Renewal theory

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Renewal processes

Let $\{N(t), t \geq 0\}$ be a counting process and let X_n denote the time between the $(n-1)$-st and the n-th event, $n \geq 1$. If $X_1, X_2, \ldots$ are i.i.d., then $\{N(t), t \geq 0\}$ is said to be a **renewal process**. When an event occurs, we say that a renewal has taken place.

- The interarrival times have the same distribution F.
- Assume $F(0) = \mathbb{P}(X_i = 0) < 1$.
- Mean time between successive renewals $\mu = \mathbb{E}[X_n] > 0, n \geq 1$.
- Define $S_0 = 0$ and, for $n \geq 1$, the time of the n-th renewal $S_n = \sum_{i=1}^{n} X_i$.

Examples: a homogeneous Poisson process, $N(t)$ is the number of lightbulbs with i.i.d. lifetimes that have failed by time t (assuming that as soon as one fails it is replaced by a new one).

Number of renewals

- The **number of renewals** in a finite amount of time is finite. i.e.,

$$\mathbb{P}(N(t) < \infty) = 1, \quad \text{for each } t.$$

Proof. Write $N(t) = \max\{n : S_n \leq t\}$. By the SLLN, $\frac{S_n}{n} \to \mu$ a.s. as $n \to \infty$. Since $\mu > 0$, then $S_n \to \infty$ as $n \to \infty$. Hence S_n can be less than or equal to t for at most a finite number of values of n. □

- The total number of renewal that occur is infinite a.s., i.e.,

$$N(\infty) = \lim_{t\to\infty} N(t) = \infty \quad \text{a.s..}$$

Proof. We have that

$$\mathbb{P}(N(\infty) < \infty) = \mathbb{P}(X_n = \infty \text{ for some } n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty}\{X_n = \infty\}\right)$$

$$\leq \sum_{n=1}^{\infty} \mathbb{P}(X_n = \infty) = 0.$$ □

Distribution of $N(t)$

Note that $N(t) \geq n$ if and only if $S_n \leq t$.

Two ways to obtain the **distribution of** $N(t)$.

- We have that

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1) \\ &= \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) \\ &= F^{*n}(t) - F^{*(n+1)}(t).\end{aligned}$$

- Alternatively, by conditioning on S_n,

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \int_0^\infty \mathbb{P}(N(t) = n \,|\, S_n = y) f_{S_n}(y)\, dy \\ &= \int_0^t \mathbb{P}(X_{n+1} > t - y \,|\, S_n = y) f_{S_n}(y)\, dy \\ &= \int_0^t (1 - F(t-y)) f_{S_n}(y)\, dy.\end{aligned}$$

The renewal equation

- The mean-value or renewal function

$$m(t) = \mathbb{E}[N(t)] = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \sum_{n=1}^{\infty} F^{*n}(t),$$

  and it uniquely determines the renewal process.
- $m(t) < \infty$ for all $t > 0$.
- **Renewal equation.** Assuming that the interarrival distribution F is continuous with density function f,

$$\begin{aligned} m(t) &= \int_0^{\infty} \mathbb{E}(N(t) \,|\, X_1 = x) f(x)\, dx \\ &= \int_0^t (1 + \mathbb{E}[N(t-x)]) f(x)\, dx \\ &= F(t) + \int_0^t m(t-x) f(x)\, dx. \end{aligned}$$

Example 7.3: Solution when F is uniform.

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Rate of the renewal process

Theorem (Rate of the renewal process)

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad a.s.\ as\ t \to \infty.$$

Proof. If $S_{N(t)}$ is the time of the last renewal prior to or at time t, while $S_{N(t)+1}$ is the time of the first renewal after time t, then

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}.$$

By $N(t) \to \infty$ and the SLLN:

- the lhs $\frac{S_{N(t)}}{N(t)} \to \mu$ a.s.;
- the rhs $\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\frac{N(t)+1}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\left(1 + \frac{1}{N(t)}\right) \to \mu$ a.s.. $\square$

Note: If $\mu = \infty$, the result is still true.

Examples

Examples 7.4 and 7.5: batteries that fail.

Example 7.6: customers arriving at a bank.

Example 7.7: sequence of toin cosses.

Stopping times

The nonnegative integer valued random variable N is said to be a **stopping time** for a sequence of i.i.d. r.v.'s $X_1, X_2, \ldots$ if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1, 2, \ldots$.

Example 7.10: gambler.

