Solutions Stochastic Processes and Simulation II, May 23, 2019

Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $0 < S_1 < S_2 < S_3 < \cdots$. Define $S_0 = 0$. In parts a)-c) of this question let $T > 0$ be a constant.

a) Provide the distribution of $N(T)$. (3p)

Solution: By the definition of a homogeneous Poisson process $N(T)$ is Poisson distributed with expectation λT. That is, for $k \in \{1, 2, \cdots\}$ we have $\mathbb{P}(N(T) = k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$.

b) Provide the distribution of $S_{N(T)+1} - T$, that is provide the distribution of the waiting time until the next arrival at time T. (2p)

Solution: Inter-arrival times in a Poisson process are exponentially distributed, and the exponential distribution is memoryless, therefore $S_{N(T)+1} - T$ is exponentially distributed with expectation $1/\lambda$. That is, $\mathbb{P}(S_{N(T)+1} - T \leq x) = 1 - e^{-\lambda x}$ for $x \geq 0$

c) Let n be a strictly positive integer. Compute for $x \in [0, T]$

$$\mathbb{P}[T - S_n > x | N(T) = n].$$

Use this (or use some other way) to compute (also for $x \in [0, T]$) $\mathbb{P}[T - S_{N(T)} > x]$. That is, provide (one minus) the distribution of the time since the last arrival at time T. (4p)

Solution: By the order statistic property the n arrivals in $[0, T]$ are distributed as i.i.d. uniform random variables on $[0, T]$, therefore the probability of having all n points before $T - x$ is $(1 - x/T)^n$. Therefore $\mathbb{P}[T - S_n > x | N(T) = n] = (1 - x/T)^n$.

Using $\mathbb{P}[T - S_n > x] = \sum_{n=0}^{\infty} \mathbb{P}[T - S_n > x | N(T) = n]\mathbb{P}[N(T) = n]$, we obtain that

$$\begin{aligned}
\mathbb{P}[T - S_n > x] &= \frac{(\lambda T)^0}{0!} e^{-\lambda T} + \sum_{n=1}^{\infty} (1 - x/T)^n \frac{(\lambda T)^n}{n!} e^{-\lambda T} \\
&= \sum_{n=0}^{\infty} (1 - x/T)^n \frac{(\lambda T)^n}{n!} e^{-\lambda T} = \sum_{n=0}^{\infty} \frac{[\lambda(T - x)]^n}{n!} e^{-\lambda T} = e^{-\lambda x} \qquad \text{for } x \in [0, T]
\end{aligned}$$

For $x > T$ we have $\mathbb{P}[T - S_n > x] = 0$.

d) Provide the distribution of $S_{N(T)+1} - S_{N(T)}$ for $T \to \infty$. That is, provide the distribution of the length of the interarrival interval at time T, for $T \to \infty$. (3p)

Solution: Note that because of the inspection paradox $S_{N(T)+1} - S_{N(T)}$ is not exponentially distributed with parameter λ.
As $T \to \infty$ we know from c) that $T - S_{N(T)}$ is exp. distributed with parameter λ, while from b) we also know that $S_{N(T)+1} - T$ is exp. distributed with parameter λ and the two exponential distributed random variables are independent (because the first depends on what happens before time T and the second on what happens after time T). Together this gives that $S_{N(T)+1} - S_{N(T)}$ is gamma distributed with parameters 2 and λ (i.e. with mean $2/\lambda$ and variance $2/\lambda^2$).

Problem 2: Renewal Theory

In this question all limits are for $t \to \infty$.

