Extra räkneövning 13/12 -'23, Analys B

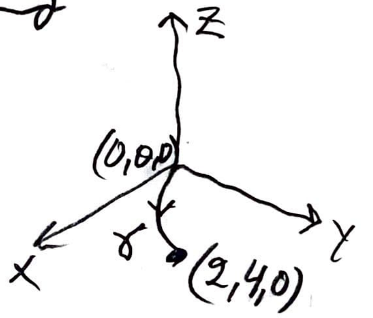
ÖPB 2
10.3

Låt $\bar{F}$ vara vektorfältet

$$\bar{F} = (y, (x+z)^2, (x-z)^2)$$

Beräkna $\int_\gamma \bar{F} \cdot d\bar{r}$ där γ: $y = x^2$, $z = 0$ från $(0,0,0)$ till $(2,4,0)$

Lösning



Parametrisera γ: $\bar{r}(t) = (t, t^2, 0)$
$t: 0 \to 2$

$\bar{r}'(t) = (1, 2t, 0)$

$$\int_\gamma \bar{F} \cdot d\bar{r} = \int_{t=0}^{2} \bar{F} \cdot \bar{r}'(t)\, dt = \int_0^2 (t^2, t^2, t^2) \cdot (1, 2t, 0)\, dt =$$

$$= \int_0^2 (t^2 + 2t^3)\, dt = \left[\frac{t^3}{3} + \frac{t^4}{2}\right]_0^2 = \frac{8}{3} + 8 = \underline{\underline{\frac{32}{3}}}$$

$$\left[\int_\gamma \bar{F} \cdot d\bar{r} = \int_\gamma (P\,dx + Q\,dy + R\,dz) = \{\bar{r}(t) = (t, t^2, 0)\} = \ldots = \underline{\underline{\frac{32}{3}}}\right]$$

10.28) a) Bestäm konstanterna a och b så att fältet

$\bar{F} = (ax^2 + xy, xy + y^2, byz + b)$ blir källfritt

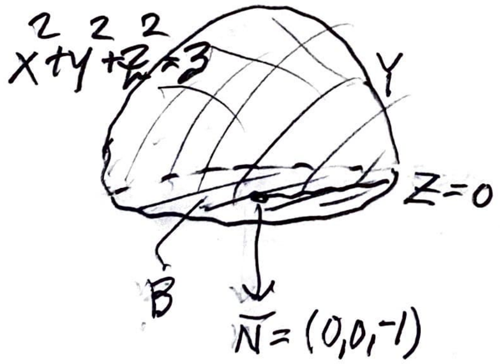
b) För dessa värden på a och b beräkna flödet av $\bar{F}$ ut genom sfären $x^2+y^2+z^2=3, z \geq 0$

Lösning:

a) $\operatorname{div}\bar{F} = (2ax + y + x + 2y + by) = 0$

$x(2a+1) + y(3+b) = 0$ om $a = -\frac{1}{2}$ och $b = -3$

b) Gauss sats:



Enligt Gauss sats:

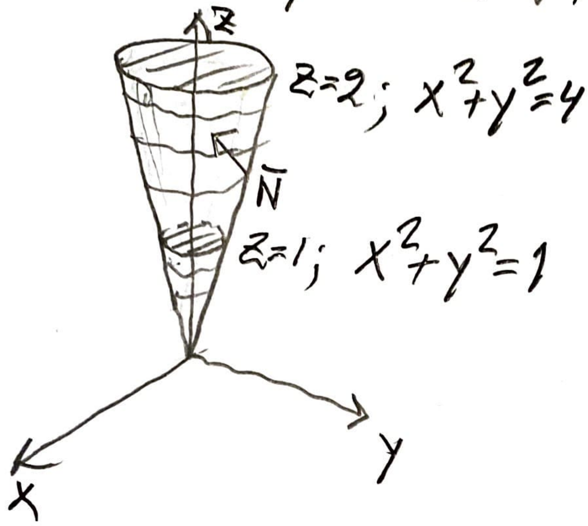
$$\iint_Y \bar{F}\cdot\bar{N}\,dS + \iint_B \bar{F}\cdot\bar{N} = 0$$

eftersom $\operatorname{div}\bar{F} = 0$

$$\iint_B \bar{F}\cdot\bar{N} = \{z=0\} = \iint_B \left(-\frac{x^2}{2} + xy, xy + y^2, -3\right)\cdot(0,0,-1)\,dS =$$

$$3\iint_{x^2+y^2\leq 3} dS = 3\cdot\pi(\sqrt{3})^2 = 9\pi \Rightarrow \iint_Y \bar{F}\cdot\bar{N}\,dS = -9\pi$$

10.32) Beräkna flödet av $\bar{u}(x,y,z) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, z\right)$ genom ytan $z = \sqrt{x^2+y^2}$ $1 \le z \le 2$ i riktning mot z-axeln.



