Chapter 7

7.1 We know that $N(t) < n$ means that the n-th arrival occurs strictly after time t, which is equivalent to $S_n > t$. So, (a) is true.

For part (b) it is good to observe that $N(t) = n$ implies that the n-th arrival occurs before or at time t, Which implies that $S_n \leq t$, so (b) is not true.

Similarly, $S_n < t$ implies that the n-th arrival occurs before time t, but the $n+1$-st arrival might be after time t, while $N(t) > n$ implies that the $n+1$-st arrival was before time t and therefore (c) is not true.

7.3 (a) Assume $t \geq y$ (otherwise the probability below is 0). Then using that $N(t) = n$ is equivalent to $S_n \leq t$ and $S_{n+1} > t$, we obtain

$$\mathbb{P}(N(t) = n | S_n = y) = \mathbb{P}(S_n \leq t < S_{n+1} | S_n = y) = \mathbb{P}(S_{n+1} - S_n > t - y) = 1 - F(t - y).$$

(b) Recall (e.g. page 33) that

$$f_{S_n}(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{(n-1)!}.$$

We obtain that

$$\begin{aligned}
\mathbb{P}(N(t) = n) &= \int_0^\infty \mathbb{P}(N(t) = n | S_n = y) f_{S_n}(y) dy \\
&= \int_0^t \mathbb{P}(N(t) = n | S_n = y) f_{S_n}(y) dy = \int_0^t (1 - F(t - y)) \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{(n-1)!} dy \\
&= \int_0^t e^{-\lambda(t-y)} \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{(n-1)!} dy = e^{-\lambda t} \frac{\lambda^n}{(n-1)!} \int_0^t y^{n-1} dt = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.
\end{aligned}$$

This is of course exactly what we want, because with the given F the renewal process has exponential inter-arrival times, which makes the process a homogeneous Poisson Process.

7.5 This exercise is heavily based on Example 7.3 Let $\{N(t); t \geq 0\}$ be a renewal process with uniform (0,1) interarrival times and let S_n be the time of the n-th arrival in this process. Then, by the definition of $N(1)$,

$$N = \min\{n \geq 0; S_n > 1\} = N(1) + 1.$$

Thus $\mathbb{E}[N] = \mu(1) + 1$, which by Example 7.3 equals $e^1 - 1 + 1 = e$.

7.6 (a) For this exercise to make sense assume that r is a positive integer. The answer follows immediately from the fact that the sum of r independent exponential (λ) distributed random variables has density function f. Therefore, $\mathbb{P}(N(t) \geq n)$ is the probability that the $n \times r$-th arrival in a homogeneous Poisson Process with rate λ is before time t, which is the probability that a Poisson random variable with expectation λt is at least nr.

Alternatively letting $f_n(x)$ be the density function of S_n, we obtain $f_1(x) = f(x)$ and $f_n(x) = f * f_{n-1}(x) = \int_0^x f(y) f_{n-1}(x-y)dy$ and by induction it is straightforward to prove that $f_n(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{nr-1}}{(nr-1)!}$. Indeed, it is trivial for $n = 1$ and using the above expression for $f_n(x)$ we obtain

$$f_{n+1}(x) = \int_0^x f(y) f_n(x-y)dy = \int_0^x \frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!} \frac{\lambda e^{-\lambda(x-y)}(\lambda(x-y))^{nr-1}}{(nr-1)!} dy$$
$$= \frac{(\lambda x)^{(n+1)r}}{((n+1)r-1)!} e^{-\lambda x} \frac{1}{x^2} \int_0^x \frac{((n+1)r-1)!}{(r-1)!(nr-1)!} (y/x)^{r-1} (1-y/x)^{nr-1} dy$$

Now performing a change of variables $z = y/x$ (and thus $dy = xdz$ and recognizing the density of a beta distribution, we obtain, that the above equals

$$\lambda \frac{(\lambda x)^{(n+1)r-1}}{((n+1)r-1)!} e^{-\lambda x} \int_0^1 \frac{((n+1)r-1)!}{(r-1)!(nr-1)!} z^{r-1} (1-z)^{nr-1} dz = \lambda \frac{(\lambda x)^{(n+1)r-1}}{((n+1)r-1)!} e^{-\lambda x}$$

as desired.