Let $\{N(t), t \geq 0\}$ be a renewal process, with interarrival distribution function $F(t)$. Assume that the interarrival time has expectation μ and variance σ^2 and that $F(0) = 0$. Let $\{S_i, i = 1, 2, \cdots\}$ be the times of the renewals in the renewal process, such that $0 < S_1 < S_2 < S_3 < \cdots$. Define $S_0 = 0$.

a) Provide the definition of a renewal process. (4p)

Solution: A non-decreasing non-negative integer valued process $\{N(t); t \geq 0\}$ is a renewal process if the random variables $X_n = \inf\{t \geq 0; N(t) \geq n\} - \inf\{t \geq 0; N(t) \geq n-1\}$ are i.i.d.

b) Define the age of the process at time t as

$$A(t) = t - S_{N(t)}.$$

Provide the almost sure limit of

$$\frac{1}{t}\int_0^t A(s)ds$$

and justify your answer. (4p)

Solution: By the theory on renewal reward processes, we know that $\frac{1}{t}\int_0^t A(s)ds$ converges almost surely to $\mathbb{E}[\int_0^T A(s)ds]/\mathbb{E}[T]$, where T is the time of the first renewal. This expression is given by

$$\frac{\mathbb{E}[\int_0^T sds]}{\mu} = \frac{\mathbb{E}[T^2/2]}{\mu} = \frac{\sigma^2 + \mu^2}{2\mu}.$$

c) Provide the almost sure limit of

$$\frac{1}{t}\int_0^t (S_{N(s)+1} - S_{N(s)})ds.$$

(4p)

Solution: Define $Y(t) = S_{N(t)+1} - t$, to be the excess of the process at time t and note that $S_{N(s)+1} - S_{N(s)} = A(s) + Y(s)$. By the theory on renewal reward processes, we know that $\frac{1}{t}\int_0^t Y(s)ds$ converges almost surely to $\mathbb{E}[\int_0^T Y(s)ds]/\mathbb{E}[T]$, where T is the time of the first renewal. This expression is given by

$$\frac{\mathbb{E}[\int_0^T (T-s)ds]}{\mu} = \frac{\mathbb{E}[T^2/2]}{\mu} = \frac{\sigma^2 + \mu^2}{2\mu}.$$

Combining this with part *b*) we obtain

$$\frac{1}{t}\int_0^t (S_{N(s)+1} - S_{N(s)})ds \to \frac{1}{t}\int_0^t A(S)ds + \frac{1}{t}\int_0^t Y(s)ds = \frac{\sigma^2 + \mu^2}{\mu}.$$

Problem 3: Queueing Theory

Consider an $M/M/2$ queue in which customers arrive according to a Poisson Process with rate λ, and customers have independent workloads which are exponentially distributed with expectation $1/\mu$. Assume that $\lambda < 2\mu$. Assume further that the number of customers in the system at time 0, follows the stationary distribution of the number of customers in the system.

a) For $k \in \mathbb{N}_{\geq 0}$, let P_k be the probability that there are k customers in the system in stationarity. Show that

$$P_0 = \frac{2\mu - \lambda}{2\mu + \lambda} \qquad \text{and} \qquad P_k = 2P_0 \left(\frac{\lambda}{2\mu}\right)^k \quad \text{for } k \in \mathbb{N}_{\geq 1}.$$

(4p)

Solution: Using the balance equations (rate of entering a state equals rate of leaving a state) we obtain $\lambda P_0 = \mu P_1$, $(\lambda+\mu)P_1 = \lambda P_0 + 2\mu P_2$ and for $k \geq 2$ we have $(\lambda+2\mu)P_k = \lambda P_{k-1} + 2\mu P_{k+1}$. With the suggested values of P_k filled in, those equations read

$$\lambda P_0 = 2\mu P_0 \frac{\lambda}{2\mu},$$

$$2(\lambda+\mu)P_0 \frac{\lambda}{2\mu} = \lambda P_0 + 4\mu P_0 \left(\frac{\lambda}{2\mu}\right)^2$$

and

$$2(\lambda + 2\mu)P_0 \left(\frac{\lambda}{2\mu}\right)^k = 2\lambda P_0 \left(\frac{\lambda}{2\mu}\right)^{k-1} + 4\mu P_0 \left(\frac{\lambda}{2\mu}\right)^{k+1},$$

which are all correct. Furthermore for the suggested values,

$$\sum_{k=0}^{\infty} P_k = P_0 + 2P_0 \sum_{k=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^k = 2P_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{2\mu}\right)^k - P_0 = P_0 \left(\frac{2}{1-\lambda/(2\mu)} - 1\right) = P_0 \frac{2\mu+\lambda}{2\mu-\lambda} = 1.$$

b) Compute the long run average time that a customer is in the queue. That is, compute the long run average time that customers are waiting before their service start. (4p)

