K4 Beräkna $\int_\Gamma \frac{3z-2}{z^2-z}\,dz$ där a) $\Gamma: |z|=\frac{1}{2}$ b) $\Gamma: |z|=2$

$$f(z)=\frac{3z-2}{z^2-z}=\frac{3z-2}{z(z-1)} \qquad \operatorname{Res}(f,0)=zf(z)\Big|_0=\frac{-2}{-1}=2$$

$$\operatorname{Res}(f,1)=(z-1)f(z)=\frac{3-2}{1}=1$$

a) $\int_\Gamma \frac{3z-2}{z^2-z}\,dz = 2\pi i \operatorname{Res}(f,0) = 4\pi i$

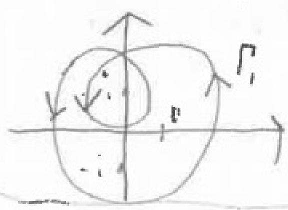
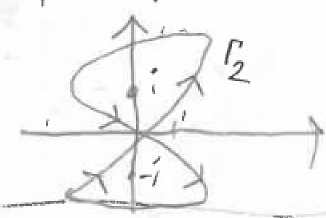
b) $\int_\Gamma \frac{3z-2}{z^2-z}\,dz = 2\pi i\left(\operatorname{Res}(f,0)+\operatorname{Res}(f,1)\right) = 6\pi i$

K6 Beräkna $\int_\Gamma \frac{\cos(z)}{z^3+9z}\,dz$ där Γ cirkeln $|z|=2$

$$f(z)=\frac{\cos(z)}{z^3+9z}=\frac{\cos(z)}{z(z^2+9)}=\frac{\cos(z)}{z(z-3i)(z+3i)}$$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f,0) = 2\pi i \frac{\cos(0)}{0^2+9} = \frac{2}{9}\pi i$$

K7 Beräkna $\int_{\Gamma_k} \frac{dz}{z^2+1}$ där

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}, \quad \operatorname{Res}(f,i)=(z-i)f(z)\Big|_i=\frac{1}{2i}$$

$$\operatorname{Res}(f,-i)=(z+i)f(z)\Big|_{-i}=\frac{-1}{2i}$$

$$\int_{\Gamma_1}\frac{dz}{z^2+1} = 2\pi i\left(2\cdot\operatorname{Res}(f,i)+\operatorname{Res}(f,-i)\right) = 2\pi i\left(\frac{2}{2i}-\frac{1}{2i}\right)=\pi$$

$$\int_{\Gamma_2}\frac{dz}{z^2+1} = 2\pi i\left(\operatorname{Res}(f,i)-\operatorname{Res}(f,-i)\right) = 2\pi i\left(\frac{1}{2i}-\frac{-1}{2i}\right)=2\pi$$

K8 Beräkna $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}$

Sätt $\Gamma_1=(-R,R)$, $\Gamma_2: |z|=R$, $\operatorname{Im}(z)\geq 0$ (positivt orienterad)

$$f(z)=\frac{1}{(z^2+1)(z^2+4)}=\frac{1}{(z-i)(z+i)(z-2i)(z+2i)} \qquad \operatorname{Res}(f,i)=(z-i)f(z)\Big|_i=\frac{1}{2i(-i)3i}=\frac{1}{6i}$$

$$\operatorname{Res}(f,2i)=(z-2i)f(z)\Big|_{2i}=\frac{1}{i\,3i\,4i}=\frac{-1}{12i}$$