Now recall that $\mathbb{P}(N(t) \geq n) = \mathbb{P}(S_n \leq t) = \int_0^t f_n(x)dx$. Integration by parts (or using your favorite mathematical software) gives the desired result.

7.9 Say that a renewal occurs when a new job starts (so either when a job is completed or at a shock). The inter"arrival" distribution is the distribution of the random variable X and we know that $\mathbb{P}(X > t)$ is the product of the probability that the first shock occurs after time t and the probability that it takes longer than t time units to finish a job. That is, $\mathbb{P}(X > t) = e^{-\lambda t}(1 - F(t))$. For ease of exposition, assume that $F(t)$ has a derrivative, that is, the time until job completion has a density (in the absence of shocks).

Also recall that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)dt = \frac{1}{\lambda} - \int_0^\infty e^{-\lambda t} F(t)dt = \frac{1}{\lambda} - \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} f(t)dt,$$

where we sed integration by parts in the last identity.

The rate at which renewals occur is $1/\mathbb{E}[X] = \frac{\lambda}{1 - \int_0^\infty e^{-\lambda t} f(t)dt}$. The rate at which shocks occur is λ. So the rate at which jobs are completed is the rate of renewals minus the rate of shocks, which is

$$\lambda \left(\frac{1}{1 - \int_0^\infty e^{-\lambda t} f(t)dt} - 1 \right) = \lambda \frac{\int_0^\infty e^{-\lambda t} f(t)dt}{1 - \int_0^\infty e^{-\lambda t} f(t)dt}.$$

7.12 Consider a renewal process, in which renewals occur if d-events occur. Let X be the time between two d events. Assume that there is an event at time 0 (for the long run rate adding a single point does not matter). Throughout what follows, note that there is a difference in meaning between "d-events" and "events".

Condition on the time of the first event to occur after time 0, call that time S_1. Then note that the time of the first d-event is equal to S_1 if the $S_1 \leq d$ and it is distributed as S_1 plus the time until the first d-event otherwise. So

$$\mathbb{E}[X] = \int_0^d t\lambda e^{-\lambda t}dt + \int_d^\infty \lambda e^{-\lambda t}(t+\mathbb{E}[X])dt = \int_0^\infty t\lambda e^{-\lambda t}dt + \int_d^\infty \lambda e^{-\lambda t}\mathbb{E}[X]dt = \frac{1}{\lambda} + \mathbb{E}[X]e^{-\lambda d}.$$

So, $\mathbb{E}[X] = \frac{1}{\lambda(1-e^{-\lambda d})}$ and the rate of d-events is $\lambda(1-e^{-\lambda d})$ and the proportion of all events which are d-events is the rate of d-events divided by the rate of events, which is $1 - e^{-\lambda d}$.

7.15 a) Let $X_1, X_2, \cdots$ be a sequence of random variables taking values 2, 4 and 6 each with probability 1/3 and let N be the first index i, for which $X_i = 2$ (this is a stopping time, since whether $N = i$ only depends on $X_1, X_2, \cdots X_i$).

So, upto and including the N-th trip out of the room, the X_i's correspond to the time of the excursion. After that the random variables have no relevant interpretation.

b) Note $\mathbb{E}[X_i] = 4$ and $\mathbb{E}[N] = 3$ So, $\mathbb{E}[T] = 12$.

c) $\mathbb{E}[\sum_{i=1}^N X_i | N = n] = \sum_{i=1}^{n-1} \mathbb{E}[X_i | X_i \neq 2] + \mathbb{E}[X_n | X_n = 2] = (n-1)5 + 2$, which is generally not equal to $4n = \mathbb{E}[\sum_{i=1}^n X_i]$.

d) $\mathbb{E}[\sum_{i=1}^N X_i] = \sum_{n=1}^\infty \mathbb{E}[\sum_{i=1}^n X_i | N = n]\mathbb{P}(N = n) = \sum_{n=1}^\infty ((n-1)5 + 2)\mathbb{P}(N = n) = 5\mathbb{E}[N-1] + 2 = 12$.

7.16 In this example $\mathbb{E}[\sum_{i=1}^N X_i] = 4$ because N is defined as the fourth time $X_i = 1$. $\mathbb{E}[X_i] = 1/13$ for all i, because the cards are ordered uniformly at random and 4 out of 52 cards are aces. So, in order for the equality to hold we need $\mathbb{E}[N] = 52$, which is clearly nonsense, because that implies that the 52-th card must be an ace with probability 1.

