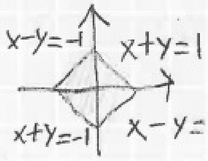
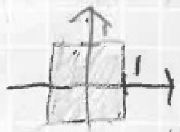
6.19 $D: |x|+|y|\le 1$



$\iint_D (x^2-y^2)^{10}\,dx\,dy = ?$



$u = x+y,\quad v = x-y$

$\frac{u+v}{2} = x \quad \frac{u-v}{2} = y$

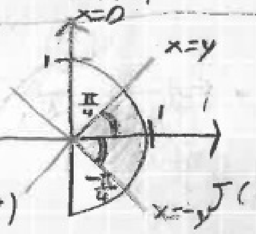
$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

$E: -1\le u\le 1,\ -1\le v\le 1$

$$\iint_D (x^2-y^2)^{10}\,dx\,dy = \iint_E (uv)^{10}\left|-\tfrac{1}{2}\right| du\,dv = \int_{-1}^{1}\int_{-1}^{1}\tfrac{1}{2}u^{10}v^{10}\,du\,dv$$

$$= \frac{1}{2}\int_{-1}^{1}u^{10}\,du\int_{-1}^{1}v^{10}\,dv = \frac{1}{2}\left[\frac{u^{11}}{11}\right]_{-1}^{1}\left[\frac{v^{11}}{11}\right]_{-1}^{1} = \frac{1}{2}\,\frac{2}{11}\,\frac{2}{11} = \frac{2}{121}$$

6.23 $D: x^2+y^2\le 1$, $-x\le y\le x$, $x\ge 0$



$\iint_D (x^2-y^2)e^{2xy}\,dx\,dy = ?$

$x = r\cos\theta,\ y = r\sin(\theta)$ $\quad J(r,\theta) = r \quad E = [0,1]\times[-\frac{\pi}{4},\frac{\pi}{4}]$

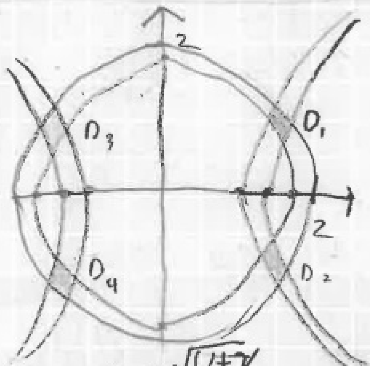
$$\iint_D (x^2-y^2)e^{2xy}\,dx\,dy = \iint_E (r^2\cos^2(\theta)-r^2\sin^2(\theta))e^{2r\cos(\theta)r\sin(\theta)}\,r\,dr\,d\theta$$

$$= \int_0^1\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos(2\theta)\,e^{r^2\sin(2\theta)}\,d\theta\,dr \overset{(*)}{=} \int_0^1\left[r\,\frac{1}{2}e^{r^2\sin(2\theta)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dr$$

$$= \int_0^1\left(\frac{r}{2}e^{r^2\sin(\frac{\pi}{2})} - \frac{r}{2}e^{r^2\sin(-\frac{\pi}{2})}\right)dr = \int_0^1\left(e^{r^2}-e^{-r^2}\right)\frac{r}{2}\,dr$$

$$= \begin{bmatrix} r^2=t \\ 2r\,dr=dt \end{bmatrix} = \int_0^1 (e^t-e^{-t})\frac{1}{2}\,\frac{dt}{2} = \frac{1}{4}\left[e^t+e^{-t}\right]_0^1 = \frac{1}{4}\left(e+e^{-1}-(e^0+e^0)\right) = \frac{e+e^{-1}-2}{4}$$

B15 $D: x^2-y^2=1$, $x^2-y^2=2$, $x^2+y^2=3$, $x^2+y^2=4$



$\iint_D (x^4-y^4)\,dx\,dy = ?$

P.G.A x^4-y^4 jämn i x, y och symetr
för D i v

$$\iint_D (x^4-y^4)\,dx\,dy = 4\iint_{D_1}(x^4-y^4)\,dx\,dy$$

Sätt $u = x^2+y^2$, $v = x^2-y^2$ $\underset{på\ D_1}{\Rightarrow}$ $x = \sqrt{\frac{u+v}{2}}$, $y = \sqrt{\frac{u-v}{2}}$