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \leq \int_{\Gamma_2}\frac{dz}{|z^2+1|\,|z^2+4|} \leq \int_{\Gamma_2}\frac{dz}{(|z|^2-1)(|z|^2-4)} = \int_{\Gamma_2}\frac{dz}{(R^2-1)(R^2-4)} = \frac{\pi R}{(R^2-1)(R^2-4)}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)} = \lim_{R\to\infty}\left(\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz\right) = \lim_{R\to\infty} 2\pi i\left(\operatorname{Res}(f,i)+\operatorname{Res}(f,2i)\right)$$

$$= 2\pi i\left(\frac{1}{6i}-\frac{1}{12i}\right)=\frac{\pi}{6}$$

K10 Beräkna $\int_0^\infty \frac{\cos(x)\,dx}{(x^2+1)(x^2+4)}$

Notera att $\int_0^\infty \frac{\cos(x)\,dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)\,dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)+i\sin(x)}{(x^2+1)(x^2+4)}\,dx$

$$f(z) = \frac{\cos(z)+i\sin(z)}{(z^2+1)(z^2+4)} = \frac{e^{iz}}{(z-i)(z+i)(z-2i)(z+2i)}$$

$$\mathrm{Res}(f,i) = \frac{e^{-1}}{2i(-i)3i} = \frac{e^{-1}}{6i} \qquad \mathrm{Res}(f,2i) = \frac{e^{-2}}{i\,3i\,4i} = -\frac{e^{-2}}{12i}$$

$\Gamma_1 = [-R,R]$, $\Gamma_2: |z|=R$, $\mathrm{Im}(z)\geq 0$ positivt orienterad

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \leq \int_{\Gamma_2}\frac{|e^{iz}|\,dz}{|z^2+1||z^2+4|} \leq \int_{\Gamma_2}\frac{e^{\mathrm{Re}(iz)}\,dz}{(|z^2|-1)(|z^2|-4)} = \int_{\Gamma_2}\frac{e^{-\mathrm{Im}(z)}\,dz}{(R^2-1)(R^2-4)} \leq \int_{\Gamma_2}\frac{dz}{(R^2-1)(R^2-4)}$$

$$= \frac{\pi R}{(R^2-1)(R^2-4)}$$

så $\int_0^\infty \frac{\cos(x)\,dx}{(x^2+1)(x^2+4)} = \lim_{R\to\infty}\frac{1}{2}\left(\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz\right) = \lim_{R\to\infty} 2\pi i\,\frac{1}{2}\left(\mathrm{Res}(f,i)+\mathrm{Res}(f,2i)\right)$

$$= \pi i\left(\frac{e^{-1}}{6i} - \frac{e^{-2}}{12i}\right) = \frac{\pi}{12}\left(2e^{-1}-e^{-2}\right)$$

K12 Undersök om funktionsföljden likformigt konvergerar i intervallet

a) $x^k(1-x^k)$ $\quad 0\leq x<1$

$x^k(1-x^k) \to 0(1-0) = 0 \quad 0\leq x<1$ $\qquad$ punktvis

$1^k(1-1^k) = 0 \to 0$

$$\frac{d}{dx}\left(x^k(1-x^k)\right) = kx^{k-1}(1-x^k) - x^k kx^{k-1} = kx^{k-1}(1-2x^k)$$

$0 = 1-2x^k \Rightarrow x = \frac{1}{2^{\frac{1}{k}}}$

$\sup_{0\leq x<1}|x^k(1-x^k)| \leq \left(\frac{1}{2^{\frac{1}{k}}}\right)^k\left(1-\left(\frac{1}{2^{\frac{1}{k}}}\right)^k\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4} \not\to 0$ $\quad$ ej likformig

b) $\frac{kx}{1+kx}$ $\quad x\geq 1$

$\frac{kx}{1+kx} = \frac{1}{\frac{1}{kx}+1} \to \frac{1}{0+1} = 1$ $\quad$ punktvis

$\sup_{x\geq 1}\left|\frac{kx}{1+kx}-1\right| = \sup_{x\geq 1}\frac{1}{1+kx} = \frac{1}{1+k} \to 0$ $\quad$ så likformig

c) $\frac{kx}{1+kx}$ $\quad x>0$

$\frac{kx}{1+kx} = \frac{1}{\frac{1}{kx}+1} \to \frac{1}{0+1} = 1$ $\quad$ punktvis

$\sup_{x>0}\left|\frac{kx}{1+kx}-1\right| = \sup_{x>0}\frac{1}{1+kx} = 1 \not\to 0$ $\quad$ ej likformig