Solutions Stochastic Processes and Simulation II, May 28, 2018

Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $S_1 < S_2 < S_3 < \cdots$. Define $S_0 = 0$.

a) Provide the distribution of S_1. (3p)

Solution: A homogeneous Poisson process with intensity λ has no points in $(0, t)$ with probability $e^{-\lambda t}$, which implies $\mathbb{P}(S_1 > t) = e^{-\lambda t}$, i.e. S_1 is exponentially distributed with expectation $1/\lambda$. Also possible to say immediately that an homogeneous Poisson process has independent exponentially distributed inter-arrival times.

b) For $x \in [0, T]$ compute $\mathbb{P}[T - S_n > x | N(T) = n]$. (4p)

Solution: By order statistic property the n points of the Poisson Process in $(0, T)$ are distributed as n independent random variables, uniformly distributed on $(0, T)$. The probability that all n of those points fall in the interval $(0, T - x)$ (in which case $T - S_N > x$ is given by $\left(\frac{T-x}{T}\right)^n$. So, $\mathbb{P}[T - S_n > x | N(T) = n] = \left(\frac{T-x}{T}\right)^n = \left(1 - \frac{x}{T}\right)^n$.

c) Compute $\mathbb{E}[T - S_n | N(T) = n]$. (2p)

Solution: Using the standard identity $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)dx$ for all non negative continuous random variables X, we obtain

$$\begin{aligned}\mathbb{E}[T - S_n | N(T) = n] = \int_0^\infty \mathbb{P}[T - S_n > x | N(T) = n]dx &= \int_0^T \left(1 - \frac{x}{T}\right)^n dx \\ &= -\frac{T}{n+1}\left(1 - \frac{x}{T}\right)^{n+1}\Big]_{x=0}^{x=T} = \frac{T}{n+1}.\end{aligned}$$

d) Compute $\mathbb{E}[T - S_{N(T)}]$. (3p)

Solution:

$$\begin{aligned}\mathbb{E}[T - S_{N(T)}] &= \mathbb{P}(N(T) = 0)T + \sum_{n=1}^{\infty} \mathbb{P}(N(T) = n)\mathbb{E}[T - S_n | N(T) = n] \\ &= e^{-\lambda T}T + \sum_{n=1}^{\infty} \frac{(\lambda T)^n}{n!}e^{-\lambda T}\frac{T}{n+1} = e^{-\lambda T}T + \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{(\lambda T)^{n+1}}{(n+1)!}e^{-\lambda T} \\ &= e^{-\lambda T}T + \frac{1}{\lambda}(1 - e^{-\lambda T} - \lambda T e^{-\lambda T}) = \frac{1 - e^{-\lambda T}}{\lambda}\end{aligned}$$

Problem 2: Renewal Theory

Alice and Bob play a match consisting of rallies, where Alice starts the first rally and the winner of a rally starts the next rally.

The probability that Alice wins a rally that she starts herself is p_a (and she loses that rally with probability $1-p_a$), while the probability that Bob wins a rally that he starts himself is p_b (and he loses the rally with probability $1-p_b$). Conditioned on who starts the rally, the outcomes of a rally is independent of the outcomes of other rallies. Assume that $0 < p_a < 1$ and $0 < p_b < 1$.

Let $\{N(t), t \in \mathbb{N}_{\geq 0}\}$ be the number of rallies won by Alice among the first t rallies, and for $n \in \mathbb{N}_{\geq 1}$ let $S_n = \min\{k \in \mathbb{N}_{\geq 1}; N(k) = n\}$ be the number of rallies Alice needs to play in order to win n rallies.

a) Provide the distribution and expectation of S_1. (4p)

Solution If Alice wins the first game, then $S_1 = 1$, so $\mathbb{P}(S_1 = 1) = p_a$. If Bob wins the first game, then the number of extra games it takes for Alice to win again is geometrically distributed with parameter $1-p_b$ (and thus expectation $1/(1-p_b)$). So, $\mathbb{P}(S_1 = k+1) = (1-p_a)(p_b)^{k-1}(1-p_b)$ for $k \geq 1$.

It follows immediately that

$$\begin{aligned}\mathbb{E}[S_1] = \mathbb{P}(S_1 = 1) + \mathbb{P}(S_1 \neq 1)(1 + \mathbb{E}[\text{Geometric r.v. with parameter } 1-p_b] \\ = p_a + (1-p_a)(1 + \frac{1}{1-p_b}) = 1 + \frac{1-p_a}{1-p_b}.\end{aligned}$$

b) Compute $\mathbb{E}[N(t)]/t$, for $t \to \infty$. (4p)

Solution $N(t)$ is a renewal process with mean interarrival time $\mathbb{E}[S_1]$. By the Elementary renewal theorem, we therefore obtain that $\mathbb{E}[N(t)]/t \to 1/\mathbb{E}[S_1] = \frac{1-p_b}{(1-p_a)+(1-p_b)}$.

