Posson processes and continuous-time Markov chains

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The exponential distribution

A continuous random variable X is said to have an **exponential distribution** with paramenter $\lambda > 0$, if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

or, equivalently, if

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)dt = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Let $X \sim \mathrm{Exp}(\lambda)$.

- $\mathbb{E}[X] = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}$.
- $M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \frac{\lambda}{\lambda - t}$, for $t < \lambda$.

Properties

- **Memoryless.** $\mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s)$, for all $s, t \geq 0$. The exponential distribution is the only one with this property.
  Example 5.2: time spent in a bank.
  Example 5.3: post office.
- **Comparing exponentials.** $\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.
- **Minimum of exponentials.** Assume that $X_1, \ldots, X_n$ are independent exponentials with parameters λ_i, respectively. Then

$$\min_i X_i \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \lambda_i\right)$$

and, for each $j = 1, \ldots, n$,

$$\mathbb{P}(X_j = \min_i X_i) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}.$$

Moreover, the value of the minimum and the rank ordering of the X_i are independent.
Example 5.8: post office.

The gamma distribution

If $X_1, \ldots, X_n$ are i.i.d. exponentials with parameter λ, then their sum $S_n = \sum_{i=1}^n X_i$ has a **gamma distribution** with parameters n and λ, and we write $S_n \sim \Gamma(n, \lambda)$.

- $f_{S_n}(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}$.
- $F_{S_n}(x) = \mathbb{P}(S_n \leq x) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$
- $\mathbb{E}[S_n] = \frac{n}{\lambda}, \qquad \mathrm{Var}(S_n) = \frac{n}{\lambda^2}$.
- $M_{S_n}(t) = \mathbb{E}[e^{tS_n}] = (1 - \frac{t}{\lambda})^{-n}$, for $t < \lambda$.

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Counting processes

A stochastic process $\{N(t), t \geq 0\}$ is said to be a **counting process** if $N(t)$ represents the total number of "events" that occur by time t. A counting process must satisfy:

(i) $N(0) = 0$;

(ii) $N(t)$ is integer valued:

(iii) if $s < t$, then $N(s) \leq N(t)$;

(iv) for $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

A counting process has:

- **independent increments**, if the numbers of events that occur in disjoint time intervals are independent;
- **stationary increments**, if the distribution of the number of events that occur in any time interval depends only on the length of the interval.

Poisson processes

Notation: The function $f(\cdot)$ is said to be $o(h)$ if $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

The counting process $\{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate $\lambda > 0$ if the following holds:

(i) $N(0) = 0$;

(ii) $N(t)$ has independent increments;

(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h)$;

(iv) $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Theorem

If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, then, for all $s, t > 0$, $N(s+t) - N(s)$ is Poisson distributed with mean λt. In other words, the number of events in any interval of length t is a Poisson r.v. with mean λt.

It follows that the Poisson process has stationary increments.

Interarrival and waiting times

- **Interarrival times.** Denote the time of the first event by T_1 and, for $n > 1$, let T_n denote the time between the $(n-1)$-st and the n-th event. The interarrival times $T_n, n = 1, 2, \dots$ are i.i.d. exponential r.v.'s with parameter λ.
(The assumption of stationary and independent increments is equivalent to the memoryless property, hence interarrival times are exponentials.)
- **Waiting times.** For $n \geq 1$, denote the waiting time until the n-th event by S_n. Note that $S_n = \sum_{i=1}^{n} T_i \sim \Gamma(n, \lambda)$. We have that $S_n \leq t$ if and only if $N(t) \geq n$.

Combining and splitting Poisson processes

- **Combining.** If $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson Processes with rates λ_1 and λ_2 respectively, then $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.
- **Splitting.** Consider a Poisson process with rate λ and classify its arrivals as type 1 with probability p or type 2 with probability $1 - p$, independently of all other events. If $N_1(t)$ and $N_2(t)$ denote respectively the number of type I and type II events occurring in $[0, t]$, then $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes with rates λp and $\lambda(1 - p)$, respectively. Moreover, the two processes are independent.
  Note that this can be generalized to more than two types.

Order statistics property

Given a collection of r.v.'s $Y_1, \dots, Y_n$, we say that $Y_{(1)}, \dots, Y_{(n)}$ are its **order statistics** if $Y_{(k)}$ is the k-th smallest value among $Y_1, \dots, Y_n$, for $k = 1, \dots, n$.

Theorem (Order statistics property)

Given that $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent $U(0, t)$ r.v.'s. In other words, the times $S_1, \dots, S_n$, considered as unordered r.v.'s, are distributed independently and uniformly in $(0, t)$.

Given S_n, the first $(n-1)$ arrival times $S_1, \dots, S_{n-1}$ are distributed as the ordered values of a set of $U(0, S_n)$ r.v.'s.

