9. $\bar F=(4xz,-y^2,yz)$ Beräkna $\iint_Y \bar F\cdot\bar N\,dS$ om;

a) Y ytan $x=1$, $0\le y\le 1$, $0\le z\le 1$, $\bar N$ i positiva x-axelns riktning

$\bar r(y,z)=(1,y,z)\quad 0\le y,z\le 1$

$\bar r'_y=(0,1,0)$, $\bar r'_z=(0,0,1)$, $\bar r'_y\times \bar r'_z=(-1,0,0)$ så $dS=dy\,dz$

$$\iint_Y \bar F\cdot\bar N\,dS=\int_0^1\int_0^1(4z,-y^2,yz)\cdot(1,0,0)\,dy\,dz=\int_0^1\int_0^1 4z\,dy\,dz=$$

$$=\int_0^1 4z\,dz=2$$

b) Y ytan som begränsar $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$, och $\bar N$ utåtriktad

$Y_1: x=1,\ 0\le y,z\le 1$, $Y_2: x=0\quad 0\le y,z\le 1$,

$Y_3: y=1\quad 0\le x,z\le 1$, $Y_4: y=0\quad 0\le x,z\le 1$

$Y_5: z=1\quad 0\le x,y\le 1$, $Y_6: z=0\quad 0\le x,y\le 1$

$$\iint_{Y_1}\bar F\cdot\bar N\,dS=2$$

$$\iint_{Y_2}\bar F\cdot\bar N\cdot dS=\int_0^1\int_0^1(0\cdot z,-y^2,yz)\cdot(-1,0,0)\,dy\,dz=\int_0^1\int_0^1 0\,dy\,dz=0$$

$$\iint_{Y_3}\bar F\cdot\bar N\,dS=\int_0^1\int_0^1(4xz,-1^2,1\cdot z)\cdot(0,1,0)\,dx\,dz=\int_0^1\int_0^1 -1\,dx\,dz=-1$$

$$\iint_{Y_4}\bar F\cdot\bar N\,dS=\int_0^1\int_0^1(4xz,-0^2,0\cdot z)\cdot(0,-1,0)\,dx\,dz=\int_0^1\int_0^1 0\,dx\,dz=0$$

$$\iint_{Y_5}\bar F\cdot\bar N\,dS=\int_0^1\int_0^1(4x\cdot 1,-y^2,y\cdot 1)\cdot(0,0,1)\,dx\,dy=\int_0^1\int_0^1 y\,dx\,dy=\frac12$$

$$\iint_{Y_6}\bar F\cdot\bar N\,dS=\int_0^1\int_0^1(4x\cdot 0,-y^2,y\cdot 0)\cdot(0,0,-1)\,dx\,dy=\int_0^1\int_0^1 0\,dx\,dy=0$$

$$\iint_Y\bar F\cdot\bar N\,dS=\iint_{Y_1}\bar F\cdot\bar N\,dS+\iint_{Y_2}\bar F\cdot\bar N\,dS+\iint_{Y_3}\bar F\cdot\bar N\,dS+\iint_{Y_4}\bar F\cdot\bar N\,dS+\iint_{Y_5}\bar F\cdot\bar N\,dS+\iint_{Y_6}\bar F\cdot\bar N\,dS$$

$$=\frac32$$

85 Beräkna flödet av $F(r)=\frac{-\bar r}{|\bar r|^4}$ in i sfären $x^2+y^2+z^2=R^2$

$\bar N=-\frac{\bar r}{|\bar r|}$ och $|\bar r|=R$ på sfären så $\bar F\cdot\bar N=\frac{-\bar r}{|\bar r|^4}\cdot\frac{\bar r}{|\bar r|}=\frac{1}{|\bar r|^3}=\frac{1}{R^3}$

så $\iint_Y\bar F\cdot\bar N\,dS=\iint_Y\frac{1}{R^3}\,dS=\frac{1}{R^3}\iint_Y dS=\frac{1}{R^3}\,\text{Area}(Y)$