Wald's equation

Theorem (Wald's equation)

If $X_1, X_2, \ldots$ are i.i.d. with finite mean $\mathbb{E}[X]$, and if N is a stopping time for this sequence s.t. $\mathbb{E}[N] < \infty$, then

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[X]\mathbb{E}[N].$$

Proof. For $n = 1, 2, \ldots$, let $I_n = 1$ if $n \leq N$, and $I_n = 0$ if $n > N$. Note that the value of I_n is determined before X_n has been observed, hence X_n is independent of I_n. Then

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n I_n] = \mathbb{E}[X] \sum_{n=1}^{\infty} \mathbb{E}[I_n]$$
$$= \mathbb{E}[X]\mathbb{E}\left[\sum_{n=1}^{\infty} I_n\right] = \mathbb{E}[X]\mathbb{E}[N]. \qquad \square$$

Elementary renewal theorem

Theorem (Elementary renewal theorem - ERT)

$$\frac{m(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty.$$

Proof of $\lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$. Consider the time $S_{N(t)+1}$ of the first renewal after t. Note that $N(t)+1$ is a stopping time, since

$$N(t)+1 = n \Leftrightarrow N(t) = n-1 \Leftrightarrow X_1 + \cdots X_{n-1} \leq t, X_1 + \cdots X_n > t.$$

Then,

$$\mathbb{E}[S_{N(t)+1}] = \mathbb{E}[X_1 + \cdots + X_{N(t)+1}] = \mathbb{E}[X]\mathbb{E}[N(t)+1] = \mu(m(t)+1).$$

Define the excess time as $Y(t) = S_{N(t)+1} - t$. Taking expectations, we get $\mu(m(t)+1) = t + \mathbb{E}[Y(t)]$, which implies

$$\frac{m(t)}{t} = \frac{1}{\mu} + \frac{\mathbb{E}[Y(t)]}{t\mu} - \frac{1}{t} \geq \frac{1}{\mu} - \frac{1}{t} \to \frac{1}{\mu}.$$

□

Application of the ERT

Assume integers interarrival times. Let $I_i = 1$ if there is a renewal at time i, and $I_i = 0$ otherwise. Then $N(n) = \sum_{i=1}^{n} I_i$, and, taking expectations,

$$m(n) = \sum_{i=1}^{n} \mathbb{P}(\text{renewal at time } i).$$

By the ERT, we get

$$\frac{m(n)}{n} = \frac{\sum_{i=1}^{n} \mathbb{P}(\text{renewal at time } i)}{n} = \frac{1}{\mu},$$

which, if the limit exists, yields to

$$\lim_{n \to \infty} \mathbb{P}(\text{renewal at time } n) = \frac{1}{\mu}.$$

Example 7.12: random walk with negative mean.

CLT for renewal processes

For large t, $N(t)$ is approximately normally distributed with mean $\frac{t}{\mu}$ and variance $\frac{t\sigma^2}{\mu^3}$, where μ and σ^2 are respectively the mean and variance of the interarrival distribution.

Theorem (CLT for renewal processes)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

Furthermore, it can be shown that $\frac{\mathrm{Var}(N(t))}{t} \to \frac{\sigma^2}{\mu^3}$.

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Renewal reward processes

Consider a renewal process $\{N(t), x \geq 0\}$ with intererarrival times $X_n, n \geq 1$. Assume that $R_n = R_n(X_n), n \geq 1$ are i.i.d. r.v.'s representing the rewards earned each time a renewal occurs. The proces $\{R(t), t \geq 0\}$, with

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

representing the total reward earned by time t, is said to be a **renewal reward process**.

- $\mathbb{E}[R] = \mathbb{E}[R_n], \quad \mathbb{E}[X] = \mathbb{E}[X_n] = \mu.$
- We say that a cycle is completed every time a renewal occurs. The reward can also be earned gradually during a cycle.

Renewal reward theorem

The long-run average reward per unit time is equal to the expected reward earned during a cycle divided by the expected length of a cycle.

Theorem (Renewal reward theorem - RRT)

If $\mathbb{E}[R] < \infty$ and $\mu < \infty$, then, as $t \to \infty$:

(a) $\frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mu}$ *a.s.;*

(b) $\frac{\mathbb{E}[R(t)]}{t} \to \frac{\mathbb{E}[R]}{\mu}$.

Proof of (a). We can write

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t}.$$

By the SLLN, $\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \to \mathbb{E}[R]$ a.s., and $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s.. □

Examples

Costs replacing rewards.

Example 7.14: changing the car when it breaks down or gets old.