$\bar{N}$ har positiv z-koordinat

Parametrisera ytan $\bar{r}(x,y) = (x, y, \sqrt{x^2+y^2})$

normalvektor $= \bar{r}_x' \times \bar{r}_y' = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right) \times \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$

$$= \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)$$

Flödet $\Phi = \iint_Y \bar{u} \cdot \bar{N}\, dS = \iint_Y \bar{u} \cdot (\bar{r}_x' \times \bar{r}_y')\, dx\,dy =$ $\quad |\bar{N}| = 1$

$$= \iint_{1 \le x^2+y^2 \le 4} \left(\frac{-x^2}{(x^2+y^2)^{3/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} + z\right) dx\,dy = \left\{ z = \sqrt{x^2+y^2} \right\} =$$

$$= \iint_{1 \le x^2+y^2 \le 4} \left(-\frac{1}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\right) dx\,dy = \{\text{Polärakoordinater}\} =$$

$$= 2\pi \int_1^2 \left(-\frac{1}{r} + r\right) r\,dr = 2\pi \left[-r + \frac{r^3}{3}\right]_1^2 = \underline{\underline{\frac{8\pi}{3}}}$$

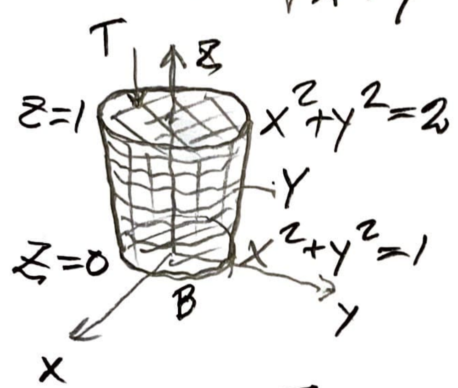
B94)

Beräkna $\iint_Y \bar{F}\cdot\bar{N}\,dS$ där $\bar{F}=(x^4+yz-x^5, 5x^4y, z)$

och Y är den del av ytan $x^2+y^2-z^2=1$ för vilken $0\le z\le 1$

Lösning:

$$z=\sqrt{x^2+y^2-1} \qquad 0\le z\le 1$$



Gauss: $\iint_Y \bar{F}\cdot\bar{N}\,dS+\iint_B \bar{F}\cdot\bar{N}\,dS+\iint_T \bar{F}\cdot\bar{N}\,dS=\iiint_K \operatorname{div}\bar{F}\,dxdydz$

K är slutna kroppen $Y \cup B \cup T$ "Union"

$$\iiint_K \operatorname{div}\bar{F}\,dxdydz=\{\operatorname{div}\bar{F}=4x^3-5x^4+5x^4+1\}=$$

$$=\iiint_K (1+4x^3)\,dxdydz=\int_{z=0}^{1}\left[\iint_{x^2+y^2\le z^2+1}(4x^3+1)\,dxdy\right]dz$$

$$=\int_{z=0}^{1}\Bigg[\underbrace{\iint_{x^2+y^2\le z^2+1}4x^3\,dxdy}_{4\int_{\theta=0}^{2\pi}\int_0^{\sqrt{z^2+1}} r^4\cos^3\theta\,drd\theta}\Bigg]dz+\int_0^1\left[\iint_{x^2+y^2\le z^2+1}dxdy\right]dy=\int_0^1\pi(z^2+1)\,dz=$$

$$\int_0^{2\pi}\cos^3\theta\,d\theta=0$$

$$=\pi\left[\frac{z^3}{3}+z\right]_0^1=\underline{\underline{\frac{4\pi}{3}}}$$

Botten B: $\bar{N} = (0,0,-1) \Rightarrow \bar{F} \cdot \bar{N} = \bar{0} \Rightarrow \iint_B \bar{F} \cdot \bar{N} \, dS = \underline{\underline{0}}$

$z = 0$

Toppen T: $\bar{N} = (0,0,1) \Rightarrow \bar{F} \cdot \bar{N} = 1 \Rightarrow$

$z = 1$

$$\Rightarrow \iint_T \bar{F} \cdot \bar{N} \, dS = \{\text{Ytan av } x^2 + y^2 = 2\} = \pi \cdot 2$$

$$\iint_Y \bar{F} \cdot \bar{N} \, dS + \underbrace{\iint_T \bar{F} \cdot \bar{N} \, dS}_{2\pi} = \underbrace{\iiint_K \operatorname{div} \bar{F} \, dx\,dy\,dz}_{\frac{4\pi}{3}}$$

$$\therefore \iint_Y \bar{F} \cdot \bar{N} \, dS = \frac{4\pi}{3} - 2\pi = \underline{\underline{-\frac{2\pi}{3}}}$$

