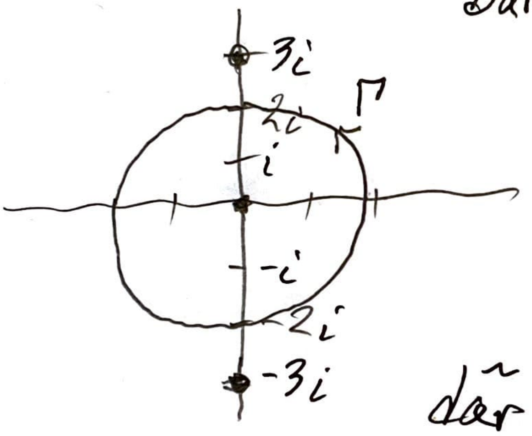
Extra räkneövning 20/12-'23, Analys B

K6) Beräkna $\int_\Gamma \underbrace{\frac{\cos z}{z^3+9z}}_{f(z)} dz$ där Γ är cirkeln $|z|=2$ orienterad moturs.

$$f(z) = \frac{\cos(z)}{z(z+3i)(z-3i)};$$

analytisk över allt utom i 0 och $\pm 3i$



Bara $z=0$ ligger innanför Γ

$$\int_\Gamma \frac{\overbrace{\cos(z)\,dz}^{h(z)}}{z(z+3i)(z-3i)} = \int_\Gamma \frac{h(z)dz}{z}$$

där $h(z) = \frac{\cos z}{(z+3i)(z-3i)}$

Cauchy's integralformel: $\int_\Gamma \frac{h(z)}{z} dz = 2\pi i h(0)$

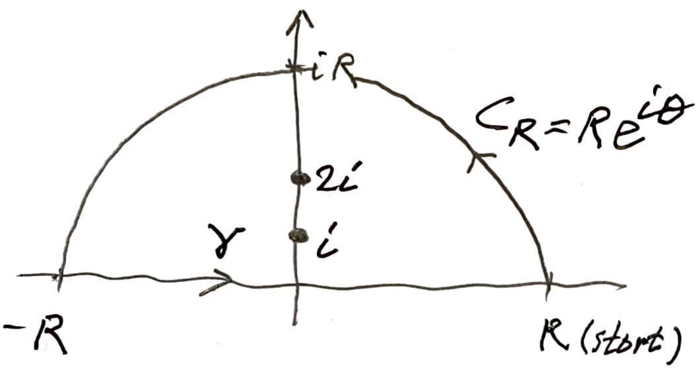
$$h(0) = \frac{\cos 0}{(3i)(-3i)} = \frac{1}{9} \Rightarrow \int_\Gamma f(z)dz = \underline{\underline{\frac{2\pi i}{9}}}$$

K8)

Beräkna den generaliserade integralen

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

Lösning: Integrera $f(z)$ längs konturen Γ i det komplex talplanet.



$$f(z) = \frac{1}{(z-i)(z+i)(z-2i)(z+2i)}$$

$f(z)$ är analytisk överallt utom i $\pm i$ och $\pm 2i$.

$\Gamma = \gamma + C_R$; $z = x$ på Re-axeln

$$\int_\Gamma f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i\left[\mathrm{Res}_f(i) + \mathrm{Res}_f(2i)\right]$$

Låt $R \to \infty \Rightarrow \underbrace{\int_{-\infty}^{\infty} f(x)dx}_{\text{Sökes}} + \underbrace{\lim_{R\to\infty} \int_{C_R} f(z)dz}_{\text{Visa att detta} \to 0} = 2\pi i\left[\mathrm{Res}_f(i) + \mathrm{Res}_f(2i)\right]$

