STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5012
TENTAMEN
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Second exam: Stochastic Processes and Simulation II

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Grading: The exam consists of five exercises, each divided into three questions. Each question is worth a maximum of 4 points, for a total of 12 points per exercise. Exercise 2 contains a question which is worth 6 points, for a total of 14 points. Exercise 5 contains four questions, each worth 3 points. The maximum score for the exam is 62 points.
In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

Grade	A	B	C	D	E
Needed points	54	48	42	36	30

Partial answers might be worth points, while answers "out of the blue" will not be rewarded. The questions differ in difficulty.

Exercise 1: Poisson processes

(i) Give the definition of a counting process and explain what it means for the process to have independent increments and stationary increments.

(ii) Give the definition of a mixed Poisson process and explain why it is not a Poisson process.

The Amazon rainforest is suffering a major deforestation problem which is drastically reducing its size year after year. The rate of deforestation has changed over the years ranging from an annual forest loss of around 25 000 km^2 in 2003 to an annual forest loss of around 9 000 km^2 in 2023. In the Brazilian Amazon rainforest, it has been observed that the leading political party has deeply influenced the rate of deforestation over the last decades. Denote by $\{N(t), t \geqslant 0\}$ a Poisson process describing the amount of land reduction per month, where, for $k \geqslant 1$, the k-th event occurs when $100 \cdot k$ km^2 of total land have been deforested. For example, if after 1 month has been deforested 750 km^2 of land, then $\mathrm{N}(1) = 7$, and if after 2 months have been deforested 1 600 km^2 of land then $\mathrm{N}(2) = 16$. Assume that the process starts at the beginning of 2024, so that time units correspond to calendar months. The rate at which events occur, i.e., the rate of deforestation, depends on the leading political party and is described by a positive random variable L with mean μ and variance σ^2.

(iii) Calculate the expected value and variance of the total land reduction of the year 2024.

Exercise 2: Renewal theory

(i) Let $\{N(t), t \geqslant 0\}$ be a renewal process with renewal function $m(t) = \mathbb{E}[N(t)]$. Assuming that the interarrival distribution F is continuous with density function f, derive the renewal equation for $m(t)$. For $t \leqslant 1$, what is the solution of the renewal equation when F is uniform on $(0, 1)$?

Consider now a new type of tree-cutting machine for deforestation that has recently started being produced. Assume that every day such machines are introduced in the Amazon rainforest one by one at random time intervals $U_1, U_2, \ldots,$ that are distributed uniformly on $(0, 1)$ independently of each other, where a unit time indicates a day. Assume also that every day the process restarts, independently of the previous day.

(ii) How many new machines are introduced on average in a day? How many in a month (30 days)?

Consider now one of the above machines and assume that a workers association has just bought it. The machine breaks down according to a renewal process with interarrival times uniformly distributed between 0 and 4 years. Assume that the cost of a new machine is 2 million SEK, while the expected cost of a new repair increases with the number of earlier repairs in such a way that the expected cost of the k-th repair is $100\ 000 \cdot k$ SEK. Assume that at the 5-th breakdown the association decides to replace the machine and buy a new one instead of repairing it.

(iii) (6 points) What is the expected cost the association has per year?

Exercise 3: Queueing theory

(i) Write the basic cost equation and describe how it can be used to derive Little's formula.

In order to contrast the Amazon deforestation, lots of initiatives are arising to re-plant trees in the forest. The Brasilian Ministry of Environment in 2017 sponsored a project aiming at planting 73 million new trees in the following 6 years. On a smaller scale, many associations allow people to donate money online to plant trees and help with the cause. To maintain biodiversity, a lot of different species need to be planted, among which are the andiroba and the Roucouyer tree, known for their medicinal properties. Consider an association specializing in medicinal plants and planting andiroba and Roucouyer trees in different areas of the Amazon forest. Assume that anyone can donate money and place an order to plant a tree through the association website and can even specify which type of tree to plant. Assume that the association receives orders for andiroba trees according to a Poisson process with rate λ_a, while it receives orders for Roucouyer trees according to a Poisson process with rate λ_r, independently of all the other orders. When an order arrives, it is normally picked up by the people in the association who process it and make sure the tree is planted. The time it takes from when the order is picked up to when the tree is planted is exponentially distributed with mean μ, independently of the other orders and the type of tree. However, the association consists of a limited number of people, hence they can only pick up and process k orders at the time. If there are more than k orders in the system, the remaining ones are placed in a queue according to their arrival time.

(ii) What type of queueing model best describes the above situation? What is the total arrival rate λ of orders in the system? What conditions, if any, must λ and μ satisfy? Write down the balance equations.

(iii) Assuming that you know the average number of orders in the system, what is the average time between the placing of an order and the planting of its tree?

Exercise 4: Simulation

(i) Describe how to use the inverse transformation method to simulate an exponential random variable with parameter λ.

(ii) Consider the tree-planting association of the previous exercise. Describe how we can simulate the arrival times of the andiroba tree orders by simulating only standard uniform random variables.

(iii) Let T_n denote the time of the n-th order placed on the website of the association. What is its distribution? How can we simulate T_n in such a way that we can tell if it is for an andiroba tree or a Roucouyer tree?

Exercise 5: Brownian motion

(i) (3 points) Give the definition of a standard Brownian motion $\{X(t), t \geqslant 0\}$. Give some intuition on why $X(t)$ is a continuous function of t and it is nowhere differentiable.

The Amazon rainforest represents over half of Earth's remaining rainforests and comprises the largest and most biodiverse tract of tropical rainforest in the world, with around 390 billion trees in about 16 000 species. Assume that we can model the number of trees in the Amazon rainforest as a standard Brownian motion $\{X(t), t \geqslant 0\}$ with drift, where $X(t)$ is the number of trees at time t. We can assume as a starting point the initial condition $X(0) = 390\ 000\ 000\ 000$ at the beginning of the year 2024. The drift is given by the estimate that deforestation is responsible for a loss of trees at an average rate of 1.5 billion per year.

(ii) (3 points) In how many years should we expect to reach the threshold of 300 billion trees in total in the Amazon rainforest?

Consider now also the action of re-planting trees and assume that various initiatives from governments and associations manage to re-plant trees at an average rate of 500 million (half billion) per year.

(iii) (3 points) In how many years should we expect to reach the threshold of 300 billion trees in total in the Amazon rainforest?

Assume that the average deforestation rate will decrease linearly in the next 4 years and then it will suddenly drop to match the average re-planting rate. More precisely, it will still be 1.5 billion trees in the year 2024, then 1.4 billion trees in 2025, 1.3 billion trees in 2026, 1.2 billion trees in 2027, 1.1 billion in 2028, and then drop to 500 million trees starting from the year 2029.

(iv) (3 points) If the number of trees in 2029 will be exactly at its mean value, what is the probability that it will reach the threshold of 350 billion trees before the year 2065? (You can leave the answer in the form of an integral, without calculating it).