Solutions Stochastic Processes and Simulation II, August 19, 2020

Problem 1: Poisson Processes

To describe the spread of an infectious disease, such as Covid-19, mathematical modellers often use SIR models. In those models people can be Susceptible, Infectious or Removed (which might mean recovered and eternally immune or dead). At time $t = 0$ there is one infectious person, Adam, who just turned Infectious (so just before $t = 0$ Adam was susceptible). All other individuals in the population are susceptible at time $t = 0$. If an infectious person makes an "infectious contact" (more about this below) with a susceptible person, the susceptible one immediately becomes infectious and stays so for a random time, which is distributed as X and has distribution function $f_X(\cdot)$. This random time is called the infectious period. After the infectious period the infected person is removed forever. The durations of the infectious periods of different people are independent.

An infectious individual makes "casual" infectious contacts during his or her infectious period according to a homogeneous Poisson Process with rate λ_c, you may assume that all infectious contacts will be with different people. Furthermore, the infectious individual attends independently "gatherings" according to a homogenous Poisson process with rate λ_g. At a gathering an infectious person makes infectious contacts with a random number of people distributed as the non-negative integer-valued random variable A. The number of people contacted at different gatherings are independent.

Set

$$\mathbb{E}[X] = \mu_X \quad \text{and} \quad \mathbb{E}[A] = \mu_A.$$

Further define $p_0 := \mathbb{P}(A = 0)$ and assume $p_0 > 0$.

a) What is the probability that Adam does not attend any gathering during his infectious period. You do not have to evaluate possible integrals involving $f_X(\cdot)$ in your answer. (4pt)

Solution: If Adam's infectious period is t then Adam does not attend any gathering with probability $e^{-\lambda_g t}$ by the definition of a Poisson Process. So the probability that Adam does not attend any gathering during his infectious period is

$$\int_0^\infty f_X(t)e^{-\lambda_g t}dt.$$

b) Provide the expectation of the number of people that Adam makes an infectious contact with during his infectious period. (4pt)

Solution: For notational convenience. Let Z_c be the number of "casual infections" and Z_g be the number of "gathering infections" caused by Adam. We are interested in

$$\mathbb{E}[Z_c + Z_g] = \mathbb{E}[Z_c] + \mathbb{E}[Z_g].$$

We compute the terms one by one conditioning on the infectious period X.

$$\mathbb{E}[Z_c] = \mathbb{E}[\mathbb{E}[Z_c|X]] = \lambda_c\mathbb{E}[X] = \lambda_c\mu_X.$$

and to deal with Z_g, also consider the number of gatherings Adam attend during his infectious period (call this Y).

$$\mathbb{E}[Z_g] = \mathbb{E}[\mathbb{E}[\mathbb{E}[Z_c|X,Y]|X]] = \mathbb{E}[\mathbb{E}[Y\mu_A|X]] = \lambda_g\mu_A\mathbb{E}[X] = \lambda_g\mu_A\mu_X.$$

So,

$$\mathbb{E}[Z_c + Z_g] = \mathbb{E}[Z_c] + \mathbb{E}[Z_g] = \mu_X(\lambda_c + \lambda_g\mu_A).$$

c) If $\mathbb{P}(X = 1) = 1$, what is the probability that Adam makes no infectious contacts at all during his infectious period? (4pt)

Solution: If we consider only the gatherings in which Adam makes infectious contacts, then we consider a thinned Poisson process, in which every point of the original Poisson process with rate λ_g is kept independently with probability $1 - p_0$. If Adam does not have any infectious contacts at all, then he does not make any casual contacts during his infectious period (with happens with probability $e^{-\lambda_c}$ by the definition of a Poisson Process) and he does not attend any gathering in which he makes infectious contacts during his infectious period (which happens with probability $e^{-\lambda_g(1-p_0)}$. The two Poisson processes are independent and therefore the probability of making no infectious contact is the product of these probabilities:

$$e^{-(\lambda_c+\lambda_g(1-p_0))}.$$

Problem 2: Renewal Theory

Let $Z_1, Z_2, \cdots$ be independent and identically exponentially distributed random variables with expectation $1/\beta$. Define

$$Y_k := \sum_{j=1}^{k} Z_j \qquad \text{for } k \in \mathbb{N} := \{1, 2, \cdots\}.$$

For convenience define $Y_0 := 0$. Consider the process $\{X(t); t \geq 0\}$ taking values in the state space $\mathbb{N}_0 := \{0, 1, 2, \cdots\}$. This process is defined as follows. Let t_0 be a strictly positive constant.

