STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5012
TENTAMEN
May 22, 2024

First exam: Stochastic Processes and Simulation II

May 22nd 2024, kl. 08–13

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Grading: The exam consists of five exercises, each divided into three questions. Each question is worth a maximum of 4 points, for a total of 12 points per exercise. Exercise 3 contains a bonus question which is worth 2 extra points, for a total of 14 points. The maximum score for the exam is then 62 points.
In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

Grade	A	B	C	D	E
Needed points	54	48	42	36	30

Partial answers might be worth points, while answers "out of the blue" will not be rewarded. The questions differ in difficulty.

Exercise 1: Poisson processes

(i) Give the definition of a continuous-time Markov chain.

The Amazon rainforest has mainly two seasons, wet and dry, providing the perfect environment for plants and wildlife to thrive. While heavy rainfalls are dominant during the wet season, during the dry season we can observe an alternation in the weather conditions, from sunny periods (S) to light rainfalls (L) to heavy rainfalls (H). During the dry season, assume that a sunny period lasts for an amount of time that can be described by an exponential random variable with parameter λ_S, after which light rainfalls occur with probability 3/4 and heavy rainfalls with probability 1/4. The length of a period with light rain can be described by an exponential random variable with parameter λ_L, while the length of a period with heavy rain by an exponential random variable with parameter λ_H. After light rainfalls, it becomes sunny again with probability 2/3 while heavy rainfalls begin with probability 1/3. After heavy rainfalls, there is an equal probability of light rainfalls and sun. Assume that these exponential random variables are all independent of each other.

(ii) Draw a Markov chain whose states represent the weather conditions during the dry season. Write the associated transition probability matrix P and transition rate matrix Q. What is the relation between the entries of the two matrices?

(iii) Write down the balance equations and calculate the proportion of time in which it rains.

Exercise 2: Renewal theory

(i) State and prove the inspection paradox.

Consider one of the cultural centers in the Amazon rainforest, where tourists arrive to visit its museum about indigenous culture and/or to explore the unique nature by a river cruise.

(ii) Assume that, during the opening hours of the cultural center, tourists arrive according to a renewal process, where the average time between two arrivals is 10 minutes and its variance is 5 minutes. Calculate, for a given time, the average time passed from the last arrival and the average time until the next arrival. Explain how these results are related to the inspection paradox.

(iii) Consider now the boats that are used for the river cruise and assume that each boat has a lifetime described by an exponential random variable with parameter λ, independently of the other boats. The cultural center has a policy for which they replace a boat as soon as it breaks or reaches the age of K years. The cost of a new boat is C_1 and an additional cost of C_2 has to be paid when a boat breaks. Assume that the cultural center owns 2 boats. What is the long-run average cost?

Exercise 3: Queueing theory

The cultural center also has a restaurant where tourists can have a break eating some local food. Assume that the restaurant has a total of 10 sitting indoor spots that can be used. Tourists arrive according to a Poisson process with rate λ and they are allowed to order and consume as long as there are free sitting spots, otherwise they wait in a queue. Assume also that the total time it takes to order and consume the food is exponentially distributed with rate μ, independently of everything else.

(i) What type of queueing model best describes the situation at the restaurant? What condition, if any, must λ and μ satisfy? Write down the balance equations.

Next, assume that next to the restaurant there is a very small bar that serves tropical drinks to go. During the wet season, tourists arrive according to a Poisson process with rate λ, but they are allowed to enter the bar only as long as there are no more than 5 tourists already inside and if they find the bar full then they leave. Assume also that the time it takes to be served in the bar is exponentially distributed with rate μ, independently of everything else.

(ii) What type of queueing model best describes the situation at the bar? What conditions, if any, must now λ and μ satisfy? Write down the balance equations and show how they can be solved to compute the limiting probability P_0 that there are no tourists inside the bar.

Assume now that, during the dry season, the bar opens an outdoor counter and removes the restriction that allows no more than 5 tourists inside, so that now there is enough space for everyone and every tourist that arrives can join the queue (if there is any) and order. Assume that tourists still arrive according to a Poisson process with rate λ. However, the time takes to be served is not anymore exponentially distributed, but is described by a continuous random variable S with mean μ and variance σ^2, independently of everything else.

(ii) What condition, if any, must now λ and μ satisfy? What is the limiting probability P_0 that there are no tourists at the bar.

(iv) Bonus (2 points): State and prove the Pollaczek-Khintchine formula to show that

$$W_Q = \frac{\lambda(\sigma^2 + \mu^2)}{2(1 - \lambda\mu)},$$

where W_Q is the average time a tourist spends in the queue at the bar.

Exercise 4: Simulation

(i) Describe and prove the rejection method.

(ii) Describe in detail how we can simulate the time of the n-th arrival of a tourist at the bar of the cultural center during the dry season. To do so, use the rejection method with trial density g being the density of an exponential random variable with parameter λ/n.

(iii) What is the average number of iterations in the most efficient case? What distribution does the number of iterations follow?

Exercise 5: Brownian motion

(i) Given a Brownian motion $\{X(t), t \geqslant 0\}$, define the hitting time T_a of barrier $a > 0$ and derive its distribution. Prove that the maximum $M(t)$ of a Brownian motion in the interval $[0, t]$ has the same distribution of $|X(t)|$.

In the 1970s the total surface of the Brazilian Amazon rainforest was around 4 million km^2, while at the end of 2023 it was estimated to be around 3 250 000 km^2. Current data show that the average rate of deforestation in the decade from 2009 to 2018 was approximately 5 000 km^2 per year, while in the past five years, from 2019 to 2023, it was 10 000 km^2 per year. Assume that we can model the deforestation process as a Brownian motion $\{X(t), t \geqslant 0\}$ with negative drift and variance parameter σ^2, where $X(t)$ is the size of the rainforest at time t. (We are not interested in $X(0)$, so we can ignore the initial condition and start looking at the process from when we have data).

(ii) Lately, due to the crucial role deforestation plays in the current climate crisis, more attention has been given to this matter. Assume that from this year, 2024, we are able to reduce the deforestation rate of 50% and bring it back to 5 000 km^2 per year. In which year should we expect to reach the threshold of 3 million km^2 for the total size of the Brazilian Amazon rainforest?

Assume now that governments will be able to find an agreement to reduce the deforestation rate by another 50% from the year 2029, and eventually bring it to zero from the year 2039.

(iii) If in 2039 the size of the rainforest will be exactly at its mean value, what is the probability that it will not reach the threshold of 3 million km^2 before the year 2100? (You can leave the answer in the form of an integral, without calculating it).