bar{F}\cdot\bar{r}'(t)\,dt=\int_{t=\pi}^{\frac{\pi}{2}} \overbrace{(4\sin^2 t,\cos^2 t+2)}^{F(\bar{r}(t))}\cdot\underbrace{(-\sin t, 2\cos t)}_{\bar{r}'(t)}\,dt$$

$$=\int_{\pi}^{\pi/2}\left(-4\sin^3 t+2\cos^3 t+4\cos t\right)dt=$$

$$-4\int_{\pi}^{\pi/2}\sin^3 t\,dt+2\int_{\pi}^{\pi/2}\cos^3 t\,dt+\underbrace{4\left[\sin t\right]_{\pi}^{\pi/2}}_{4}$$

$$\int_{\pi}^{\pi/2}\sin^3 t\,dt=\int_{\pi}^{\pi/2}\sin^2 t\sin t\,dt=\int_{\pi}^{\pi/2}(1-\cos^2 t)\sin t\,dt=$$

$$=\left\{\begin{array}{l}\cos t=u\\ -\sin t\,dt=du\end{array}\right\}=\int_{-1}^{0}(1-u^2)(-du)=\left[u-\frac{u^3}{3}\right]_0^{-1}=-\frac{2}{3}$$

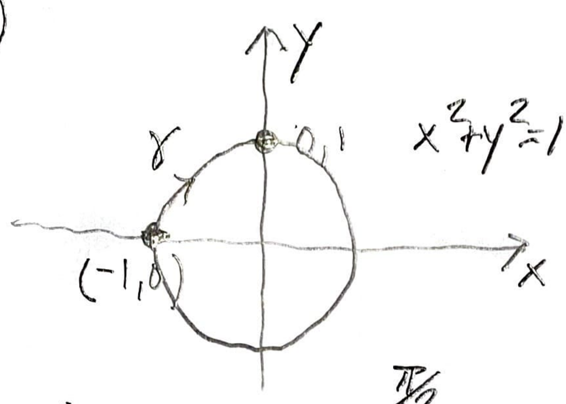
På samma sätt: $\int_{\pi}^{\pi/2}\cos^3 t\,dt=\int_{\pi}^{\pi/2}\cos^2 t\cos t\,dt=\int_{\pi}^{\pi/2}(1-\sin^2 t)\cos t\,dt=$

$$=\left\{\begin{array}{l}\sin t=u\\ \cos t\,dt=du\end{array}\right\}=\int_0^1(1-u^2)\,du=\left[u-\frac{u^3}{3}\right]_0^1=\frac{2}{3}$$

$$\therefore I=-4\cdot\left(-\frac{2}{3}\right)+2\cdot\frac{2}{3}+4=\underline{\underline{8}}$$

$\therefore$ <u>Kurvintegralen är 8</u>

b)



$x^2+y^2=1$

$x=\cos t$
$y=\sin t$
$t: \pi \to \pi/2$

$$\int_\gamma \bar{F} \cdot \bar{r}'(t)\,dt = \int_{t=\pi}^{\pi/2} (\sin^2 t, \cos^2 t + 2)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_{\pi}^{\pi/2} (-\sin^3 t + \cos^3 t + 2\cos t)\,dt =$$

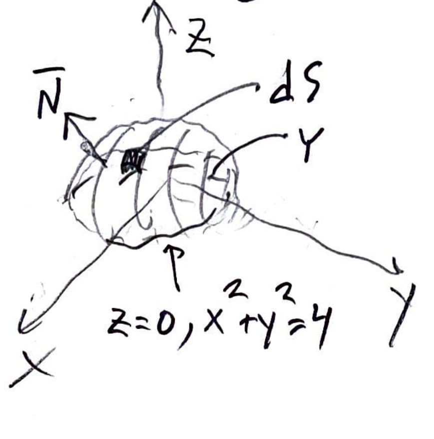
$$= -\underbrace{\int_{\pi}^{\pi/2} \sin^3 t\,dt}_{-2/3} + \underbrace{\int_{\pi}^{\pi/2} \cos^3 t\,dt}_{2/3} + \underbrace{2\left[\sin t\right]_{\pi}^{\pi/2}}_{2} =$$

$$= -\left(-\frac{2}{3}\right) + \frac{2}{3} + 2 = \underline{\underline{\frac{10}{3}}}$$