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Continuous-time Markov chains

A stochastic process $\{X(t), t \geq 0\}$ is a **continuous-time Markov chain** if for all $s, t \geq 0$ and nonnegative integers $i, j, x(u), 0 \leq u < s$,

$$\mathbb{P}(X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s)$$
$$= \mathbb{P}(X(t+s) = j \mid X(s) = i).$$

In other words, the process has the Markovian property that the conditional distribution of the future given the present and the past depends only on the present and is independent of the past.

- If $\mathbb{P}(X(t+s) = j \mid X(s) = i)$ is independent of s, then the process is said to have stationary (homogeneous) transition probabilities.
- If T_i denotes the amount of time that the process stays in state i before jumping into a different state, then $\mathbb{P}(T_i > t+s \mid T_i > s) = \mathbb{P}(T_i > t)$. We have that T_i is memoryless, hence exponentially distributed.

Alternative definition

A stochastic process $\{X(t), t \geq 0\}$ is a **continuous-time Markov chain** if each time it enters state i the following holds:

(i) the amount of time it spends in that state before jumping into a different state is exponentially distributed wih parameter v_i;

(ii) when the process leaves state i, it enters state j with probability P_{ij}, such that $P_{ii} = 0$ and $\sum_j P_{ij} = 1$.

In other words, the process moves from state to state in accordance with a discrete-time Markov chain, but the amount of time it spends in each state, before jumping to the next state, is exponentially distributed.

- The amount of time the process spends in state i and the next state visited are independent r.v.'s.

Birth and death processes

A **birth and death process** is a continuous-time Markov chain with states $\{0, 1, \dots\}$ for which transitions from state i can go only to either state $i-1$ or state $i+1$.
For example, let the states represent the number of people in a system. If from state i people arrive at an exponential rate λ_i and leave at an exponential rate μ_i, then the parameters $\{\lambda_i\}_{i=0}^{\infty}$ and $\{\mu_i\}_{i=0}^{\infty}$ rare called the birth and death rates, respectively.

- Note that if, for all $i \geq 0$, $\lambda_i = \lambda$ and $\mu_i = 0$, then the process is a Poisson process.
- If T_i denotes the time it takes for the process to reach state $i+1$ starting from state i, then $\mathbb{E}[T_0] = \frac{1}{\lambda_0}$ and $\mathbb{E}[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\mathbb{E}[T_{i-1}]$.

The transition probabilities

Let $P(t)$ be the matrix with entries

$$P_{ij}(t) = \mathbb{P}(X(t+s) = j \,|\, X(s) = i)$$

which represent the **transition probabilities** that a process now in state i will be in state j at a time t later.

For any pair of states i and j, let $q_{ij} = v_i P_{ij}$ be the rates at which the process from state i jumps to state j. Recall that v_i is the rate at which the process makes a transition when in state i.

Let Q the **transition rate matrix** with diagonal elements $-v_i$ and off-diagonal elements q_{ij}. Note that Q determines the Markov chain:

- $v_i = \sum_j v_i P_{ij} = \sum_j q_{ij}$;
- $P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$.

Kolmogorov's equations

Theorem (Kolmogorov's backward equations)

For all states i, j and times $t \geq 0$, $P'(t) = QP(t)$, i.e.,

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

Theorem (Kolmogorov's forward equations)

Under suitable regularity conditions, for all states i, j and times $t \geq 0$, $P'(t) = P(t)Q$, i.e.,

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

The forward equations hold for all birth and death processes and all processes with a finite number of states.

Stationary distribution

Assume that, for each state j, the limiting probability $P_j = \lim_{t\to\infty} P_{ij}(t)$ exists and is independent of the initial state i. Then the vector of $(P_j)_j$ is a **stationary distribution** and it satisfies the **balance equations**

$$v_j P_j = \sum_{k\neq j} q_{kj} P_k, \qquad \text{for all states } j.$$

Note that P_j represents the long-run proportion of time that the process is in state j.

- A sufficient condition for the stationary distribution to exist is that the chain is irreducible (all the states communicate) and positive recurrent (the mean return time to each state is finite).
- In a continuous-time Markov chain, if a stationary distribution exists, then it is unique.

Example

Examples 6.1 and 6.15: a shoe shine shop.

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Nonhomogeneous Poisson processes

The counting process $\{N(t), t \geq 0\}$ is said to be a **nonhomogeneous Poisson process** with intensity function $\lambda(t) > 0$, $t \geq 0$ if the following holds:

(i) $N(0) = 0$;

(ii) $N(t)$ has independent increments;

(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$;

(iv) $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$.

We no longer require stationary increments. Instead, we allow for the possibility that events may be more likely to occur at certain times.

Properties

The **mean value function** is defined by $m(t) = \int_0^t \lambda(y)\, dy$.