Example 7.15: people arriving at a bus stop.

Average age of a renewal process

Example 7.18.

Define $A(t) = t - S_{N(t)}$ to be time at t since the last renewal (age of an item). We want to compute the **average age**

$$\lim_{t\to\infty} \frac{\int_0^t A(s)\,ds}{t}.$$

- If $A(t)$ represents the reward at time t, then the long-run average reward is given by $\lim_{t\to\infty} \frac{\int_0^t A(s)\,ds}{t}$.
- By the RRT, a.s.

$$\lim_{t\to\infty} \frac{\int_0^t A(s)\,ds}{t} \to \frac{\mathbb{E}[R]}{\mu} = \frac{\mathbb{E}[\text{reward during a cycle}]}{\mathbb{E}[\text{length of the cycle}]}$$

$$= \frac{\mathbb{E}\left[\int_0^X s\,ds\right]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}.$$

Average excess of a renewal process

Example 7.19.

Define $Y(t) = S_{N(t)+1} - t$ to be time from t until the next renewal (remaining life of an item). We want to compute the **average excess**

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)\,ds}{t}.$$

- If $Y(t)$ represents the reward at time t, then the long-run average reward is given by $\lim_{t\to\infty} \frac{\int_0^t Y(s)\,ds}{t}$.
- By the RRT, a.s.

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)\,ds}{t} \to \frac{\mathbb{E}[R]}{\mu} = \frac{\mathbb{E}[\text{reward during a cycle}]}{\mathbb{E}[\text{length of the cycle}]}$$
$$= \frac{\mathbb{E}\left[\int_0^X (X-s)\,ds\right]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}.$$

Example

Example 7.20: people and buses arriving at a bus stop.

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Regenerative processes

A stochastic process $\{X(t), t \geq 0\}$ is said to be a **regenerative process** if there exist a.s. time points at which the process probabilistically restarts itself, i.e., there exist a.s. finite times $T_1, T_2, \ldots,$ such that $\{X(T_n + t), t \geq 0\}$ is distributed as $\{X(t), t \geq 0\}$, for $n \geq 1$. Note that $T_1, T_2, \ldots,$ constitute the arrival times of a renewal process.

Examples:

- A renewal process is not regenerative, because $N(t)$ is strictly increasing and does not regenerate; instead, the age of a renewal process $A(t)$ is regenerative and regenerates at renewal times.
- A recurrent Markov chain (where each state is visited either infinitely many times or never) is regenerative and T_1 is the time of the first return to the initial state.

Proportion of time in a state

We want to compute the long-run **proportion of time spent in a state**.

- If we earn a reward at rate 1 per unit time when the process is in state j, i.e., at rate $I(t) = 1$ if $X(t) = j$ and $I(t) = 0$ otherwise, then the long-run average reward is given by $\lim_{t\to\infty} \frac{\int_0^t I(s)\,ds}{t}$.
- Since the long-run average reward is equal to the proportion of time spent in state j, by the RRT, a.s.

$$\text{proportion of time in state } j = \frac{\mathbb{E}[\text{amount of time in } j \text{ during a cycle}]}{\mathbb{E}[\text{length of a cycle}]}.$$

Example 7.21: continuous-time Markov chain.

- Key renewal theorem: if the length of a cycle is a continuous r.v., then

$$\lim_{t\to\infty} \mathbb{P}(X(t) = j) = \frac{\mathbb{E}[\text{amount of time in } j \text{ during a cycle}]}{\mathbb{E}[\text{length of a cycle}]}.$$

Alternating renewal processes

An **alternating renewal process** is a regenerative process describing a system that can be in two states, on or off, s.t. the following holds:

(i) during the n-th cycle, for $n \geq 1$, starting from the on state, the system goes off after the time Z_n and it goes on after the time Y_n;

(ii) we assume both the sequences $\{Z_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ to be i.i.d. and, for each $n \geq 1$, we allow Z_n and Y_n to be dependent.

Note that the process starts over again afer a complete cycle consisting of an on and an off interval, i.e., $X_n = Z_n + Y_n$, $n \geq 1$.