- $X(0) = 0$.
- Let $k, \ell \in \mathbb{N}_0$. If $X(Y_k) = \ell$ then $X(t) = \ell$ for $t \in [Y_k, Y_{k+1})$. That is $X(t)$ is constant between Y_k and $Y_{k+1} = Y_k + Z_{k+1}$
- Let $k, \ell \in \mathbb{N}_0$. If $X(Y_k) = \ell$ then

$$X(Y_{k+1}) = \begin{cases} \ell + 1 & \text{if } Z_{k+1} := Y_{k+1} - Y_k > t_0, \\ 0 & \text{if } Z_{k+1} := Y_{k+1} - Y_k \leq t_0. \end{cases}$$

That is, $X(t)$ can be seen as the number of "events" taken place since the last event which came less than t_0 time units after its predecessor.

a) Let

$$J := \min\{j \in \mathbb{N}; Z_j \leq t_0\}$$

be the index of the first "Z random variable" which is less than t_0. Provide the distribution of J and compute $\mathbb{E}[Y_J] = \mathbb{E}[\sum_{j=1}^{J} Z_j]$. (6pt)

Solution: Since the Z random variables are independent and identically distributed they all have probability of being less than t_0 (that probability is $q := 1 - e^{-\beta t_0}$). $\mathbb{P}(J = j)$ is therefore given by the probability that the first $j-1$ Z_j's are all larger that t_0 (with probability $(1-q)^{j-1}$ time the probability that Z_j is less than t_0 (probability q). So $\mathbb{P}(J = j) = (1-q)^j q$.

Furthermore, J is a stopping time with finite expectation (it is a geometric random variable with expectation $1/q$) and we can use Wald's equation to obtain

$$\mathbb{E}[Y_J] = \mathbb{E}[\sum_{j=1}^{J} Z_j] = \mathbb{E}[J]\mathbb{E}[Z_1] = \frac{1}{q}\frac{1}{\beta}.$$

b) Compute the (almost sure) long run fraction of time that $\{X(t); t \geq 0\}$ is in state 0. That is, compute the almost sure limit of

$$\frac{\int_0^t \mathbb{1}(X(s) = 0)ds}{t},$$

for $t \to \infty$. (6pt)

Solution: Use a renewal reward process, where a cycle starts at time 0 and at every time the process enters state 0. The expected duration of a cycle is computed in a): $1/(\beta q)$. The expected time the process is in state 0 during a cycle is the expectation of the first Z random variable in the cycle, which is $1/\beta$. By the Renewal Reward theorem, the long run fraction of time the process is in state 0 is therefore

$$\frac{1/\beta}{1/[\beta q]} = q.$$

Problem 3: Queueing Theory

Consider the following queing system. Customers arrive at the system according to a homogeneous Poisson Process with rate λ. Independently of other customers and time of arrival a customer is either Easy (with probability p) or Hard (with probability $1-p$). Whether a custormer is Easy or Hard is known at his or her entrance in the system. Easy customers have a workload (i.e. the time that a unit speed server needs to work on that customer when in service) which is exponentially distributed with expectation $1/\mu_E$ and Hard customers have a workload which is exponentially distributed with expectation $1/\mu_H$. Workloads are independent of each other.

Suppose that there are two queues, one for the Easy customers and one for the Hard customers. There are also two servers both working at unit speed. One server serves the queue with Easy customers only, while the other serves the queue with Hard customers only.

a) Provide necessary and sufficient relations between λ, p, μ_E and μ_H for neither of the queue length to go to infinity? (2pt)

Solution: The arrival processes of easy and hard customers can be seen as two independent Poisson processes (See Prop. 5.2 on page 304). So using that proposition the easy queue is an $M/M/1$ queue with arrival rate $p\lambda$ and departure rate μ_E, which does not explode if

$$p\lambda < \mu_E,$$

while the hard queue is an $M/M/1$ queue with arrival rate $(1-p)\lambda$ and departure rate μ_H, which does not explode if

$$(1-p)\lambda < \mu_H.$$

Assume for the remainder of the problem that the condition of part a) is satisfied.

