MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Mathematics Methods for Economist MM3001
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**Instructions:**

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material apart from the formula sheet given to you.
- The text is written in both English and Swedish, in case of discrepancies between the two the English version is the official one.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- You can use the formula sheet that come with the exam.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear and wrong arguments, even if the final answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) (5pt) Compute the degree 3 Taylor polynomial of the function $f(x) = (x-2)e^{x^2-4}$, around the point $x_0 = 2$, and use it to give an approximation of $f(2.1)$.

**Solution:** We have $f(2) = 0$. We compute the first derivative:

$$\begin{aligned} f'(x) &= e^{x^2-4} + (x-2)(2x)e^{x^2-4} \\ &= (2x^2 - 4x + 1)e^{x^2-4}. \end{aligned}$$

We get that $f'(2) = e^0 = 1$. We compute the second derivative:

$$\begin{aligned} f''(x) &= (4x-4)e^{x^2-4} + 2x(2x^2 - 4x + 1)e^{x^2-4} \\ &= (4x^3 - 8x^2 + 2x + 4x - 4)e^{x^2-4} \\ &= (4x^3 - 8x^2 + 6x - 4)e^{x^2-4} \end{aligned}$$

We get $f''(2) = 8$. Finally we compute the third derivative

$$\begin{aligned} f'''(x) &= (12x^2 - 16x + 6)e^{x^2-4} + 2x(4x^3 - 8x^2 + 6x - 4)e^{x^2-4} \\ &= (8x^4 - 16x^3 + 12x^2 - 8x + 12x^2 - 16x + 6)e^{x^2-4} \\ &= (8x^4 - 16x^3 + 24x^2 - 24x + 6)e^{x^2-4} \end{aligned}$$

we have that $f'''(2) = 54$. In conclusion

$$\begin{aligned} p(x) &= 0 + (x-2) + \frac{8}{2}(x-2)^2 + \frac{54}{3!}(x-2)^3 \\ &= \boxed{(x-2) + 4(x-2)^2 + 9(x-2)^3} \end{aligned}$$

We have that

$$\boxed{p(2.1) = 0.149}$$

(2) Implicit differentiation: Let $C$ a real number. The equation

$$y^2e^{2x} = 3x + y^2 + C$$

defines $y$ as a function of $x$

(a) (1 pt) Determine the value of $C$ for which $y(0) = 3$

(b) (3pt) With $C$ as in point (a), derive $y(x)$ implicitly and compute $y'(0)$. Observe that we have $y(0) = 3$.

(c) With $C$ as in point $(a)$, find the equation of the tangent line to the curve

$$y^2e^{2x} = 3x - y^2 + C$$

in the point $(0, 3)$.

**Solution:** For (a) we have to set $x = 0$ and $y = 3$. We get $9 = -9 + C$, so $C = 18$. For (b), we rewrite the equation as

$$y^2e^{2x} - 3x + y^2 = 18.$$

We derive both sides and we get

$$2yy'e^{2x} + 2y^2e^{2x} - 3 + 2yy' = 0$$

Now we set $x = 0$ and $y = 3$ and we get

$$6y' + 18 - 3 + 6y' = 0.$$

We solve for $y'$ and get $12y' = -15$ which yields

$$\boxed{y'(0) = -\frac{5}{4}}$$

For (c), we use the previous computation to see that the tangent line has slope $-\frac{5}{4}$. In addition the tangent passes through $(0, 3)$. We conclude that

$$t : y = -\frac{5}{4}x + 3$$

(3) Consider the function $f(x) = (x^2 - 1)e^x$.

(a) (2pt) Find where the function is increasing or decreasing, concave or convex.

(b) (1pt) Find all the critical points and determine their type.

(c) (1pt) Find the max and min value of the function on the interval $[0, 2]$.