Där Y sfären $x^2+y^2+z^2=R^2$ så $\text{Area}(Y)=4\pi R^2$

Alltså är flödet $\frac{1}{R^3}4\pi R^2=\frac{4\pi}{R}$

10.16 Beräkna divergensen av a) $\bar{v}=(x^2,3y,x^3)$ b) $\bar{v}=\bar{r}$ c) $v=\frac{\bar{r}}{r^3}$

a) $\operatorname{div}\bar{v}=\frac{\partial}{\partial x}x^2+\frac{\partial}{\partial y}3y+\frac{\partial}{\partial z}x^3=2x+3$

b) $\operatorname{div}\bar{v}=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$

c) $\operatorname{div}\bar{v}=\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2+z^2}^3}+\frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2+z^2}^3}+\frac{\partial}{\partial z}\frac{z}{\sqrt{x^2+y^2+z^2}^3}$

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2+z^2}^3}=\frac{\sqrt{x^2+y^2+z^2}^3-x\cdot 2x\,\frac{3}{2}\sqrt{x^2+y^2+z^2}}{\left(\sqrt{x^2+y^2+z^2}^3\right)^2}$$

$$=\frac{x^2+y^2+z^2-3x^2}{\sqrt{x^2+y^2+z^2}^5}$$

så $\operatorname{div}\bar{v}=\frac{x^2+y^2+z^2-3x^2}{\sqrt{x^2+y^2+z^2}^5}+\frac{x^2+y^2+z^2-3y^2}{\sqrt{x^2+y^2+z^2}^5}+\frac{x^2+y^2+z^2-3z^2}{\sqrt{x^2+y^2+z^2}^5}=0$

0.20 Beräkna flödet av $F=(4xz,-y^2,yz)$ ut genom kuben $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$

$$\iint_{\partial([0,1]^3)}F\cdot N\,dS=\iiint_{[0,1]^3}\operatorname{div}F=\int_0^1\int_0^1\int_0^1(4z-2y+y)\,dx\,dy\,dz$$

$$=\int_0^1\int_0^1(4z-y)\,dy\,dz=\int_0^1\left(4z-\frac{1}{2}\right)dz=2-\frac{1}{2}=\frac{3}{2}$$

.93 $F=(x^3,y^3,1)$ Y ytan $z=\cos(x^2+y^2)$ $x^2+y^2\le\frac{\pi}{2}$

Beräkna $\iint_Y F\cdot N\,dS$ (normalen uppåt)

$D=\{(x,y,z)\mid x^2+y^2\le\frac{\pi}{2},\ z=0\}$ $K=\{(x,y,z)\mid x^2+y^2\le\frac{\pi}{2},\ 0\le z\le\cos(x^2+y^2)\}$

$\iint_Y F\cdot N\,dS+\iint_D F\cdot N\,dS=\iiint_K\operatorname{div}F\,dx\,dy\,dz$ (normalen ut)

$\iint_D F\cdot N\,dS=\iint_{x^2+y^2\le\frac{\pi}{2}}(x^3,y^3,1)\cdot(0,0,-1)\,dx\,dy=-\iint_{x^2+y^2\le\frac{\pi}{2}}dx\,dy=-\frac{\pi^2}{2}$

$$\iiint_K\operatorname{div}F\,dx\,dy\,dz=\iint_{x^2+y^2\le\frac{\pi}{2}}\int_0^{\cos(x^2+y^2)}(3x^2+3y^2+0)\,dz\,dx\,dy$$

$$=\iint_{x^2+y^2\le\frac{\pi}{2}}3\cos(x^2+y^2)(x^2+y^2)\,dx\,dy$$

$$=\int_{-\pi}^{\pi}\int_0^{\sqrt{\frac{\pi}{2}}}3\cos(r^2)\,r^2\,r\,dr\,d\theta$$

$$=2\pi\int_0^{\sqrt{\frac{\pi}{2}}}3\cos(r^2)\,r^2\,\underbrace{r\,dr}_{=\frac{dt}{2}}=\pi\int_0^{\frac{\pi}{2}}3\cos(t)\,t\,dt$$

$$=\pi\left[3\sin(t)t\right]_0^{\frac{\pi}{2}}-\pi\int_0^{\frac{\pi}{2}}3\sin(t)\,dt$$

$$=\frac{3}{2}\pi^2+\pi\left[3\cos(t)\right]_0^{\frac{\pi}{2}}=\frac{3}{2}\pi^2-3\pi$$

$$\iint_Y F\cdot N\,dS=\frac{3}{2}\pi^2-3\pi-\left(-\frac{\pi^2}{2}\right)=2\pi^2-3\pi$$