Assume now that Alice and Bob get points for their "winning streaks" (rows of consecutive wins). If Alice wins k rallies in a row then the points for that streak are k^2.

c) Provide the (almost-sure) long run average number of points per rally for Alice. (4p)

Solution Consider a new renewal process $\{N'(t), t \in \mathbb{N}_{\geq 0}\}$, where $N'(t)$ is the number of finished winning streaks of Bob, where a winning streak of Bob ends at rally k if Bob wins rally k and Alice wins rally $k+1$. $N'(t)$ also constitutes a renewal process. We use now theory on renewal reward processes, where the duration of a cycle is the duration of a winning streak of Alice (which has a geometrically distributed length with expectation $1/(1-p_a)$) plus the duration of winning streak of Bob (which has a geometrically distributed length with expectation $1/(1-p_b)$).

The expected number of points Alice gets during a cycle is $\sum_{k=1}^{\infty} k^2 (p_a)^{k-1}(1-p_a)$, which is the variance plus the square of the expectation of a Geometric distributed random variable with parameter $(1-p_a)$. That is, $\sum_{k=1}^{\infty} k^2 (p_a)^{k-1}(1-p_a) = \frac{p_a}{(1-p_a)^2} + \frac{1}{(1-p_a)^2}$. So, the long run average number of points per rally for Alice is $\left(\frac{p_a}{(1-p_a)^2} + \frac{1}{(1-p_a)^2}\right) / \left(\frac{1}{1-p_a} + \frac{1}{1-p_b}\right)$.

Problem 3: Queueing Theory

Consider an $M/M/\infty$ queue in which customers arrive according to a Poisson Process with rate λ, and customers have independent workloads which are exponentially distributed with expectation $1/\mu$. Because there are infinitely many servers every customer will enter service immediately upon entering.

a) For $k \in \mathbb{N}_{\geq 0}$. Let P_k be the probability that there are k customers in the system if the number of customers in the systems starts in the stationary distribution. Show that $P_k = \frac{(\lambda/\mu)^k}{k!}e^{-\lambda/\mu}$ for $k \in \mathbb{N}_{\geq 0}$. (4p)

Solution The rate at which the number of customers increases by 1 is λ, while the rate at which a customer leaves if there are k customers in the system is $k\mu$. So, using balance equations we obtain $\lambda P_0 = \mu P_1$ and for $k \geq 1$, we have $(\lambda + k\mu)P_k = \lambda P_{k-1} + (k+1)\mu P_{k+1}$. We then note that $P_k = \frac{(\lambda/\mu)^k}{k}e^{-\lambda/\mu}$ satisfies those equations and that $\sum_{k=0}^{\infty} P_k = 1$.

Now assume that there is one very friendly server (say Claire). If she is finished serving a customer while there are still other customers in the system, she takes over the service on one of those customers (and sends the server who was working on that customer for a coffee). If a customer arrives when Claire is idle, that customer will start service with Claire.

b) In the long run, what fraction of arriving customers will start service with Claire? (2p)

Solution Claire is only idle when there are no customers in the system. Because of PASTA, a fraction P_0 of the arriving customers will find Claire idle and therefore start their service with Claire.

c) In the long run, what fraction of customers will finish their time in the system being served by Claire. (6p)

Solution As long as there are customers in the system, customers leave from Claire at rate μ. So the long run average number of customers that leave Claire per time unit is $(1 - P_0)\mu$ (where $1 - P_0$ is the proportion of the time that the system is not empty. In the long run the number of customers that leave per time unit is λ (because this should equal the long run rate at which customers arrive). So, the fraction of customers that finish their service with Claire is $(1 - P_0)\mu/\lambda$.

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and for $t > 0$, let $M(t) := \max_{0 \leq s \leq t} B(s)$, be the maximum of the Brownian motion up to time t.

a) For $y > 0$ and $x > 0$, argue that $\mathbb{P}(M(t) > y, B(t) < y - x) = \mathbb{P}(B(t) > y + x)$. (4p)

Solution Let T_y be $\inf\{t \geq 0; B(t) > y\}$ be the hitting time of y for this Brownian Motion. Now use the reflection principle and note that $\{B(t) - y, t \geq T_y\}$ has by symmetry of the Brownian motion the same distribution as $\{-(B(t) - y), t \geq T_y\}$. Now note that

$$\mathbb{P}(M(t) > y, B(t) < y - x) = \mathbb{P}(T_y < t, B(t) < y - x) = \mathbb{P}(T_y < t, B(t) - B(T_y) < -x),$$

by symmetry the latter term is then equal to

$$\mathbb{P}(T_y < t, B(t) - B(T_y) > x) = \mathbb{P}(T_y < t, B(t) > x + y) = \mathbb{P}(M(t) > y, B(t) > x + y).$$

Since if $B(t) > x + y$ then $M(t) = \max_{0 \leq s \leq t} B(s) \geq B(t) > x$ the desired result follows.

