6.36 $\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+y^2)^\alpha} \overset{\text{polärα}}{=} \iint_{[0,\infty)\times[-\pi,\pi]} \frac{r\,dr\,d\theta}{(1+r^2)^\alpha} = \int_0^\infty \int_{-\pi}^{\pi} \frac{r\,d\theta}{(1+r^2)} dr =$

$= \int_0^\infty \frac{2\pi r\,dr}{(1+r^2)^\alpha} = \begin{bmatrix} r^2 = t \\ 2r\,dr = dt \end{bmatrix} = \int_0^\infty \frac{\pi\,dt}{(1+t)^\alpha}$

$\alpha = 1:\ \int_0^\infty \frac{dt}{1+t} = \ln(\infty) - \ln(1) = \infty$

då $R \to \infty$

$\alpha \neq 1:\ \int_0^\infty \frac{dt}{(1+t)^\alpha} = \left[\frac{1}{1-\alpha}\left((1+\infty)^{1-\alpha} - 1\right)\right] \xrightarrow[R\to\infty]{} \begin{cases} \frac{1}{1-\alpha}(0-1) = \frac{1}{\alpha-1} & \alpha > 1 \\ \infty & \alpha < 1 \end{cases}$

Så konvergerar om $\alpha > 1$. Då konvergerar den till $\frac{\pi}{\alpha-1}$

326 a) $\iint_D \frac{dx\,dy}{1+x^3+y^3} = \iint_{[0,\infty)\times[0,\frac{\pi}{2}]} \frac{r\,dr\,d\theta}{1+r^3\cos^3(\theta)+r^3\sin^3(\theta)}$ D: första kvadranten

$f(\theta) = \cos^3(\theta) + \sin^3(\theta)$, $f'(\theta)\ 3\cos^2(\theta)(-\sin(\theta)) + 3\sin^2(\theta)\cos(\theta)$
$= 3\cos(\theta)\sin(\theta)\left(\cos(\theta) - \sin(\theta)\right)$

$f'(\theta) = 0 \iff \theta = 0, \frac{\pi}{2}, \frac{\pi}{4}$
$0 \le \theta \le \frac{\pi}{2}$

$f(0) = 1$
$f(\frac{\pi}{2}) = 1$
$f(\frac{\pi}{4}) = \left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$

Alltså är $f(\theta) \ge \frac{1}{\sqrt{2}}$ för $0 \le \theta \le \frac{\pi}{2}$

så $\int_0^{\frac{\pi}{2}} \frac{r\,d\theta}{1+r^3\cos^3(\theta)+r^3\sin^3(\theta)} \le \int_0^{\frac{\pi}{2}} \frac{r\,d\theta}{1+r^3\frac{1}{\sqrt{2}}} = \frac{\pi r}{2+\sqrt{2}r^3}$ och $\int_0^\infty \frac{\pi r}{2+\sqrt{2}r^3}dr < \infty$

7.3 $D: x^2+y^2 \le z^2$, $0 \le z \le 1$ $\quad D_z = \{(x,y): x^2+y^2 \le z^2\}$

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \int_0^1 \iint_{D_z} (x^2+y^2)\,dx\,dy\,dz = \int_0^1 \iint_{[0,z]\times[-\pi,\pi]} r^2\, r\,dr\,d\theta\,dz$$

$$= \int_0^1 \int_0^z \int_{-\pi}^{\pi} r^3\,d\theta\,dr\,dz = \int_0^1 \int_0^z 2\pi r^3\,dr\,dz = \int_0^1 \left[2\pi \frac{r^4}{4}\right]_0^z dz$$

$$= \int_0^1 \pi \frac{z^4}{2}\,dz = \left[\pi \frac{1}{2}\frac{z^5}{5}\right]_0^1 = \frac{\pi}{10}$$

36 $D: 0 \le x \le y+z \le z \le 1 \Leftrightarrow \begin{cases} 0 \le z \le 1 \\ 0 \le x \le y+z \le z \end{cases} \Leftrightarrow \begin{cases} 0 \le z \le 1 \\ 0 \le y+z \le z \\ 0 \le x \le y+z \end{cases} \Leftrightarrow \begin{cases} 0 \le z \le 1 \\ -z \le y \le 0 \\ 0 \le x \le y+z \end{cases}$

$$\iiint_D x^2yz\,dx\,dy\,dz =$$

$$= \int_0^1 \int_{-z}^0 \int_0^{y+z} x^2yz\,dx\,dy\,dz = \int_0^1 \int_{-z}^0 \left[\frac{x^3}{3}yz\right]_0^{y+z} dy\,dz$$

$$= \int_0^1 \int_{-z}^0 \frac{(y+z)^3}{3} yz\,dy\,dz = \int_0^1 \left(\left[\frac{(y+z)^4}{12}yz\right]_{-z}^0 - \int_{-z}^0 \frac{(y+z)^4}{12} z\,dy\right)dz$$

$$= \int_0^1 -\left[\frac{(y+z)^5}{60}z\right]_{-z}^0 dz = \int_0^1 \frac{-z^6}{60}\,dz = \left[\frac{-z^7}{420}\right]_0^1 = \frac{-1}{420}$$

39 a) $$\iiint_{\mathbb{R}^3} \frac{dx\,dy\,dz}{1+x^2+y^2+z^2} \overset{\text{sfäriska}}{=} \iiint_{[0,\infty)\times[-\pi,\pi]\times[0,\pi]} \frac{r^2\sin(\theta)\,dr\,d\varphi\,d\theta}{1+r^2} = \int_0^\infty \int_0^\pi \frac{2\pi r \sin(\theta)}{1+r^2}\,d\theta\,dr$$

$$= \int_0^\infty \left[-\frac{2\pi r\cos(\theta)}{1+r^2}\right]_0^\pi r = \int_0^\infty \frac{4\pi r^2}{1+r^2}\,dr = \infty = +$$

eftersom $\frac{r^2}{1+r^2} \to 1$ då $r \to \infty$

b) $$\iiint_{\mathbb{R}^3} \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)} \overset{\text{sfäriska}}{=} \iiint_{[0,\infty)\times[-\pi,\pi]\times[0,\pi]} \frac{r^2\sin(\theta)}{(1+r^2)^2}\,dr\,d\varphi\,d\theta = \int_0^\infty \int_0^\pi \frac{2\pi r^2 \sin(\theta)}{(1+r^2)^2}\,d\theta\,dr$$

$$= \int_0^\infty \frac{4\pi r^2}{(1+r^2)^2}\,dr = \int_0^\infty r\,\frac{4\pi r}{(1+r^2)^2}\,dr$$

Notera att $\int \frac{4\pi r}{(1+r^2)^2}\,dr = [r^2 = t] = \int \frac{2\pi}{(1+t)^2}\,dt = \frac{-2\pi}{1+t} = \frac{-2\pi}{1+r^2}$

så $\int_0^\infty r \frac{4\pi r}{(1+r^2)^2}\,dr = \left[r\,\frac{-2\pi}{1+r^2}\right]_0^\infty - \int_0^\infty \frac{-2\pi}{1+r^2}\,dr = \int_0^\infty \frac{2\pi}{1+r^2}\,dr = \left[2\pi \arctan(r)\right]_0^\infty = \pi^2$