b) Compute the (almost sure) long run average ammount of time that a customer are in the system. That is, compute the cummulative ammount of time customers have spent in the system up to time t divided by the number of customers which arrived before time t for $t \to \infty$. (4pt)

Solution: Using that the two queues are $M/M/1$ queues, we know (using page 490 of Ross) that the average time an easy (resp. hard) customer is in the system is $\frac{1}{\mu_E - p\lambda}$ (resp. $\frac{1}{\mu_H-(1-p)\lambda}$). A fraction p of the customers is Easy and a fraction $1-p$ of the customers is hard, so the average ammount of time a customer is in the system is

$$p\frac{1}{\mu_E - p\lambda} + (1-p)\frac{1}{\mu_H - (1-p)\lambda}.$$

Now assume that the two servers are replaced by one single server who works at double speed. That is a customer with workload w needs $w/2$ units of time of service from this server.

c) Compute the (almost sure) long run average ammount of time that a customer is in this system? (6pt)

Hint: What kind of queueing system is this single queue system?

Solution: The system with one queue is an $M/G/1$ queue. With expected needed service time

$$\mathbb{E}[S] = p\frac{1}{2\mu_E} + (1-p)\frac{1}{2\mu_H} = \frac{p\mu_H + (1-p)\mu_E}{2\mu_E\mu_H},$$

where the factor 2 in the denominator is because the server is working at double speed. Similarly we can compute

$$\mathbb{E}[S^2] = p\frac{2}{(2\mu_E)^2} + (1-p)\frac{2}{(2\mu_H)^2} = \frac{p(\mu_H)^2 + (1-p)(\mu_E)^2}{2(\mu_E\mu_H)^2}.$$

Using equation (8.34) from Ross we obtain that the average time a customer is in the system is:

$$\begin{aligned}
&\frac{\lambda\mathbb{E}[S^2]}{2(1-\lambda\mathbb{E}[S])} + \mathbb{E}[S] \\
= \;& \frac{\lambda(p(\mu_H)^2 + (1-p)(\mu_E)^2)}{2(\mu_E\mu_H)^2 2(1-\lambda\frac{p\mu_H+(1-p)\mu_E}{2\mu_E\mu_H})} + \frac{p\mu_H + (1-p)\mu_E}{2\mu_E\mu_H} \\
= \;& \frac{\lambda(p(\mu_H)^2 + (1-p)(\mu_E)^2)}{(2\mu_E\mu_H)[(2\mu_E\mu_H) - \lambda(p\mu_H + (1-p)\mu_E)]} + \frac{p\mu_H + (1-p)\mu_E}{2\mu_E\mu_H} \\
= \;& \frac{\lambda(p(\mu_H)^2 + (1-p)(\mu_E)^2) + 2\mu_E\mu_H(p\mu_H + (1-p)\mu_E) - \lambda(p\mu_H + (1-p)\mu_E)^2}{(2\mu_E\mu_H)[(2\mu_E\mu_H) - \lambda(p\mu_H + (1-p)\mu_E)]}
\end{aligned}$$

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t); t \geq 0\}$ be a standard Brownian motions satisfying $B(0) = 0$ and having variance parameter 1. Let

$$\{B_\mu(t); t \geq 0\} := \{B(t) + \mu t; t \geq 0\}$$

be a Brownian Motion with drift, where μ is a strictly negative constant. Let

$$\{M_\mu(t); t \geq 0\} := \{\max_{0 \leq s \leq t} B_\mu(s); t \geq 0\}$$

be the process providing the maximu of the Brownian motion up to time t. Define the random variable

$$M_\mu := \max_{t>0} B_\mu(t).$$

You may assume without further proof that $\mathbb{P}(M_\mu \text{ exists and is finite}) = 1$.

a) Provide the density function of M_μ. (3pt)