Wald's identity does not hold here, because the X_i's are not independent. One can see this by $\mathbb{P}(X_2 = 1 | X_1 = 1) = 3/51$, while $\mathbb{P}(X_2 = 1 | X_1 = 0) = 4/51$.

7.19 Use Example 7.3 and equation (7.9) on page 422. Where $Y(t)$ is the time from time t until the next renewal. In this question $\mu = 1/2$, $m(1) = e - 1$ by Example 7.3 and $t = 1$. Filling that in in (7.9). We obtain $e/2 = 1 + \mathbb{E}[Y(1)]$. So the expected time until the next arrival at time 1 is given by $\mathbb{E}[Y(1)] = e/2 - 1$.

7.20 In this example we use the Strong law of large numbers and note that

$$W_n = \frac{n^{-1}\sum_{i=1}^n R_i}{n^{-1}\sum_{j=1} X_j}.$$

Because the R_i's are independent and identically distributed (i.i.d.) the numerator converges almost surely to $\mathbb{E}[R]$ by the strong law of large numbers. The X_j's are also i.i.d. and therefore, again by the strong law of large numbers, the denominator converges almost surely to $\mathbb{E}[X]$. Which implies that W_n converges almost surely to the desired limit.

7.26 In this problems, the renewals are arrivals of trains. We use renewal-reward theory. We first look at what happens upto the arrival of the N-th customer. The cost upto the arrival of the first customer is 0. Then the expected cost between the first and second arrival is c times the expected time between the first and second arrival, which is c/λ (recall the arrivals occur according to a Poisson process) and in general for $0 \le k \le N-1$, the expected cost between the k-th and $k+1$-st arrival is kc times the expected duration between those arrivals, which is k/λ. So the total expected cost up to the arrival of the N-th customer is $\sum_{k=0}^{N-1} kc/\lambda = \frac{cN(N-1)}{2\lambda}$.

In this problem the total cost a customer brings with it is c times the time he is waiting for the train. So the N customers waiting contribute cKN.

Then in the interval between the arrival of the N-th customer and K time units later, arrive a Poisson number (say M) of customers with expectation λK. Say that there arrival times (measured from the time of arrival of the N-th customer are $T_1, T_2, \cdots, T_M$. The expected extra cost those customers bring with them is $c\mathbb{E}[\sum_{i=1}^{M}(K - T_i)]$. Note

$$\mathbb{E}[\sum_{i=1}^{M}(K - T_i)] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{M}(K - T_i)|M]].$$

Using the order statistic property $\mathbb{E}[\sum_{i=1}^{M}(K - T_i)|M] = \mathbb{E}[\sum_{i=1}^{M}(K - U_i)|M]$, where the U_is are independent and all uniformly distributed on the interval $[0, K]$. So,

$$\mathbb{E}[\sum_{i=1}^{M}(K - U_i)|M] = MK - MK/2 = MK/2.$$

Using this, we obtain that the expected extra cost those customers bring with them is $c\mathbb{E}[M]K/2 = c\lambda K^2/2$. So the total expected cost until a bus arrives is

$$\frac{cN(N-1)}{2\lambda} + cKN + \frac{c\lambda K^2}{2}.$$

The expected duration until a bus arrives (a renewal) is K plus the expected time until the N-th arrival, which is $K + N/\lambda$.

So, using the theory on Renewal-Reward systems the long run average cost is given by

$$\frac{\frac{cN(N-1)}{2\lambda} + cKN + \frac{c\lambda K^2}{2}}{K + N/\lambda} = \frac{cN(N-1) + 2c\lambda KN + c\lambda^2 K^2}{2(K\lambda + N)}.$$

7.22 Let X be the time until the first replacement of the car (the renewal), Y is the time until the first breakdown. Note that if $Y > T$ then $X = Y$, while if $Y \leq T$ then X is T plus the time until the first arrival in a Poisson Process with rate μ. So,

$$\mathbb{E}[X] = \mathbb{E}[X|Y \leq T]\mathbb{P}(Y \leq T) + \mathbb{E}[X|Y > T]\mathbb{P}(Y > T),$$

where $\mathbb{E}[X|Y \leq T] = (T + 1/\mu)$, $\mathbb{E}[X|Y > T] = (T + 1/\lambda)$ and
$\mathbb{P}(Y > T) = 1 - \mathbb{P}(Y \leq T) = e^{-\lambda t}$. So,

$$\mathbb{E}[X] = (T + 1/\mu)(1 - e^{-\lambda T}) + (T + 1/\lambda)e^{-\lambda t} = T + 1/\mu + (1/\lambda - 1/\mu)e^{-\lambda T}.$$

The rate at which new cars are bought is $1/\mathbb{E}[X]$.