$E:\ 1\le v\le 2,\ 3\le u\le 4$

$$J(u,v) = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{2}\sqrt{u+v}} & \frac{1}{2}\frac{1}{\sqrt{2}\sqrt{u+v}} \\ \frac{1}{2}\frac{1}{\sqrt{2}\sqrt{u-v}} & -\frac{1}{2}\frac{1}{\sqrt{2}\sqrt{u-v}} \end{vmatrix}$$

$$= -\frac{1}{4}\frac{1}{2\sqrt{(u+v)(u-v)}} - \frac{1}{4}\frac{1}{2\sqrt{(u+v)(u-v)}} = -\frac{1}{4\sqrt{u^2-v^2}}$$

$$\iint_D (x^4-y^4)\,dx\,dy = 4\iint_{D_1}(x^4-y^4)\,dx\,dy = 4\iint_E uv\,\frac{1}{\sqrt{u^2-v^2}}\,\frac{1}{4}\,du\,dv$$

$$= \int_1^2\left(\int_3^4\frac{u\,du}{\sqrt{u^2-v^2}}\right)v\,dv \overset{(**)}{=} \int_1^2\left[\sqrt{u^2-v^2}\right]_3^4 v\,dv$$

$$= \int_1^2\left(\sqrt{4^2-v^2}-\sqrt{3^2-v^2}\right)v\,dv$$

$$= \frac{1}{2}\int_1^2\left(\sqrt{16-v^2}-\sqrt{9-v^2}\right)2v\,dv$$

$$(***) = \frac{1}{2}\left[-\frac{2}{3}(16-v^2)^{\frac{3}{2}}+\frac{2}{3}(9-v^2)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{1}{2}\left(-\frac{2}{3}(16-4)^{\frac{3}{2}}+\frac{2}{3}(9-4)^{\frac{3}{2}}-\left(-\frac{2}{3}(16-1)^{\frac{3}{2}}+\frac{2}{3}(9-1)^{\frac{3}{2}}\right)\right)$$

$$= \frac{1}{3}\left(15^{\frac{3}{2}}+5^{\frac{3}{2}}-12^{\frac{3}{2}}-8^{\frac{3}{2}}\right)$$

Primitiva funktioner

För samtliga primitiva funktioner använder jag samma trick
om $f \in C^1$, $g \in C$ så är

$$\int f'(x) g(f(x)) dx = \begin{bmatrix} t = f(x) \\ dt = f'(x) dx \end{bmatrix} = \int g(t)\, dt$$

(*) $$\int r^3 \cos(2\theta) e^{r^2 \sin(2\theta)} d\theta = \begin{bmatrix} t = \sin(2\theta) \\ dt = 2\cos(2\theta) d\theta \end{bmatrix} = \int r^3 e^{r^2 t} \frac{dt}{2}$$

$$= r^3 \frac{1}{r^2} e^{r^2 t} \frac{1}{2} + C = \frac{r}{2} e^{r^2 \sin(2\theta)} + C$$

(**) $$\int \frac{u\, du}{\sqrt{u^2 - v^2}} = \begin{bmatrix} t = u^2 \\ dt = 2u\, du \end{bmatrix} = \int \frac{\frac{dt}{2}}{\sqrt{t - v^2}} = \int \frac{1}{2} (t - v^2)^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \frac{1}{1 - \frac{1}{2}} (t - v^2)^{1 - \frac{1}{2}} + C = \sqrt{u^2 - v^2} + C$$

(***) $$\int \left(\sqrt{16 - v^2} - \sqrt{9 - v^2}\right) v\, dv = \begin{bmatrix} t = v^2 \\ dt = 2v\, dv \end{bmatrix} = \int \left(\sqrt{16 - t} - \sqrt{9 - t}\right) \frac{dt}{2}$$

$$= \left(\frac{-1}{3/2} (16 - t)^{\frac{3}{2}} + \frac{1}{3/2} (9 - t)^{\frac{3}{2}}\right) \frac{1}{2} + C$$