Theorem

If $\{N(t), t \geq 0\}$ is a nonstationary Poisson process with intensity function $\lambda(t)$, $t \geq 0$, then $N(t+s) - N(s)$ is a Poisson r.v. with mean $m(t+s) - m(s) = \int_s^{t+s} \lambda(y)\, dy$

- If $\{N(t), t \geq 0\}$ is an ordinary Poisson process with rate 1, and $\{\hat{N}(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)$, then $\{\hat{N}(t), t \geq 0\}$ is distributed as $\{N(m(t)), t \geq 0\}$.
- Conditioned on $\hat{N}(t) = n$, the points of $\{\hat{N}(s), s \in (0, t)\}$ are i.i.d. on $(0, t)$ with distribution function $\frac{m(s)}{m(t)}$.

Time sampling

Time sampling an ordinary Poisson process generates a nonhomogeneous Poisson process.

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Suppose that an event occurring at time t is counted with probability $p(t)$, independently of the past.
- If $N_c(t)$ denotes the number of counted events by time t, then $\{N_c(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t) = \lambda p(t)$.

Note: This can be used when simulating nonhomogeneous Poisson processes.

Example

Example 5.24: food stand.

Compound Poisson processes

A stochastic process $\{X(t), t \geq 0\}$ is said to be a **compound Poisson process** if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0,$$

where $\{N(t), t \geq 0\}$ is a Posson process, and $\{Y_i, i \geq 1\}$ is a family of i.i.d. r.v.'s independent of $\{N(t), t \geq 0\}$.

- $\mathbb{E}[X(t)] = \lambda t \mathbb{E}[Y_1], \quad \mathrm{Var}(X(t)) = \lambda t \mathbb{E}[Y_1^2]$.

Properties

- **Countably many values.** Suppose that Y_i can only take countably many values α_j, $j \geq 1$, and $\mathbb{P}(Y_i = \alpha_j) = p_j$. If $N_j(t)$ denotes the number of type j events by time t, then $N_j(t)$, $j \geq 1$ are independent Poisson r.v.'s with respective means $\mathbb{E}[N_j(t)] = \lambda p_j t$. Hence, the cumulative sum at time t can be expressed as $X(t) = \sum_j \alpha_j N_j(t)$.

  As a consequence, as t grows large, the distribution of $X(t)$ converges to the normal distribution.
- **Sum of independent compound Poisson processes.** If $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are independent compound Poisson processes with respective parameters and distributions λ_1, F_1 and λ_2, F_2, then $\{X(t) + Y(t), t \geq 0\}$ is a compound Poisson process with parameter $\lambda_1 + \lambda_2$ and distribution $F = \frac{\lambda_1}{\lambda_1+\lambda_2} F_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} F_2$.

Example

Example 5.27: busy periods in a single-server queue.

Mixed Poisson processes

For a positive r.v. L, the counting process $\{N(t), t \geq 0\}$ is said to be a **mixed Poisson process** if, conditional on $L = \lambda$, it is a Poisson process with rate λ.

A mixed Poisson process has stationary increments: if L is continuous with density function g, then

$$\begin{aligned}\mathbb{P}(N(t+s) - N(s) = n) &= \int_0^\infty \mathbb{P}(N(t+s) - N(s) = n \,|\, L = \lambda) g(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(\lambda)\, d\lambda.\end{aligned}$$

In general, a mixed Poisson process does not have independent increments: knowing how many events occur in an interval gives information about the possible value of L, which affects the distribution of the number of events in any other interval. Hence, it is not a Poisson process or a Markov process.

Mean and variance

From the laws of total expectation and variance, we get that the **mean**

$$\mathbb{E}[N(t)] = \mathbb{E}_L[\mathbb{E}[N(t) \mid L]] = \mathbb{E}_L[Lt] = t\mathbb{E}[L]$$

and the **variance**

$$\begin{aligned}\mathrm{Var}(N(t)) &= \mathbb{E}_L[\mathrm{Var}(N(t) \mid L)] + \mathrm{Var}_L(\mathbb{E}[N(t) \mid L]) \\ &= \mathbb{E}_L[Lt] + \mathrm{Var}_L(Lt) \\ &= t\mathbb{E}[L] + t^2\mathrm{Var}(L).\end{aligned}$$

The conditional density function of L given that $N(t) = n$ is

$$f_{L|N(t)=n}(\lambda) = \frac{e^{-\lambda t}\lambda^n g(\lambda)}{\int_0^\infty e^{-\lambda t}\lambda^n g(\lambda)\, d\lambda}, \qquad \lambda \geq 0.$$

Example

Example 5.30: insurance company.

Exercises

Session 1. Chapter 5: 3, 8, 36, 40, 45, 49, 60.

Session 2. Chapter 5: 46, 78, 81b (assume that the result of 81a is given), 95.