Stochastic Processes and Simulation II

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- Book: S.M. Ross, Introduction to Probability Models, 11th edition, Academic Press, 2014.
- Homepage on kurser.math.su.se, course MT5012 (VT24).
  Register also in Ladok.
- Schedule: Monday and Thursday (not on 01/04, 09/05).
  09:15-10:00, discussion of exercises,
  10:15-11:00, lecture,
  11:15-12:00, lecture.
  Computer labs on 29/04 and 02/05.
- Written exam + report of lab exercises (with deadlines).
  Exam on 22/05, 08:00-13:00.
  Re-exam on 20/08, 08:00-13:00.

Contents

- Poisson processes and continuous-time Markov chains (Chapters 5-6), Sessions 1-2.
- Renewal theory (Chapter 7), Sessions 3-5.
- Queueing theory (Chapter 8), Sessions 6-7.
- Simulation (Chapter 11), Sessions 8-10.
- Brownian motion and stationary processes (Chapter 10), Sessions 11-12.

Poisson processes

Let $N(t)$ be the number of arrivals by time t, with $N(0) = 0$.

A stochastic process $\{N(t), t \geq 0\}$ is said to be a (homogeneous) **Poisson process** with rate $\lambda > 0$ if is independent of the past (independent increments) and the number of arrivals in any interval of length t is $\mathrm{Po}(\lambda t)$ (stationary increments).

Interarrival times are $\mathrm{Exp}(\lambda)$.

- **Nonhomogeneous Poisson process**: arrival rate $\lambda = \lambda(t)$; no longer stationary increments, arrivals may be more likely to occurr at certain times.
- **Compound Poisson process**: groups arrive according to a Poisson process and their sizes are i.i.d..
- **Mixed Poisson process**: conditional on a r.v. $L = \lambda$, we have a Poisson process with rate λ; increments are stationary, but not independent, so it is not a Poisson process.

Continuous-time Markov chains

A stochastic process $\{X(t), t \geq 0\}$ is a **continuous-time Markov chain** if the future depends only on the present and not on the past, i.e., if

$$\mathbb{P}(X(t+s) = j \,|\, X(s) = i, X(u) = x(u), 0 \leq u < s) \\ = \mathbb{P}(X(t+s) = j \,|\, X(s) = i).$$

- The process moves between states according to a discrete-time Markov chain, but the amount of time it spends in each state, before jumping to the next state, is exponentially distributed.
- The **transition probabilities** that a process now in state i will be in state j at a time t later

$$P_{ij}(t) = \mathbb{P}(X(t+s) = j \,|\, X(s) = i).$$

- If the limiting probabilities $P_j = \lim_{t\to\infty} P_{ij}$ exist (sufficient conditions), the vector $(P_j)_j$ is a **stationary distribution** and P_j represents the long-run proportion of time that the process is in state j.

Renewal theory

Let $N(t)$ be the number of arrivals by time t. If the interarrival times X_n are i.i.d., then $\{N(t), t \geq 0\}$ is a **renewal process**.

The time of the n-th renewal is $S_n = \sum_{i=1}^{n} X_i$ and $N(t) \geq n$ iff $S_n \leq t$.

Limit theorems: $\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[X_n]}$ and (ERT) $\frac{\mathbb{E}[N(t)]}{t} \to \frac{1}{\mathbb{E}[X_n]}$ a.s..

- **Reward renewal process** $R(t) = \sum_{i=1}^{N(t)} R_n$, with R_n i.i.d. rewards at each renewal.
  Limit theorem (RRT): $\frac{R(t)}{t} \to \frac{\mathbb{E}[R_n]}{\mathbb{E}[X_n]}$ and $\frac{\mathbb{E}[R(t)]}{t} \to \frac{\mathbb{E}[R_n]}{\mathbb{E}[X_n]}$ a.s..
- **Regenerative process**: there exist time points at which the process restarts itself; they consitute the arrival times of a renewal process.
- **Semi-Markov process**: as a continuous-time Markov chain, but the time it spends in a state before jumping is not exponential.

Inspection paradox: the length of a renewal interval containing a specific time point tends to be larger than an ordinary renewal interval.

Queueing theory

Queueing system: customers arrive randomly; they might have to wait some time in queue; once served they leave the system.

Condition: mean departure time $<$ mean arrival time.

What is the **average number of customers** in the system? What is the **average waiting time**?

- $M/M/1$ queue: Poisson (memoryless) arrivals, exponential (memoryless) service times, 1 server.
- $M/G/1$ queue: Poisson (memoryless) arrivals, general service times, 1 server.
- $M/M/k$ queue: Poisson (memoryless) arrivals, exponential (memoryless) service times, k servers.

The PASTA principle: the proportion of arrivals that find n customers equals the proportion of time that the system contains n customers.

Simulation

Given a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with density $f(x_1, \ldots, x_n)$ and some n-dimensional function g, how do we compute

$$\mathbb{E}[g(\mathbf{X})] = \int \int \cdots \int g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n,$$

if analytic computation is not possible?

Monte Carlo simulation.

1. Simulate r independent random vectors $\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_n^{(i)})$, $i = 1, \ldots, r$, having density $f(x_1, \ldots, x_n)$.
2. Compute $Y_i = g(\mathbf{X}^{(i)})$.
3. SLLN: $\lim_{r\to\infty} \frac{\sum_{i=1}^r Y_i}{r} = \mathbb{E}[Y_i] = \mathbb{E}[g(\mathbf{X})]$ a.s..

How to simulate random vectors with specified joint distribution? How to reduce the variance of $\frac{\sum_{i=1}^r Y^{(i)}}{r}$? How to simulate Poisson processes? How to simulate Markov chains?

Brownian motion

Consider a **symmetric random walk**, which at each time is equally likely to go one step up or down. Take smaller and smaller steps in smaller and smaller time intervals, and eventually take the limit.

The **standard Brownian motion** $\{B(t), t \geq 0\}$ is a scaled symmetric random walk defined as

$$B_{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \to B(t) \sim \mathcal{N}(0, t) \qquad \text{as } n \to \infty.$$

where $\{S_{i+1} - S_i, i \geq 0\}$ take values $+1$ and -1 with equal probability.

- The Brownian motion $\{B(t), t \geq 0\}$ has **independent and stationary increments**.
- Brownian motion **with drift** μ and variance σ^2: $X(t) = \sigma B(t) + \mu t$.

Gaussian process: if $X(t_1), \ldots, X(t_n)$ have joint normal distribution.

Stationary process: if $X(t_1), \ldots, X(t_n)$ and $X(t_1 + s), \ldots, X(t_n + s)$ have the same joint distribution.