(d) (1pt) Compute $\lim_{x\to+\infty}$ and determine if the function has a max and/or a min in the interval $[0, +\infty)$

**Solution:** For (a) we have to find the sign of the first and second derivative of $f(x)$. We have that

$$f'(x) = 2xe^x + (x^2 - 1)e^x = (x^2 + 2x - 1)e^x.$$

Since $e^x$ is always positive, we have that $f'(x) > 0$ if, and only if, $x^2 + 2x - 1 > 0$. The polynomial $x^2 + 2x - 1$ has roots $x_\pm = -1 \pm \sqrt{2}$. As the coefficient of $x^2$ is positive we have that $f'(x)' \geq 0$ whenever $x \leq -1 - \sqrt{2}$ or $x \geq -1 + \sqrt{2}$. We conclude that $f(x)$ is increasing when $x \leq -1 - \sqrt{2}$ or $x \geq -1 + \sqrt{2}$ and decreasing when $-1 - \sqrt{2} \leq x \leq -1 + \sqrt{2}$. For convexity we have to compute the second derivative

$$f''(x) = (2x + 2)e^x + (x^2 + 2x - 1)e^x = (x^2 + 4x + 1)e^x.$$

We have, again, that $f''(x) \geq 0$ whenever $x^2 + 4x + 1$ is. Reasoning as above, we get that $f''(x) \geq 0$ whenever $x \leq -2 - \sqrt{3}$ or $x \geq -2 + \sqrt{3}$. We conclude that $f$ is convex when $x \leq -2 - \sqrt{3}$ or $x \geq -2 + \sqrt{3}$ and concave when $-2 - \sqrt{3} \leq x \leq -2 + \sqrt{3}$.

For (b): whe have that the critical points are exactly the points where $f'(x) = 0$ which we computed in (a) to be $-1 \pm \sqrt{2}$. To find their type, we either use the second derivative test and compute $f''(-1 \pm \sqrt{2})$. We find that $f''(-1-\sqrt{2}) < 0$ so that this is a local max, and $f''(-1+\sqrt{2}) > 0$ and this is a local min. Alternatively we use the first derivative test:

| x | | -1$-\sqrt{2}$ | | $-1+\sqrt{2}$ | |
|---|---|---|---|---|---|
| $f'(x)$ | + | 0 | - | 0 | + |
| $f(x)$ | $\nearrow$ | max | $\searrow$ | min | $\nearrow$ |

Now we attak (c). The exteme values are taken either in the extreme of the interval or in the point inside the interval where the derivative vanishes. Thus candidate for extreme point are 0, 2, and $-1+\sqrt{2}$ we compute $f(x)$ in these three points and we get that maximum value of the function is $f(2) = 3e^2$ and the minimum value is $f(-1+\sqrt{2})$.

We proceed with (d). As $x$ grows both $x^2 - 1$ and $e^x$ grows, so the limit is $+\infty$. This means that the function has no max between $[0, +\infty)$. On the other side, due to the increasing/decreasing pattern the function cannot go below $f(-1+\sqrt{2})$ in the given interval. Thus this is a minimum.

(4) Compute the following integrals:
   (a) (2pt) $\int \left(t^2 \ln(2t) + e^{3t}\right) dt$,
   (b) (3pt) $\int_0^1 \frac{3y}{(y^2-1)^3} dy$.
   (**Attention:** The function $1/(y^2-1)^3$ is not defined in $x = 1$ this is an improper integral)

**Solution**

$$\begin{aligned}
\int t^2 \ln(2t) + e^{3t} dt &= \int t^2 \ln(2t) dt + \int e^{3t} dt \\
&= \int \left(\frac{u}{2}\right)^2 \ln(u) \frac{du}{2} + \int e^v \frac{dv}{3} \\
&= \frac{1}{8} \int (u)^2 \ln(u) du + \frac{e^v}{3} + C_1 \\
&= \frac{1}{8} \left( \ln(u) \frac{u^3}{3} - \int ( \frac{u^3}{3} \cdot \frac{1}{u} du \right) + \frac{e^{3t}}{3} + C_1 \\
&= \frac{1}{8} \left( \ln(2t) \frac{8t^3}{3} - \frac{u^3}{9} + C_2 \right) + \frac{e^{3t}}{3} + C_1 \\
&= \boxed{\frac{t^3}{3} \ln(2t) - \frac{t^3}{9} + \frac{e^{3t}}{3}}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{3y}{(y^2-1)^3} dy &= \lim_{b \to 1} \int_0^b \frac{3y}{(y^2-1)^3} dy \\
&= \lim_{b \to 1} \frac{3}{2} \int_{-1}^{b^2-1} \frac{1}{u^3} du \\
&= \lim_{b \to 1} \frac{3}{2} \left[ -\frac{u^{-2}}{2} \right]_{-1}^{b^2-1} \\
&= \frac{3}{2} \left( \lim_{b \to 1} -\frac{(b^2-1)^{-2}}{2} + \frac{1}{2} \right) = +\infty
\end{aligned}$$