b) For $0 < t < 1$, show that

$$\begin{aligned}
\mathbb{P}(M(1) > M(t)) = & \int_0^\infty \int_{-\infty}^y \frac{2}{\pi\sqrt{t(1-t)}} \frac{2y-x}{t} e^{-\frac{(2y-x)^2}{2t}} \left(\int_{y-x}^\infty e^{-\frac{z^2}{2(1-t)}} dz \right) dxdy \\
= & \frac{2}{\pi\sqrt{t(1-t)}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{y'+x'}{t} e^{-\frac{(y'+x')^2}{2t}} e^{-\frac{(z'+x')^2}{2(1-t)}} dz'dx'dy',
\end{aligned}$$

where you may use the change of variables, $z' = z - y + x$, $x' = y - x$ and $y = y'$ for the last identity. (4p)

Solution $M(1) > M(t)$ if

$$\max_{t \leq s \leq 1} B(s) > \max_{0 \leq s \leq t} B(s). \qquad \text{That is, if} \qquad \max_{t \leq s \leq 1} (B(s) - B(t)) > M(t) - B(t).$$

Also note that by the independent increment property $\max_{t \leq s \leq 1}(B(s) - B(t))$ is independent of $M(t) - B(t)$ and distributed as $M(1 - t)$. So, we obtain that

$$\mathbb{P}(M(1) > M(t)) = \int_0^\infty \int_{-\infty}^y f_{M(t),B(t)}(y, x)\mathbb{P}(M(1 - t) > y - x)dxdy$$

We know that $f_{M(t),B(t)}(y, x) = \frac{2}{\sqrt{2\pi t}} \frac{2y-x}{t} e^{-\frac{(2y-x)^2}{2t}}$, and from the cheat-sheet that $\mathbb{P}(T_{y-x} < t) = \mathbb{P}(M(1 - t) > y - x) = \int_{y-x}^\infty \frac{2}{\sqrt{2\pi(1-t)}} e^{-\frac{z^2}{2(1-t)}} dz$. and the first equality follows.

The second equality follows by using the change of variables as described and checking that the combined conditions $0 < y = y' < \infty$, $-\infty < x = y' - x' < y'$ and $x' = y - x < z = z' + x' < \infty$ is equivalent to x', y' and z' all being in the interval $(0, \infty)$.

c) Let $T_{\max}(1) = \{t \in (0, 1); B(t) = M(1)\}$ be the time when the Brownian Motion takes its maximum on the interval $(0, 1)$. Provide the distribution function of $T_{\max}(1)$. (4p)

Solution $\mathbb{P}(T_{\max}(1) \leq t) = \mathbb{P}(M(t) = M(1)) = 1 - \mathbb{P}(M(t) < M(1)) = 1 - \frac{2}{\pi} \arccos(\sqrt{t})$. Where we have used the note in the last equality.

Problem 5: Simulation

Consider $\{N(t); t \geq 0\}$, a non-homogeneous Poisson Process on $[0, \infty)$ with intensity function

$$\lambda(x) = ce^{-x}.$$

Note that the number of points in this Poisson process on the interval $[0, t]$ is denoted by $N(t)$ and let $N(\infty)$ be the number of points on $[0, \infty)$.

a) What is the distribution of $N(\infty)$? (4p)

Solution Because we are considering a Poisson process, the number of points in $[0, \infty)$ is Poisson distributed with expectaction $\int_0^\infty \lambda(x)dx = c$.

b) Generate the random variable N' as follows.

- Generate U_1, which is a random variable which is uniformly distributed on the interval $(0, 1)$. If $-\log[U_1] \geq c$, then set $N' = 0$.
- If $-\log[U_1] < c$ then generate U_2, which is independent of U_1 and also uniformly distributed on $(0, 1)$. If $-(\log[U_1] + \log[U_2]) \geq c$ set $N' = 1$.
- Continue this way: If $-\sum_{k=1}^{n} \log[U_k] < c$, then generate U_{n+1} which is independent of $U_1, U_2, \cdots, U_n$ and uniformly distributed on $(0, 1)$. If then $-\sum_{k=1}^{n+1} \log[U_k] \geq c$, set $N' = n$.

What is the density of $-\log[U_1]$? and what is the distribution of N'? (4p)

Solution Using inverse function method, we know that $-\log[U]$, where U is a uniform on $(0, 1)$ is exponentially distributed with expectation 1. So $-\log[U_1] \sim Exp[1]$.

By definition for $n \in \mathbb{N}_{\geq 0}$ the points $\sum_{k=1}^{n} - \log[U_k]$ are the points of a point process with intensity 1 and N' is the number of those points which are on the interval $(0, c)$. By the definition (or standard properties) of a Poisson Process this number is Poisson distributed with expectation c.

c) Provide a way to simulate the process $\{N(t); t \geq 0\}$. (4p)

Solution First generate N' as in question b and note that this number is distributed as $N(\infty)$. Say $N(\infty) = n$. Then by the order statistic property of Poisson processes, the n points of the (inhomogeneous) Poisson process can be generated as n independent and identically distributed random variables with density $\lambda(x)/\int_0^\infty \lambda(y)dy = e^{-x}$. That is the points are n i.i.d. exponential random variables with expectation 1. Denote those points by $X_1, X_2, \cdots, X_n$ Now let $V_1, V_2, \cdots, V_n$ be i.i.d. Uniform random variables on $(0, 1)$, which are independent of the $U's$ in b. Now, as in b, X_k is distributed as $-\log[V_k]$.