---

2) Beräkna flödet av vektorfältet $\bar{u}(x,y,z) = (yz, xz, y^2+x^2+z)$ ut ur den halva ellipsoiden $\Sigma = \{(x,y,z) \in R3;\ x^2+y^2+3z^2=4,\ z>0\}$

Lösning



Alt 1) Parametrisera ytan: $r(x,y) = \left(x, y, \sqrt{\frac{4-x^2-y^2}{3}}\right)$ $z>0$

Flödet $\phi = \iint_Y u(\bar{r}) \cdot \bar{N}(\bar{r})\, dS$

Normalvektor: $\bar{r}_x' \times \bar{r}_y' = \left(1, 0, \frac{-x}{\sqrt{3(4-x^2-y^2)}}\right) \times \left(0, 1, \frac{-y}{\sqrt{3}\sqrt{4-x^2-y^2}}\right)$

$= \left(\frac{x}{\sqrt{3(4-x^2-y^2)}}, \frac{y}{\sqrt{3}\sqrt{4-x^2-y^2}}, 1\right)$

pos z-koordinat! (ut ur ellipsoiden)

$\bar{N}(\bar{r}) = \frac{\bar{r}_x' \times \bar{r}_y'}{|\bar{r}_x' \times \bar{r}_y'|}$ normerad normalvektor

$dS =$ ytelementet $= |\bar{r}_x' \times \bar{r}_y'|\, dx\, dy$

$\left.\right\} \bar{N}(r)\, dS = \bar{r}_x' \times \bar{r}_y'\, dx\, dy$

$\therefore \Phi = \iint_Y \bar{u} \cdot \bar{N}\, dS = \iint_{0 \le x^2+y^2 \le 4} (yz, xz, y^2+x^2+z) \cdot (\bar{r}_x' \times \bar{r}_y')\, dx\, dy$

$= \left\{ z = \sqrt{\frac{4-x^2-y^2}{3}} \text{ på } Y \right\} =$

$= \iint_{0 \le x^2+y^2 \le 4} \left(\frac{y}{\sqrt{3}}\sqrt{4-x^2-y^2}, \frac{x}{\sqrt{3}}\sqrt{4-x^2-y^2}, x^2+y^2+\frac{\sqrt{4-x^2-y^2}}{\sqrt{3}}\right) \cdot (\bar{r}_x' \times \bar{r}_y')\, dx\, dy$

$$\phi = \iint_{0\le x^2+y^2\le 4} \Big(\underbrace{\frac{xy}{3} + \frac{xy}{3}}_{\frac{2xy}{3}} + x^2+y^2 + \frac{\sqrt{4-x^2-y^2}}{\sqrt{3}} \Big) dxdy =$$

$$= \{\text{Polära koord.}\} = \int_0^2\int_0^{2\pi} \Big(\underbrace{\frac{2r^2\cos\theta\sin\theta}{3}}_{=0 \text{ då } \int_0^{2\pi}\sin 2\theta\, d\theta = 0} + r^2 + \frac{\sqrt{4-r^2}}{\sqrt{3}} \Big) r\,dr\,d\theta$$

$$= 2\pi \int_0^2 \Big(r^3 + \frac{r}{\sqrt{3}}\sqrt{4-r^2} \Big) dr = 2\pi \Big[\frac{r^4}{4} - \frac{1}{3\sqrt{3}}(4-r^2)^{3/2} \Big]_0^2$$

$$2\pi \Big[\frac{16}{4} + \frac{8}{3\sqrt{3}} \Big] = \underline{\underline{8\pi \Big[1 + \frac{2}{3\sqrt{3}} \Big]}}$$