- $\mathbb{E}[\text{on}] = \mathbb{E}[Z] = \mathbb{E}[Z_n]$, $\quad \mathbb{E}[\text{off}] = \mathbb{E}[Y] = \mathbb{E}[Y_n]$.
- The long-run proportion of time that the system is on/off is given by

$$P_{\text{on}} = \frac{\mathbb{E}[\text{on}]}{\mathbb{E}[\text{on}] + \mathbb{E}[\text{off}]}, \qquad P_{\text{off}} = \frac{\mathbb{E}[\text{off}]}{\mathbb{E}[\text{on}] + \mathbb{E}[\text{off}]}.$$

Continuous-time Markov chains

- Recall that a **continuous-time Markov chain** on $\mathcal{S}$ can be characterized by the departure rates $\{v_i, i \in \mathcal{S}\}$ and the transition probabilities $\{P_{ij}, (i,j) \in \mathcal{S}^2\}$.
- If the Markov chain is irreducible and positive recurrent, a unique stationary distribution ρ exists, satisfying $\rho_i v_i = \sum_{j \in \mathcal{S}} v_j P_{ji} \rho_j$, for all states $i \in \mathcal{S}$, and $\sum_{i \in \mathcal{S}} \rho_i = 1$.
- Let π be the stationary distribution of the embedded discrete-time Markov chain. i.e., $\pi_i = \sum_{j \in \mathcal{S}} \pi_j P_{ji}$, for all $i \in \mathcal{S}$, and $\sum_{i \in \mathcal{S}} \pi_i = 1$.
- The long-run proportion of time P_i spent in state i is given by $P_i = \rho_i$ and is proportional to π_i / v_i.

Semi-Markov processes

A **semi-Markov process** is a process on $\mathcal{S} = \{1, \ldots, N\}$ that evolves as a continuous-time Markov chain, with the difference that, for all $i \in \mathcal{S}$, the amount of time it spends in state i before jumping into a different state is a r.v. (not exponential) with mean μ_i.

- Consider the embedded discrete-time Markov chain $\{X_n, n \geq 0\}$, where X_n denotes the state of the process after the n-th jump.
- Let π be its stationary distribution and π_i the proportion of jumps that take the process into state i.
- Since the process spends an expected time μ_i in state i whenever it visits that state, the long-run proportion of time P_i spent in state i is given by the weighted average

$$P_i = \frac{\pi_i \mu_i}{\sum_{j \in \mathcal{S}} \pi_j \mu_j}.$$

- If the time in each state during a visit is a continuous r.v., then P_i also represents the limiting probability that the process will be in state i at time t.

Example

Example 7.30: machine that can be good, fair, or broken.

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The inspection paradox

- **Inspection paradox.** Let $X_{N(t)+1} = S_{N(t)+1} - S_{N(t)}$ denote the length of the renewal interval containing the time point t. Then

$$\mathbb{P}(X_{N(t)+1} > x) \geq \mathbb{P}(X > x) = 1 - F(x).$$

  In other words, the length of the renewal interval containing the point t tends to be larger than an ordinary renewal interval.

Size biasing: if the whole line is covered by intervals, is it not more likely that a larger interval covers the point t?

Since $X_{N(t)+1} = A(t) + Y(t)$ (age + excess), the average length of a renewal interval containing a specified point is

$$\lim_{t\to\infty} \frac{\int_0^t X_{N(s)+1}\, ds}{t} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} + \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]} > \mathbb{E}[X].$$

Proof. By conditioning of the time of the last renewal prior to or at t,

$$\mathbb{P}(X_{N(t)+1} > x) = \mathbb{E}[\mathbb{P}(X_{N(t)+1} > x \mid S_{N(t)} = t - s)].$$

If $s > x$, since there are no renewals between $t - s$ and t, then

$$\mathbb{P}(X_{N(t)+1} > x \mid S_{N(t)} = t - s) = 1.$$

If $s \leq x$, no renewals should occur for an additional time $x - s$, hence

$$\begin{aligned}\mathbb{P}(X_{N(t)+1} > x \mid S_{N(t)} = t - s) = \mathbb{P}(X > x \mid X > s) &= \frac{\mathbb{P}(X > x)}{\mathbb{P}(X > s)} \\ &= \frac{1 - F(x)}{1 - F(s)} \geq 1 - F(x).\end{aligned}$$

Hence, $\mathbb{P}(X_{N(t)+1} > x \mid S_{N(t)} = t - s) \geq 1 - F(x)$ for all s, and we conclude by taking expectations. $\square$

Exercises

Session 3. Chapter 7: 1, 3, 5, 6a, 9, 12.

Session 4. Chapter 7: 15, 16, 19, 20 (convergence is a.s.), 26.

Session 5. Chapter 7: 22, 31, 38, 46, 47. For 46 assume that jumps are independent of waiting times, while for 47 allow for dependence.