Solution: We know that the expected number of customers in the system is given by

$$\begin{aligned} L = \sum_{k=1}^{\infty} kP_k &= P_0 \frac{\lambda}{\mu} \left(1 - \frac{\lambda}{2\mu}\right)^{-1} \sum_{k=1}^{\infty} k \left(1 - \frac{\lambda}{2\mu}\right) \left(\frac{\lambda}{2\mu}\right)^{k-1} \\ &= P_0 \frac{\lambda}{\mu} \left(1 - \frac{\lambda}{2\mu}\right)^{-1} \left(1 - \frac{\lambda}{2\mu}\right)^{-1} = \frac{4\mu\lambda}{(2\mu-\lambda)(2\mu+\lambda)}. \end{aligned}$$

We then deduce from the cheat sheet that the average time a customer spends in the system is $W = L/\lambda = \frac{4\mu}{(2\mu-\lambda)(2\mu+\lambda)}$ and the average time a customer spends in the queue is given by

$$W_Q = W - \mathbb{E}[S] = \frac{4\mu}{(2\mu-\lambda)(2\mu+\lambda)} - \frac{1}{\mu} = \frac{\lambda^2}{\mu(2\mu-\lambda)(2\mu+\lambda)}.$$

c) Assume that Adam is the first customer arriving (strictly) after time 0, what is the Probability that Adam find no customers in the queue at his arrival? (4p)

Solution: Since Adam is the first customer to arrive, his arrival is not independent of what happened before (no other customer arrived before) so we cannot use PASTA. We do know however, that the process starts in stationarity.

Let X_A be the number of other customers in the system as Adam arrives and X_0 the number of customers present at time 0. We are interested in $\mathbb{P}(X_A \leq 2)$ (at most two customers may be in the system, which then are in service).

Then

$$\mathbb{P}(X_A \leq 2) = \sum_{j=0}^{\infty} \mathbb{P}(X_A \leq 2 | X_0 = j)\mathbb{P}(X_0 = j).$$

We note that for $j \leq 2$, $\mathbb{P}(X_A \leq 2|X_0 = j) = 1$, while for $j \geq 3$, $\mathbb{P}(X_A \leq 2|X_0 = j)$ is the probability that the first $j-2$ events (departures or arrivals) are all departures. From the theory of Markov Processes we know that if there are at least 2 customers present the probability that the next event is a departure is $\frac{2\mu}{2\mu+\lambda}$. Therefore,

$$\begin{aligned}
\mathbb{P}(X_A \leq 2) &= \sum_{j=0}^{\infty} \mathbb{P}(X_A \leq 2|X_0 = j)\mathbb{P}(X_0 = j) \\
&= P_0 + P_1 + P_2 + \sum_{j=3}^{\infty} \left(\frac{2\mu}{\lambda + 2\mu}\right)^{j-2} P_j \\
&= P_0 \left(1 + 2\frac{\lambda}{2\mu} + 2\left(\frac{\lambda}{2\mu}\right)^2\right) + \frac{2\mu}{\lambda + 2\mu} \sum_{j=3}^{\infty} \left(\frac{2\mu}{\lambda + 2\mu}\right)^{j-3} 2P_0 \left(\frac{\lambda}{2\mu}\right)^j \\
&= P_0 \left(1 + 2\frac{\lambda}{2\mu} + 2\left(\frac{\lambda}{2\mu}\right)^2 + 2\frac{2\mu}{\lambda + 2\mu}\left(\frac{\lambda}{2\mu}\right)^3 \sum_{j=3}^{\infty} \left(\frac{\lambda}{\lambda + 2\mu}\right)^{j-3}\right) \\
&= P_0 \left(1 + 2\frac{\lambda}{2\mu} + 2\left(\frac{\lambda}{2\mu}\right)^2 + 2\frac{2\mu}{\lambda + 2\mu}\left(\frac{\lambda}{2\mu}\right)^3 \frac{\lambda + 2\mu}{2\mu}\right) \\
&= P_0 \left(1 + 2\frac{\lambda}{2\mu} + 2\left(\frac{\lambda}{2\mu}\right)^2 + 2\left(\frac{\lambda}{2\mu}\right)^3\right).
\end{aligned}$$