Alt 2: Gauss sats: $\phi = \iiint_K \operatorname{div} u \, dxdydz =$

$$\operatorname{div} \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0 + 0 + 1 = \underline{1}$$

$$\iint_Y \bar{u}\cdot\bar{N}\,dS + \iint_B \bar{u}\cdot\bar{N}\,dS = \iiint_K dxdydy = \mathrm{Vol}(K)$$

Volym ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $V = \frac{4\pi}{3}\cdot abc$

Volym $K = \underbrace{\frac{1}{2}}_{z>0} \cdot \frac{4\pi}{3} \cdot 2 \cdot 2 \cdot \frac{2}{\sqrt{3}} = \underline{\frac{16\pi}{3\sqrt{3}}}$

eller

$$\iiint dx\,dy\,dz = \iint_{0\le x^2+y^2\le 4}\left[\int_{z=0}^{\frac{1}{\sqrt{3}}\sqrt{4-x^2-y^2}} dz\right]dx\,dy =$$

$$= \iint_{x^2+y^2\le 4} \frac{1}{\sqrt{3}}\sqrt{4-x^2-y^2}\,dx\,dy = \{\text{Polära}\} =$$

$$= \int_0^2\int_0^{2\pi} \frac{1}{\sqrt{3}}\sqrt{4-r^2}\,r\,dr\,d\theta = \frac{2\pi}{\sqrt{3}}\int_0^2 r\sqrt{4-r^2}\,dr =$$

$$= \frac{2\pi}{\sqrt{3}}\left[-\frac{1}{3}(4-r^2)^{3/2}\right]_0^2 = \underline{\underline{\frac{16\pi}{3\sqrt{3}}}}$$

$$\iint_B \bar{u}\cdot\bar{N}\cdot dS = \{z=0 \text{ på } B,\ \bar{N}=(0,0,-1)\} =$$

$$= \iint_{0\le x^2+y^2\le 4} (0,0,x^2+y^2)\cdot(0,0,-1)\,dx\,dy = \iint_{0\le x^2+y^2\le 4} -(x^2+y^2)\,dx\,dy =$$

$$\{\text{Polära}\} = 2\pi\int_0^2(-r^3)\,dr = 2\pi\left[-\frac{r^4}{4}\right]_0^2 = \underline{\underline{-8\pi}}$$

$$\therefore \underbrace{\iint_Y \bar{u}\cdot\bar{N}\,dS}_{\phi} - 8\pi = \frac{16\pi}{3\sqrt{3}} \Rightarrow$$

$$\Rightarrow \phi = 8\pi\left(1+\frac{2}{3\sqrt{3}}\right)$$

3a) Beräkna rotationen av vektorfältet $\tilde{u}$

$$\bar{u} = \left(2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3 + \pi\sin(\pi z)\right)$$

$$\text{rot}\,\bar{u} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 + \pi\sin(\pi z) \end{vmatrix} =$$

$$= \bar{e}_x\left(12x^2y^2z^3 - 12x^2y^2z^3\right) - \bar{e}_y\left(8xy^3z^3 - 8xy^3z^3\right) +$$

$$+ \bar{e}_z\left(6xy^2z^4 - 6xy^2z^4\right) = \bar{0}$$

---

b) Beräkna kurvintegralen av $\bar{u}$ längs en rät linje från $(1,2,2)$ till $(3,5,-2)$