(5) Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ a & -1 & 1 \end{pmatrix}$$

(a) (2 pt) Compute the determinant of $A$, $|A|$ as a function of $a$.
(b) (1 pt) Find all the values of $a$ for which $A$ is not invertible.
(c) (2 pt) Set Now $a = -1$ and find the solution to the following linear system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

**Solution:**

$$\begin{vmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ a & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 \\ +1 & -1 & -1 \\ a+1 & -1 & 1 \end{vmatrix} = -2(1 + a + 1) = -2(a + 2)$$

Given (a) we have that $\det(A) = 0$ if, and only if $a = -2$. We conclude that the matrix is not invertible when $a = -2$.
For (c) we run Gauss–Jordan elimination

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & -3 \\ 0 & -1 & -1 & 5 \\ -1 & -1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & -1 & 1 & 4 \\ 0 & -1 & -1 & 5 \\ 2 & 2 & 0 & -3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & -1 & 1 & 4 \\ 0 & -1 & -1 & 5 \\ 0 & 0 & 2 & 5 \end{array}\right)$$

From the last equation we get $z = \frac{5}{2}$. The second equation is $-y - z = 5$. We plug in the result from efore and get $y = -5 - \frac{5}{2} = \frac{-15}{2}$. Finally we have that the first equation is $-x - y + z = 4$. Thus $x = -4 + \frac{15}{2} + \frac{5}{2=6}$. The final answer is

$$\boxed{(w, y, z) = \left(6, -\frac{15}{2}, \frac{5}{2}\right)}$$

(6) Consider the two variables function

$$f(x, y) = x^2 - xy + y$$

defined on the rectangle

$$D = \{(x, y) \mid 2 \geq x \geq 0,\ 3 \geq y \geq 0, \}$$

(a) (2pt) Find all the critical points of $f(x, y)$ - even those lying outside $D$ and determine their type.
(b) (2pt) Determine the maximum and minimum values of $f$ on *boundary* of $D$. (In order to get credit you have to explain what you are doing, the correct answer without the right explanation will not be accepted)
(c) (1 pt) Determine the minimum and the maximum value of $f(x, y)$ on $D$. (In order to get credit you have to explain what you are doing, the correct answer without the right explanation will not be accepted)

**Solution:** To answer (a) we have to see where the partial derivatives of $f(x,y)$ vanish simultaneously.. We have that

$$\frac{\partial}{\partial x} f(x,y) = 2x - y$$

and

$$\frac{\partial}{\partial y} f(x,y) = -x + 1$$

We set them both equal to 0 and we get that there is only one critical point

$$\boxed{(x_0, y_0) = (1,2)}$$

To determin its type we have to compute the Hessian of $f(x,y)$. We have that

$$\begin{aligned} H(x,y) &= \begin{vmatrix} \frac{\partial^2}{\partial x^2} f(x,y) & \frac{\partial^2}{\partial x \partial x} f(x,y) \\ \frac{\partial^2}{\partial x \partial y} f(x,y) & \frac{\partial^2}{\partial y^2} f(x,y) \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} \\ &= 2 \cdot 0 - (-1) \cdot (-1) = -1 < 0 \end{aligned}$$

As the Hessian is always negative we have that the critical point is a saddle point.
We proceed now with point (b). The domain $D$ consists of four sides:

(a) $x = 0$ and $y \in [0,3]$
(b) $x = 2$ and $y \in [0,3]$
(c) $y = 0$ and $x \in [0,2]$
(d) $y = 3$ and $x \in [0,2]$

We have to restrict $f(x,y)$ to each side and find its global extreme points.