Pol $z = i$

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{h(z)}{z-i}dz = \left\{ h(z) = \frac{1}{(z+i)(z^2+4)} \right\} = 2\pi i\, h(i)$$

$$= 2\pi i\left[\frac{1}{2i(3)}\right] = \underline{\underline{\frac{\pi}{3}}}$$

<u>Pol $z=2i$:</u>

$$\int_\Gamma f(z)\,dz = \int_\Gamma \frac{g(z)}{z-2i}\,dz = \left\{ g(z) = \frac{1}{(z^2+1)(z+2i)} \right\} =$$

$$= 2\pi i\, g(2i) = 2\pi i \left[\frac{1}{(-3)(4i)} \right] = -\underline{\underline{\frac{\pi}{6}}}$$

$$\therefore \int_\Gamma f(z)\,dz = \frac{\pi}{3} - \frac{\pi}{6} = \underline{\underline{\frac{\pi}{6}}}$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \int_{C_R} |f(z)|\,|dz| = \int_{C_R} \frac{|dz|}{|z^2+1|\,|z^2+4|} \leq$$

$$\left\{ z = Re^{i\theta} \text{ på } C_R,\ dz = Rie^{i\theta}\,d\theta,\ |dz| = R\,d\theta \right\}$$

$$\leq \int_0^\pi \frac{R\,d\theta}{\big||z|^2-1\big|\,\big||z|^2-4\big|} = \left\{ |z| = R \text{ på } C_R \right\} =$$

$$= \int_0^\pi \frac{R\,d\theta}{(R^2-1)(R^2-4)} = \frac{\pi R}{(R^2-1)(R^2-4)} \to 0 \text{ då } R\to\infty$$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{6}$$

K19 Beräkna $\sum_{1}^{\infty} \frac{k^2}{2^k}$

Låt $S(x) = \sum_{1}^{\infty} \underbrace{\frac{k^2 x^k}{2^k}}_{a_k} \Rightarrow S(1) = \sum \frac{k^2}{2^k}$

$$\sqrt[k]{|a_k|} = \frac{|x|}{2} k^{2/k} \Rightarrow \lim_{k\to\infty} \sqrt[k]{|a_k|} = \frac{|x|}{2} \lim_{k\to\infty} k^{2/k}$$

$$= \frac{|x|}{2} \lim_{k\to\infty} e^{\overset{\to 0}{\frac{2}{k}\ln k}} = \frac{|x|}{2} \Rightarrow$$

$S(x)$ konvergerar om $\frac{|x|}{2} < 1 \Rightarrow \underline{|x| < 2}$

$$S(x) = \sum \frac{k}{2^k} \cdot x \overset{D(x^k)}{k x^{k-1}} = x \sum_{1}^{\infty} \frac{k}{2^k} D(x^k) = x D\left[\sum_{1}^{\infty} \frac{k x^k}{2^k}\right]$$

$$= xD\left[\sum_{1}^{\infty} \frac{x}{2^k} \underset{D(x^k)}{k x^{k-1}}\right] = xD\left[xD \sum_{1}^{\infty} \frac{x^k}{2^k}\right] = xD\left[xD \sum_{1}^{\infty} \left(\frac{x}{2}\right)^k\right] =$$

$$= xD\left[xD\left(\frac{x/2}{1-x/2}\right)\right] = xD\left[xD\left(\frac{x}{2-x}\right)\right] = xD\left[x \frac{(2-x)+x}{(2-x)^2}\right] =$$

$$= xD\left[\frac{2x}{(2-x)^2}\right] = 2x\left[\frac{(2-x)^2 + x \cdot 2(2-x)}{(2-x)^4}\right] = \frac{2x(2-x+2x)}{(2-x)^3}$$

$$S(x) = \frac{2x(2+x)}{(2-x)^3} \Rightarrow \underline{S(1) = \sum_{1}^{\infty} \frac{k^2}{2^k} = \underline{\underline{6}}}$$

K20) Beräkna $\sum_{1}^{\infty} \frac{x^k}{k+3}$, Bestäm konvergensradien

a_k

Lösning:

Konvergens om $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$

$$\lim_{k\to\infty} \frac{|x^{k+1}|}{k+4} \cdot \frac{k+3}{|x^k|} = |x| < 1$$

$x=-1 \Rightarrow \sum_{3}^{\infty} \frac{(-1)^k}{k+3}$ konvergent enligt Leibnitz