Vi söker en potential U till $\bar{u}$

$$\text{grad}\,U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = \bar{u} = \left(2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3 + \pi\sin(\pi z)\right)$$

$$\frac{\partial U}{\partial x} = 2xy^3z^4 \Rightarrow U(x,y,z) = x^2y^3z^4 + \varphi(y,z)$$

$$\frac{\partial U}{\partial y} = 3x^2y^2z^4 + \frac{\partial}{\partial y}\varphi(y,z) = 3x^2y^2z^4 \Rightarrow \varphi(x,y) = \theta(z)$$

$$\therefore U(x,y,z) = x^2y^3z^4 + \theta(z)$$

$$\frac{\partial U}{\partial z} = 4x^2y^3z^3 + \theta'(z) = 4x^2y^3z^3 + \pi\sin(\pi z)$$

$$\theta'(z) = \pi \sin(\pi z) \Rightarrow \theta(z) = -\cos(\pi z)$$

$$U(x,y,z) = x^2y^3z^4 - \cos(\pi z)$$

$$\int_\gamma \bar{u}\cdot d\bar{r} = U(3,5,-2) - U(1,2,2) =$$

$$= \underbrace{3^2\cdot 5^3\cdot(-2)^4 - \overbrace{\cos(-2\pi)}^{=1}}_{U(3,5,-2)} - \underbrace{\left[1^2\cdot 2^3\cdot 2^4 - \overbrace{\cos(2\pi)}^{=1}\right]}_{U(1,2,2)} =$$

$$= \underline{17872}$$

---

4) Funktionen $u(x,y) = x^3 - 3xy^2 - x$ är realdelen av en analytisk funktion $f(z)$. Vilken av följande funktionen kan vara imaginärdel till $f(z)$?

a) $v(x,y) = -y^3 + 3x^2y^2 - y$

b) $v(x,y) = -y^3 + 3x^2y^2 + y$

c) $v(x,y) = -y^3 + 3x^2y - y$

d) $v(x,y) = x^{10}y^{100} - xy + \sin(x+y^9)$

En analytisk funktion $f(z)=u+iv$ uppfyller Cauchy-Riemanns ekv: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ och $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}=3x^2-3y^2-1$, $\frac{\partial u}{\partial y}=-6xy$

a) $\frac{\partial v}{\partial x}=6xy^2$; $\frac{\partial v}{\partial y}=-3y^2+6x^2y-1$

b) $\frac{\partial v}{\partial x}=6xy^2$; $\frac{\partial v}{\partial y}=-3y^2+6x^2y+1$

$\boxed{\text{c) } \frac{\partial v}{\partial x}=6xy \; ; \; \frac{\partial v}{\partial y}=-3y^2+3x^2-1}$ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

d) $\frac{\partial v}{\partial x}=10x^9y^{100}-y+\cos(x+y^9)$; $\frac{\partial v}{\partial y}=100x^{10}y^{99}-x+9y^8\cos(x+y^9)$

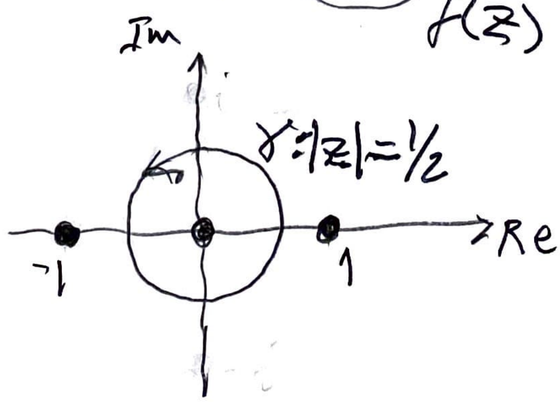
Alternativ c) $v(x,y)=-y^3+3x^2y-y$ kan vara imaginärdel till $f(z)$

---

b) Beräkna kurvintegralen $\int_\gamma \underbrace{\frac{e^z}{z^3-z}}_{f(z)}dz$ där

$\gamma=\{z\in\mathbb{C};\ |z|=1/2\}$

$f(z)=\frac{e^z}{z(z^2-1)}$



Endast polen $z=0$ ligger innanför γ.

$$\int_\gamma \frac{e^z}{z(z^2-1)}\,dz = \int_\gamma \frac{g(z)}{z}\,dz = 2\pi i g(0)$$

$g(z)$

$$g(0) = \frac{e^0}{-1} = -1 \Rightarrow \int_\gamma f(z)\,dz = \underline{\underline{-2\pi i}}$$

Lars M.