(a) We set $g(y) = f(0,y) = y$. We have to find max and min value of $g(y)$ when $y \in [0,3]$. Observe that $g$ is linear so has no critical point. Thus the min value is 0 and the max value is 3, The min value is 0 and the max value is 3 taken in $y = 0$ and $y = 3$ respectively.
(b) We set $g(y) = f(2,y) = 4 - y$. We have to find max and min value of $g(y)$ when $y \in [0,3]$. Observe that $g$ is linear so has no critical point. Thus the max value is 4 - take in in $y = 0$ and the min value is 1 taken in $y = 3$.
(c) We set $g(x) = f(x,0) = x^2$. We have to find max and min value of $g(x)$ when $x \in [0,2]$. The only critical point of $g$ is $x = 0$ which is not in the interior of the interval. Thus the max and min value are taken in the extremes. Thus the max value is 4 - take in in $x = 2$ and the min value is 0 taken in $x = 0$.
(d) We set $g(x) = f(x,3) = x^2 - 3x3$. We have to find max and min value of $g(x)$ when $x \in [0,2]$. The only critical point of $g$ is $2x - 3 = 0$, that is $x = \frac{3}{2}$ which is in the interior of the interval. We have that $g(0) = 3$, $g(2) = 1$, and $g\left(\frac{3}{2}\right) = \frac{3}{4}$. We deduce that the minimum of $g$ is $\frac{3}{4}$ taken in $x = \frac{3}{2}$ and the max is 3 -taken in $x = 0$.

In order to find the max and min of $f(x,y)$ on the boundary we have to compare the values that we got on the four sides. We have that the minimum value for $f(x,y)$ on the boundary is 0 - taken in the point $(0,0)$. The maximum value of $f(x,y)$ on the boundary is 4, taken at the point $(2,0)$.
Finally, to answer (c) we have to compare the values obtained in (b) with the value of $f(x,y)$ computed at any critical point in the interior of $D$. The

only critical point we found in (a) is $(1, 2)$ which is indeed in the interior of $D$. We have that $f(1, 2) = 1$. Thus we have that the maximum and minimum of $f(x, y)$ in $D$ are 0, taken in $(0, 0)$ and 4, taken in $(2, 0)$

GOOD LUCK!!!

**Senska texten, (formular finns ovanför)**

(1) (5pt) Beräkna grad 3 Taylor polinom till funktioner $f(x) = (x-2)e^{x^2-4}$, omkring punkten $x_0 = 2$, och använda det för approximera $f(2.1)$.

(2) La $C$ et reelt tal. Ekvationen
$$y^2 e^{2x} = 3x + y^2 + C$$
definerar $y$ som en funktion av $x$
 (a) (1 pt) Beräkna värden till $C$ så där $y(0) = 3$
 (b) (3pt) Med $C$ som i (a), deriverar $y(x)$ och räkna $y'(0)$. Observera att $y(0) = 3$.
 (c) Med $C$ som i (a), hitta ekvationent till linjen som är tangenten till curvan
$$y^2 e^{2x} = 3x + y^2 + C$$
 i $(0, 3)$.

(3) Betrakta funktionen $f(x) = (x^2 - 1)e^x$.
 (a) (2pt) Bestämm var funktionen är vaxande och avtagende, concave eller convex
 (b) (1pt) Hitta alla kritiska punkter och bestäm deras typ.
 (c) (1pt) Hitta den största och den minsta värden till funktionen i $[0, 2]$.
 (d) (1pt) Räkna $\lim_{x\to\pm\infty} f(x)$ och saga om funktionen har en max våren i intervallen $[0, +\infty]$

(4) Räkna de följande integralerna:
 (a) (2pt) $\int \left(t^2 \ln(2t) + e^{3t}\right) dt$,
 (b) (3pt) $\int_0^1 \frac{3y}{(y^2-1)^3} dy$.
 (**Attention:** Funktionen $1/(y^2-1)^3$ är icke-definerad i $x = 1$)

(5) Betrakta matrisen
$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ a & -1 & 1 \end{pmatrix}$$
 (a) (2 pt) Räkna determinanter till $A$, $|A|$ som en funktion av $c$.
 (b) (1 pt) Hitta alla värder $c$ sådan att $A$ inte är invertebär.
 (c) (2 pt) Set $a = -1$ och hitta lösningent till systemet
$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

(6) Betrakta den foljande funktionen av två variabler
$$f(x, y) = x^2 - xy + y$$
som defineras i
$$D = \{(x, y) \mid 2 \geq x \geq 0,\ 3 \geq y \geq 0, \}$$
 (a) (2pt) Hitta alla kritiska punkter till $f(x, y)$ - punkter som ligger utanför $D$ också behövs att hitta.
 (b) (2pt) Hitta den storsta och den minsta värden till $f$ på *gränsen* av $D$.

(c) (1 pt) Beräkna den storsta och den minst värden till $f$ på på $D$.

LYCKA TILL!!!