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and for $t > 0$, let

$$M(t) := \max_{0 \leq s \leq t} B(s),$$

be the maximum of the Brownian motion up to time t. Here we assume that $B(0) = 0$ and that the variance parameter $\sigma^2 = 1$ is part of the definition of a standard Brownian motion.

a) For $y > 0$ and $x > 0$, argue that

$$\mathbb{P}(M(t) > y, B(t) < y - x) = \mathbb{P}(B(t) > y + x).$$

Hint: A picture tells a thousand words. (4p)

Solution: We use the reflection principle, with T_y de first hitting time of y

$$\begin{aligned}\mathbb{P}(M(t) > y, B(t) < y - x) &= \mathbb{P}(T_y < t, B(t) < y - x) = \mathbb{P}(T_y < t, B(t) - B(T_y) < y - x - y) \\ &= \mathbb{P}(T_y < t, B(t) - B(T_y) < -x) = \mathbb{P}(T_y < t, B(t) - B(T_y) > x),\end{aligned}$$

where we have used symmetry of the Brownian motion started in y at time T_y. Note that $\mathbb{P}(T_y < t, B(t) - B(T_y) > x)$ is the probability that the Brownian motion hits y before time t and then from then makes an additional increase of at least x at time t. So

$$\mathbb{P}(T_y < t, B(t) - B(T_y) > x) = \mathbb{P}(T_y < t, B(t) > y + x) = \mathbb{P}(B(t) > y + x),$$

because $\{B(t) > y + x\}$ implies $\{T_y < t\}$.

One can deduce from part a) that the joint density function of $M(t)$ and $B(t)$ is given by

$$f_{M(t),B(t)}(y, x) = \frac{2}{\sqrt{2\pi t}} \frac{2y - x}{t} e^{-\frac{(2y-x)^2}{2t}},$$

for $y > 0$ and $x < y$. In part b) you may use this without further proof.

b) For $t \in (0, 1)$ and $x \in \mathbb{R}$, provide the density of $M(t)$ conditioned on $B(t) = x$. (4p)

Solution:

$$f_{M(t)|B(t)}(y|x) = \frac{f_{M(t)|B(t)}(y|x)}{f_{B(t)}(x)} = \frac{\frac{2}{\sqrt{2\pi t}} \frac{2y-x}{t} e^{-\frac{(2y-x)^2}{2t}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}} = 2\frac{2y - x}{t} e^{-\frac{(2y-x)^2 - x^2}{2t}} = 2\frac{2y - x}{t} e^{-2\frac{y(y-x)}{t}}$$

In part c) you me use without further proof that for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $t \in (0,1)$ we have

$$\mathbb{P}\left(\max_{t\le s\le 1} B(s) > y | B(t) = x, B(1) = 0\right) = \mathbb{P}\left(M(1-t) > y | B(1-t) = x\right).$$

c) For $t \in (0,1)$, compute $\mathbb{P}(M(t) < M(1) | B(t) = x, B(1) = 0)$. That is, compute

$$\mathbb{P}\left(\max_{0\le s\le t} B(s) < \max_{t\le s\le 1} B(s) \mid B(t) = x, B(1) = 0\right).$$

(4p)

Solution: First condition on $M(t)$ and note that for $t \in (0,1)$ conditioned on $B(t)$, $M(t)$ is independent of $B(1)$.

