10/17 L stympad sfär med ekvation $x^2+y^2+z^2=2,\ z\le 1$

a) Parameterframställning av L med sfäriska koordinater

$$\bar{r}(\varphi,\theta)=\left(\sqrt{2}\sin(\theta)\cos(\varphi), \sqrt{2}\sin(\theta)\sin(\varphi), \sqrt{2}\cos(\theta)\right)$$

$$-\pi\le\varphi\le\pi$$

$$\sqrt{2}\cos(\theta)=z\le 1 \Rightarrow \cos(\theta)\le\frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4}\le\theta\le\pi$$

b) Bestäm energiflödet ut genom L från ljuskälla med intensitet $\bar{U}(\bar{r})=c\frac{\bar{r}}{r^3}$

$$\bar{r}'_\varphi\times\bar{r}'_\theta=\sqrt{2}\begin{pmatrix}-\sin(\theta)\sin(\varphi)\\ \sin(\theta)\cos(\varphi)\\ 0\end{pmatrix}\times\sqrt{2}\begin{pmatrix}\cos(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi)\\ -\sin(\theta)\end{pmatrix}=2\begin{pmatrix}-\sin^2(\theta)\cos(\varphi)\\ -\sin^2(\theta)\sin(\varphi)\\ -\sin(\theta)\cos(\theta)\end{pmatrix}$$

$$\iint_L \bar{U}\cdot\bar{N}\,dS=\int_{-\pi}^{\pi}\int_{\frac{\pi}{4}}^{\pi}\frac{c}{(\sqrt{2})^3}\begin{pmatrix}\cancel{\sqrt{2}}\sin(\theta)\cos(\varphi)\\ \cancel{\sqrt{2}}\sin(\theta)\sin(\varphi)\\ \cancel{\sqrt{2}}\cos(\theta)\end{pmatrix}\cdot\cancel{2}\begin{pmatrix}\sin^2(\theta)\cos(\varphi)\\ \sin^2(\theta)\sin(\varphi)\\ -\sin(\theta)\cos(\theta)\end{pmatrix}d\theta\,d\varphi$$

$$=\int_{-\pi}^{\pi}\int_{\frac{\pi}{4}}^{\pi}c\left(\sin^3(\theta)\cos^2(\varphi)+\sin^3(\theta)\sin^2\varphi+\sin(\theta)\cos^2(\theta)\right)d\theta\,d\varphi$$

$$=\int_{-\pi}^{\pi}\int_{\frac{\pi}{4}}^{\pi}c\sin(\theta)\left(\sin^2(\theta)+\cos^2(\theta)\right)d\theta\,d\varphi=\int_{-\pi}^{\pi}\int_{\frac{\pi}{4}}^{\pi}c\sin(\theta)\,d\theta\,d\varphi$$

$$=\int_{-\pi}^{\pi}\left[c\cos(\theta)\right]_{\frac{\pi}{4}}^{\pi}d\varphi=\int_{-\pi}^{\pi}c\left(1+\frac{1}{\sqrt{2}}\right)=c\,2\pi\left(1+\frac{1}{\sqrt{2}}\right)$$

19.17 Bestäm α så att $\bar{V}=(x+3y, y-2z, x+\alpha z)$ är källfritt

$$0=\operatorname{div}\bar{V}=1+1+\alpha=2+\alpha \Leftrightarrow \alpha=-2$$

B96 $\quad F=(y,\,x,\,1+x^2z)$, $Y:\ (x-z)^2+(y-z)^2=1+z^2 \quad 0\le z\le 1$

Beräkna $\iint_Y F\cdot N\,dS$ där N riktad utåt

$D_0:\ x^2+y^2\le 1 \quad z=0$ $\qquad K:\ (x-z)^2+(y-z)^2\le 1+z^2$

$D_1:\ (x-1)^2+(y-1)^2\le 2 \quad z=1$ $\qquad 0\le z\le 1$

$$\iint_Y F\cdot N\,dS+\underbrace{\iint_{D_0} F\cdot N\,dS}_{\text{riktad ner}}+\iint_{D_1} F\cdot \overset{\text{riktad upp}}{N}\,dS=\iiint_K \operatorname{div} F\,dx\,dy\,dz$$

$$\iiint_K \operatorname{div} F\,dx\,dy\,dz=\iiint_K (0+0+x^2)\,dx\,dy\,dz=\int_0^1\iint_{(x-z)^2+(y-z)^2\le 1+z^2} x^2\,dx\,dy\,dz$$

$$=\begin{bmatrix}x=r\cos(\theta)+z\\ y=r\sin(\theta)+z\end{bmatrix}=\int_0^1\int_0^{\sqrt{1+z^2}}\int_{-\pi}^{\pi}(r\cos(\theta)+z)^2\,r\,d\theta\,dr\,dz$$

$$=\int_0^1\int_0^{\sqrt{1+z^2}}\int_{-\pi}^{\pi}\Big(r^3\overbrace{\cos^2(\theta)}^{=\frac{1}{2}+\frac{\cos(2\theta)}{2}}+2zr^2\cos(\theta)+z^2r\Big)\,d\theta\,dr\,dz$$

$$=\int_0^1\int_0^{\sqrt{1+z^2}}\Big[r^3\Big(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\Big)+2zr^2\sin(\theta)+z^2r\theta\Big]_{-\pi}^{\pi}\,dr\,dz$$

$$=\int_0^1\int_0^{\sqrt{1+z^2}}\big(\pi r^3+2\pi z^2 r\big)\,dr\,dz=\int_0^1\Big(\pi\frac{(1+z^2)^2}{4}+\pi(1+z^2)z^2\Big)\,dz$$

$$=\pi\int_0^1\Big(\frac{5}{4}z^4+\frac{3}{2}z^2+\frac{1}{4}\Big)\,dz=\pi\Big(\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\Big)=\pi$$

$$\iint_{D_0}F\cdot N\,dS=\iint_{D_0}(y,x,1)\cdot(0,0,-1)\,dS=-\iint_{D_0}dS=-\pi$$

$$\iint_{D_1}F\cdot N\,dS=\iint_{D_1}(y,x,1+x^2)\cdot(0,0,1)\,dS=\iint_{D_1}(1+x^2)\,dS$$

$$=\iint_{(x-1)^2+(y-1)^2\le 2}(1+x^2)\,dx\,dy=\begin{bmatrix}x=r\cos(\theta)+1\\ y=r\sin(\theta)+1\end{bmatrix}=\int_0^{\sqrt{2}}\int_{-\pi}^{\pi}\big(1+(r\cos(\theta)+1)^2\big)r\,d\theta\,dr$$

$$=\int_0^{\sqrt{2}}\int_{-\pi}^{\pi}\big(r^3\cos^2(\theta)+2r^2\cos(\theta)+2r\big)\,d\theta\,dr=\int_0^{\sqrt{2}}\Big[r^3\Big(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\Big)+2r^2\sin(\theta)+2r\theta\Big]_{-\pi}^{\pi}dr$$

$$=\int_0^{\sqrt{2}}\big(r^3\pi+4r\pi\big)\,dr=\frac{\sqrt{2}^4}{4}\pi+\frac{4\sqrt{2}^2}{2}\pi=5\pi$$

Så $\iint_Y F\cdot N\,dS=\pi-(-\pi)-(5\pi)=-3\pi$