Note that the number of repairs before a new car is bought depends on Y, if $Y < T$, then the number of repairs is 1 plus a Poisson number of further repairs with expectation $\mu(T - Y)$, while if $Y > T$ the number of repairs is 0. So, the expected number of repairs before buying a new car is

$$\int_0^T \lambda e^{-\lambda s}(1 + \mu(T - s))ds = (1 + \mu T)\int_0^T \lambda e^{-\lambda s}ds - \int_0^T \lambda\mu s e^{-\lambda s}ds$$
$$= (1 + \mu T)(1 - e^{-\lambda T}) - \mu\left(\int_0^T e^{-\lambda s}ds - Te^{-\lambda T}\right) = (1 - \frac{\mu}{\lambda})(1 - e^{-\lambda T}) + \mu T,$$

where the second identity is obtained by integration by parts. If the cost of a repair is r and the cost of a new car is C, then the expected cost until buying a new car is $C + \left((1 - \frac{\mu}{\lambda})(1 - e^{-\lambda T}) + \mu T\right) r$ and by renewal reward theory, the expected cost per time unit is this number divided by $\mathbb{E}[X]$

7.31 Define $X(t) = A(t) + Y(t)$, that is $X(t)$ is the time between the last renewal before time t and the first renewal after time t. Note that the only relevant (fore this question) information we get from $A(t) = s$ is that $X(t) \geq s$.

$$\mathbb{P}(Y(t) > x|A(t) = s) = \mathbb{P}(X(t) > x+s|A(t) = s) = \mathbb{P}(X(t) > x+s|X(t) > s) = \frac{1 - F(x + s)}{1 - F(s)}.$$

7.38 Assume that the distance (in miles) between A and B is given by D. The expected time (in hours) driving from A to B is then

$$\int_{40}^{60} \frac{1}{20}\frac{D}{v}dv = \frac{D}{20}(\log 60 - \log 40) = \frac{D}{20}\log(3/2).$$

The expected time driving from B to A is $\frac{1}{2}\frac{D}{40} + \frac{1}{2}\frac{D}{60} = \frac{5D}{240}$. By theory on regenerative processes we obtain that the long run fraction of time spent driving from A to B is

$$\frac{\text{expected time from A to B}}{\text{expected return time}} = \frac{\frac{D}{20}\log(3/2)}{\frac{D}{20}\log(3/2) + \frac{5D}{240}} = \frac{12\log(3/2)}{12\log(3/2) + 5}.$$

Similarly, the long run fraction driving 40 mph is

$$\frac{\frac{1}{2}\frac{D}{40}}{\frac{D}{20}\log(3/2) + \frac{5D}{240}} = \frac{3}{12\log(3/2) + 5}.$$

7.46 If the destination of a jump is independent of the inter-jump time, then this is a Markov Process.

7.47 Let X_i be distributed as the random time the process stays in i. So μ_i is $\mathbb{E}[X_i]$. Let J be the random variable describing the state to which the process jumps when leaving state i. Now observe

$$\mu_i = \mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|J]] = \sum_j P_{ij}\mathbb{E}[X_i|J=j]) = \sum_j P_{ij}t_{ij}.$$

For the second part of the problem let a cycle start whenever the process enters state i. We observe that P_i is the expected fraction of time the process spends in state i. So, P_i is the expected time per cycle that the process spends in state i per cycle (which is by definition μ_i) divided by the expected cycle length. So the expected cycle length is μ_i/P_i.

The expected time per cycle that the process is in state i on its way to state j is $P_{ij}t_{ij}$. So the long run fraction of time spent in state i on the way to state j is given by the above, divided by the expected cycle length as desired.