$x=1 \Rightarrow \sum_{1}^{\infty} \frac{1}{k+3}$ som är divergent

$\sum_{1}^{\infty} \frac{x^k}{k+3}$ konvergerar för $1 \le x < 1$

$$\sum_{1}^{\infty} \frac{x^k}{k+3} = \sum_{1}^{\infty} \frac{1}{x^3} \left(\frac{x^{k+3}}{k+3}\right) \leftarrow I[x^{k+2}] = \frac{1}{x^3} I\left(\sum_{1}^{\infty} x^{k+2}\right) =$$

$$= \{ \nu = k+2 \} = \frac{1}{x^3} I\left(\sum_{3}^{\infty} x^{\nu}\right) = \frac{1}{x^3} I\left[\sum_{0}^{\infty} x^{\nu} - 1 - x - x^2\right] =$$

$$\frac{1}{x^3} I\left[\frac{1}{1-x} - 1 - x - x^2\right] = \frac{1}{x^3}\left[-\ln(1-x) - x - \frac{x^2}{2} - \frac{x^3}{3}\right] =$$

$$= -\frac{\ln(1-x)}{x^3} - \frac{1}{x^2} - \frac{1}{2x} - \frac{1}{3}$$

Tentan 25/5-'23
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Använd metoden med potensserier för att lösa differentialekvationen

$$(x-1)y'(x)+2y(x)=0, \quad y(0)=2$$

Lösning Ansätt $y(x)=\sum_{0}^{\infty} a_k x^k$ som antas konvergera i en omgivning till origo.

$y'(x)=\sum_{1}^{\infty} a_k k x^{k-1}$; Insättning i diff.ekv. ger

$$\sum_{1}^{\infty} a_k k x^k - \underbrace{\sum_{1}^{\infty} a_k k x^{k-1}}_{a_1+\sum_{2}^{\infty} a_k k x^{k-1} = a_1+\sum_{1}^{\infty} a_{k+1}(k+1)x^k} + 2\sum_{0}^{\infty} a_k x^k = 0$$

$$\therefore \sum_{1}^{\infty} a_k k x^k - a_1 - \sum_{1}^{\infty} a_{k+1}(k+1)x^k + 2\left[a_0+\sum_{1}^{\infty} a_k x^k\right]=0$$

$$\underbrace{-a_1+2a_0}_{=0} + \sum_{1}^{\infty}\left[a_k \cdot k - a_{k+1}(k+1)+2a_k\right]x^k = 0$$

$a_1 = 2a_0$ och $a_k(k+2)-a_{k+1}(k+1)=0$

$$\therefore \quad a_{k+1} = a_k \frac{(k+2)}{(k+1)} \; ; \; y(0)=2 \Rightarrow 2 = a_0+a_1x+a_2x^2 \dots \Big|_{x=0}$$

$$\Rightarrow \underline{a_0 = 2}$$

$a_1 = a_0 \cdot 2 = 4$; $a_4 = a_3 \cdot \frac{5}{4} = 10$

$a_2 = a_1 \cdot \frac{3}{2} = 6$

$a_3 = a_2 \cdot \frac{4}{3} = 8$ Gissar: $a_k = 2k+2 = 2(k+1)$

$$a_{k+1} = a_k \frac{(k+2)}{(k+1)} = \{a_k = 2(k+1)\} = 2(k+1+1)$$

$a_1 = 2 \cdot 1 + 2$ dvs. sann för $k=1$

a_k sann $\Rightarrow a_{k+1}$ sann allså är

$a_k = 2(k+1)$ för alla $k \geq 0$ (Induktionsbevis)

$$\therefore y(x) = 2\sum_0^\infty (k+1)x^k = 2\sum_0 D(x^{k+1}) =$$

$$= 2D\left[\sum_0^\infty x^{k+1}\right] \underset{|x|<1}{=} 2D\left[\frac{x}{1-x}\right] = 2\left[\frac{1-x-x(-1)}{(1-x)^2}\right]$$

$$y(x) = \frac{2}{(1-x)^2} \quad (\Rightarrow y(0) = 2)$$

Konvergent för $|x| < 1$

Check: $(x-1)y' + 2y = 0 \Rightarrow y' + \frac{2}{x-1}y = 0$ Int.faktor $e^{2\ln(x-1)} = (x-1)^2$

$$\frac{d}{dx}\left((1-x)^2 y\right) = 0 \Rightarrow (1-x)^2 y = C \Rightarrow y = \frac{C}{(1-x)^2}$$

$y(0) = 2 \Rightarrow C = 2$; $\therefore y(x) = \frac{2}{(1-x)^2}$