ÖPB 2 10.62

Är $\bar{u}(x,y,z) = (2xy^2z, 2x^2yz, x^2y^2 - 2z)$ konservativt? Beräkna $\int_\gamma \bar{u}\cdot d\bar{r}$ längs kurvan γ given av $\bar{r}(t) = (\cos t, \sin t, \sin t)$, $0 \le t \le \frac{\pi}{2}$

Lösning: $\bar{u} \in C^\infty \Rightarrow \bar{u}$ konservativt om $\text{rot}\,\bar{u} = \bar{0}$

$$\text{rot}\,\bar{u} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z & 2x^2yz & x^2y^2 - 2z \end{vmatrix} = (\underbrace{2x^2y - 2x^2y}_{=0}, \underbrace{-2xy^2 + 2xy^2}_{=0}, \underbrace{4xyz - 4xyz}_{=0}) = \bar{0}$$

$\therefore$ $\bar{u}$ är konservativt

Potentialen U uppfyller $\nabla U = \bar{u} \Rightarrow$

$$\underbrace{\frac{\partial U}{\partial x} = 2xy^2z}, \quad \frac{\partial U}{\partial y} = 2x^2yz \ ; \quad \frac{\partial U}{\partial z} = x^2y^2 - 2z$$

$$\Rightarrow U(x,y,z) = x^2y^2z + \varphi(y,z) \Rightarrow 2x^2yz + \frac{\partial\varphi}{\partial y}(y,z) = 2x^2yz$$

$$\therefore \varphi(y,z) = \theta(z) \Rightarrow U(x,y,z) = x^2y^2z + \theta(z) \Rightarrow$$

$$\frac{\partial U}{\partial z} = x^2y^2 + \theta'(z) = x^2y^2 - 2z \Rightarrow \theta'(z) = -2z \Rightarrow \theta(z) = -z^2 \ (+C)$$

$$\therefore \underline{U(x,y,z) = x^2y^2z - z^2}$$

$$\bar{r}(0) = (1,0,0), \ \bar{r}(\tfrac{\pi}{2}) = (0,1,1) \Rightarrow \int_\gamma \bar{u}\cdot d\bar{r} = U(0,1,1) - U(1,0,0) = \underline{\underline{-1}}$$

Analytiska funktioner

Realdelen av en analytisk funktion $f(z)$ är $x^3-3xy^2+x^2-y^2+x+1$. Vilka av följande funktioner kan vara imaginärdel till $f(z)$?

a) $y^3+3x^2y+3xy+y$

b) $-y^3+3x^2y-2xy+y$

c) $-y^3+3x^2y+2xy+y$

d) $-y^3+3x^2y+2xy-y$

Använd Cauchy-Riemanns ekv.

$f(z)=u(x,y)+iv(x,y)$;

f analytisk $\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ och $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}=3x^2-3y^2+2x+1$

$\frac{\partial v}{\partial y}$: a) $3y^2+3x^2+3x+1 \neq \frac{\partial u}{\partial x}$

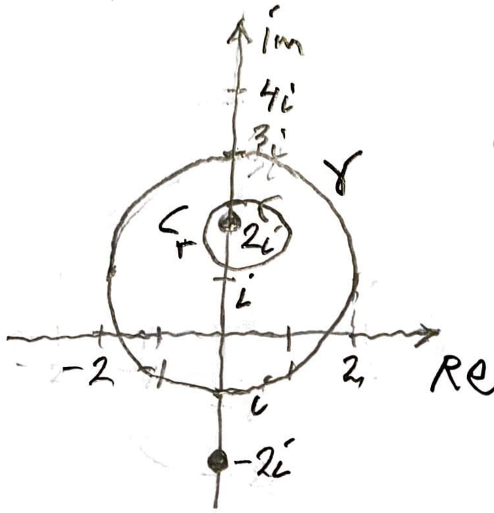
b) $-3y^2+3x^2-2x+1 \neq \frac{\partial u}{\partial x}$

c) $-3y^2+3x^2+2x+1 = \frac{\partial u}{\partial x} \longleftarrow$

d) $-3y^2+3x^2+2x-1 \neq \frac{\partial u}{\partial x}$

Svar: c)