$$\begin{aligned}
&\mathbb{P}\left(\max_{0\le s\le t} B(s) \le \max_{t\le s\le 1} B(s) \mid B(t) = x, B(1) = 0\right) \\
&\quad = \int_x^\infty \mathbb{P}\left(y \le \max_{t\le s\le 1} B(s) \mid M(t) = y, B(t) = x, B(1) = 0\right) f_{M(t)|B(t)}(y|x)dy \\
&\qquad = \int_x^\infty \mathbb{P}\left(M(1-t) > y \mid B(1-t) = x\right) f_{M(t)|B(t)}(y|x)dy,
\end{aligned}$$

where we have used the hint and that conditioned on $B(t)$, $\max_{t\le s\le 1} B(s)$ is independent of $M(t)$. Filling in what we have from part *b)* we obtain that

$$\begin{aligned}
&\mathbb{P}\left(\max_{0\le s\le t} B(s) \le \max_{t\le s\le 1} B(s) \mid B(t) = x, B(1) = 0\right) \\
&\quad = \int_x^\infty \left(\int_y^\infty f_{M(1-t)|B(1-t)}(z|x)dz\right) 2\frac{2y-x}{t} e^{-2\frac{y(y-x)}{t}} dy \\
&\quad = \int_x^\infty \left(\int_y^\infty 2\frac{2z-x}{1-t} e^{-2\frac{z(z-x)}{1-t}} dz\right) 2\frac{2y-x}{t} e^{-2\frac{y(y-x)}{t}} dy \\
&\quad = \int_x^\infty \left(e^{-2\frac{y(y-x)}{1-t}}\right) 2\frac{2y-x}{t} e^{-2\frac{y(y-x)}{t}} dy = \int_x^\infty 2\frac{2y-x}{t} e^{-2\frac{2y(y-x)}{t(1-t)}} dy = 1-t.
\end{aligned}$$

Problem 5: Simulation

Let $U_1, U_2, \cdots$ be independent random variables taking uniform values between 0 and 1.

a) Explain in detail that $-\log(U_1)$ has density $f_1(t) = e^{-t}$ for $t \geq 0$. (4p)

Solution:

$$\mathbb{P}(-\log(U_1) \leq t) = \mathbb{P}(\log(U_1) \geq -t) = \mathbb{P}(U_1 \geq e^{-t}) = 1 - e^{-t}.$$

So, $-\log(U_1)$ is exponentially distributed with mean 1, and has density e^{-t}.

b) Show that $-\sum_{k=1}^{n} \log(U_k)$ has density $f_n(x) = \frac{x^{n-1}}{(n-1)!}e^{-x}$. (4p)

Solution: This follows immediately from the fact that the sum of n independent exponential distributed random variables with mean 1 is gamma distributed with mean and variance both equal to n. Such a Gamma distribution has indeed density $f_n(t)$ (see cheat-sheet).

c) Define

$$N = \min\left\{k \in \mathbb{N}_{\geq 0}; \prod_{j=1}^{k+1} U_j \leq e^{-\lambda}\right\}.$$

What is the distribution of N?

Hint: Note that N can also be defined as

$$\min\left\{k \in \mathbb{N}_{\geq 0}; -\log\left(\prod_{j=1}^{k+1} U_j\right) \geq \lambda\right\}.$$

(4p)

Solution: Observe, that using the hint

$$N = \min\left\{k \in \mathbb{N}_{\geq 0}; \prod_{j=1}^{k+1} U_j \leq e^{-\lambda}\right\} = \min\left\{k \in \mathbb{N}_{\geq 0}; \sum_{j=1}^{k+1} -(\log(U_j)) \geq \lambda\right\}.$$

So, using part a, we observe that N is the minimum number of i.i.d. exponential random variables with mean 1, one should add up to get above λ. Which we know from the theory of Poisson processes to be distributed as $1 + X$, where X is Poisson distributed with expectation λ.