Exempel: Beräkna $\int_\gamma \underbrace{\frac{1-z}{z^2+4}}_{f(z)} dz$ där γ är cirkeln $|z-i|=2$ genomlöpt moturs



$$f(z) = \frac{1-z}{(z+2i)(z-2i)}$$

Enkelpoler i $\pm 2i$

(i) $\int_\gamma f(z)dz = \int_{C_r} f(z)dz = \int_{C_r} \frac{\overbrace{\frac{1-z}{(z+2i)}}^{g(z)}}{(z-2i)}dz = \int_{C_r} \frac{g(z)}{z-2i}$

Cauchy $\Rightarrow$ $\int_{C_r} \frac{g(z)}{z-2i}dz = 2\pi i\, g(2i) = \frac{2\pi i(1-2i)}{4i} = \frac{\pi}{2}(1-2i)$

$\therefore \int_\gamma f(z)dz = \frac{\pi}{2}(1-2i)$

K18) Beräkna $\sum_0^\infty \underbrace{k(k+1)x^k}_{a_k}$. För vilka x konvergerar

Serier.

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1 \Rightarrow \text{Konvergens}$$

$$\lim_{k\to\infty} |x| \sqrt[k]{k(k+1)} = |x| \lim_{k\to\infty} e^{\overbrace{\frac{1}{k}\ln k(k+1)}^{\to 0\ (e^0=1)}} = |x| < 1$$

$\therefore$ $|x|<1 \Rightarrow$ Serien är konvergent

$$\sum_0^\infty k(k+1)x^k = \{D^2(x^{k+1}) = k(k+1)x^{k-1}\} =$$

$$= \sum_0^\infty x D^2(x^{k+1}) = x D^2\left[\sum_0^\infty x^{k+1}\right] = \{|x|<1\} = x D^2\left[\frac{x}{1-x}\right]$$

$$D\left[\frac{x}{1-x}\right] = \frac{(1-x)-x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \Rightarrow D^2\left[\frac{x}{1-x}\right] = -2(1-x)^{-3}(-1)$$

$$= \frac{2}{(1-x)^3} \Rightarrow \sum_0^\infty k(k+1)x^k = xD^2\left(\frac{x}{1-x}\right) = \frac{2x}{(1-x)^3}$$

Svar: $\sum_0^\infty k(k+1)x^k = \frac{2x}{(1-x)^3}$; Konvergerar för $|x|<1$

K19 Beräkna $\sum_1^{\infty} \frac{k^2}{2^k}$

Låt $S(x) = \sum_1^{\infty} \underbrace{\frac{k^2 x^k}{2^k}}_{a_k} \Rightarrow S(1) = \sum \frac{k^2}{2^k}$

$\sqrt[k]{|a_k|} = \frac{|x|}{2} k^{2/k} \Rightarrow \lim_{k\to\infty} \sqrt[k]{|a_k|} = \frac{|x|}{2} \lim_{k\to\infty} k^{2/k}$

$= \frac{|x|}{2} \lim_{k\to\infty} e^{\overset{\to 0}{\frac{2}{k}\ln k}} = \frac{|x|}{2} \Rightarrow$

$S(x)$ konvergerar om $\frac{|x|}{2} < 1 \Rightarrow \underline{|x| < 2}$

$S(x) = \sum \frac{k}{2^k} \cdot x \overset{D(x^k)}{k x^{k-1}} = x \sum_1^{\infty} \frac{k}{2^k} D(x^k) = xD\left[\sum_1^{\infty} \frac{k x^k}{2^k}\right]$

$= xD\left[\sum_1^{\infty} \frac{x}{2^k} \underset{D(x^k)}{k x^{k-1}}\right] = xD\left[xD \sum_1^{\infty} \frac{x^k}{2^k}\right] = xD\left[xD \sum_1^{\infty} \left(\frac{x}{2}\right)^k\right] =$

$= xD\left[xD\left(\frac{x/2}{1-x/2}\right)\right] = xD\left[xD\left(\frac{x}{2-x}\right)\right] = xD\left[x \frac{(2-x)+x}{(2-x)^2}\right] =$

$= xD\left[\frac{2x}{(2-x)^2}\right] = 2x\left[\frac{(2-x)^2 + x \cdot 2(2-x)}{(2-x)^4}\right] = \frac{2x(2-x+2x)}{(2-x)^3}$

$S(x) = \frac{2x(2+x)}{(2-x)^3} \Rightarrow \underline{S(1) = \sum_1^{\infty} \frac{k^2}{2^k} = \underline{\underline{6}}}$