Solution: We can now use results from Section 10.5 of Ross in many ways. Here I use the the last line of the Section. Noting that if the maximum of a Brownian Motion with drift is at least y if and only if the hitting time of y is finite, we obtain (using that μ is strictly negative

$$\mathbb{P}(\max_{t>0} B_\mu(t) \geq y) = \mathbb{P}(T_y \leq \infty) = e^{2y\mu}\bar{\Phi}(-\infty) + \bar{\Phi}(+\infty) = e^{-2y|\mu|}$$

Here we have used that $\bar{\Phi}(x)$ is the probability that a standard normal random variable is larger than x.

b) Let $T_{max} := \min\{t > 0; B_\mu(t) = M_\mu\}$ be the first (and almost surely only) time that the Brownian Motion with (negative) drift takes its overall maximum. Argue that $\mathbb{P}(T_{max} \leq t) = \mathbb{P}(M_\mu(t) - B_\mu(t) > \tilde{M}_\mu)$, where $\tilde{M}_\mu$ is a random variable which is independent of $\{B_\mu(t); t \geq 0\}$ and distributed as M_μ. (4pt)

Solution: If $T_{\max} \leq t$, then $M_\mu(t) \geq \max_{s>t} B_\mu(s)$ and thus $M_\mu(t) - B_\mu(t) \geq \max_{s>t}[B_\mu(s) - B_\mu(t)]$. By independent increments and noting that the left hand side only depends on what happens before time t and the right hand side only depends on what happens after time t the left and right hand side are independent of each other. Furthermore,

$$\max_{s>t}[B_\mu(s) - B_\mu(t)] = \max_{s>t}[B(s) + \mu s - B(t) - \mu t] = \max_{s>t}[B(s) - B(t) + \mu(s-t)].$$

Then note that $B(s) - B(t)$ is distributed as $B(s-t)$ and thus that $\max_{s>t}[B_\mu(s) - B_\mu(t)]$ is distributed as $\max_{s>t}[B(s-t) - \mu(s-t)]$, which by properties of the Brownian Motion is distributed as $\max_{s>0}[B(s) - \mu(s)] = \max_{s>0} B_\mu(s) = M_\mu$. This finishes the argument.

c) For $t > 0$, compute $\mathbb{P}(B_\mu(s) < 0 \text{ for all } s > t)$. That is, compute the probability that all zeros of $\{B_\mu(s); s \geq 0\}$ are before time t. (4pt)

Solution:

$$\begin{aligned}\mathbb{P}(B_\mu(s) < 0 \text{ for all } s > t) &= \mathbb{P}(\max_{s>t} B_\mu(s) < 0) = \mathbb{P}(\max_{s>t}[B_\mu(s) - B_\mu(t)] < -B_\mu(t)) \\ &= \int_{-\infty}^{\infty} f_{B_\mu(t)}(x)\mathbb{P}(\max_{s>t}[B_\mu(s) - B_\mu(t)] < -x | B_\mu(t) = x)dx\end{aligned}$$

Arguing as in part b) we know that $\mathbb{P}(\max_{s>t}[B_\mu(s) - B_\mu(t)] < -x | B_\mu(t) = x)$ does not depend on x and therefore is equal to $\mathbb{P}(\max_{s>t}[B_\mu(s) - B_\mu(t)] < -x)$, which in turn is equal to 0 for $x > 0$ and equal to $1 - e^{-2|\mu||x|}$ for $x < 0$ by part a. So, we obtain

$$\begin{aligned}\mathbb{P}(B_\mu(s) < 0 \text{ for all } s > t)) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi t}} e^{-(x+|\mu|t)^2/(2t)}[1 - e^{-2|\mu||x|}]dx \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi t}} e^{-(x+|\mu|t)^2/(2t)}dx - \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi t}} e^{-(x+|\mu|t)^2/(2t)} e^{-2|\mu||x|}dx \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-|\mu|t)^2/(2t)}dx - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-|\mu|t)^2/(2t)} e^{-4|\mu|tx/(2t)}dx \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-|\mu|t)^2/(2t)}dx - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x+|\mu|t)^2/(2t)}dx \\ &= \int_{-\mu t}^{\mu t} \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}dx\end{aligned}$$

Problem 5: Simulation

Consider a Poisson Process with batch arrivals. At the points of a homogeneous Poisson Process with rate λ buses arrive. The number of passengers per bus are independent of the arrival times and independent and identically distributed and distributed as the random variable B. With

$$\mathbb{P}(B = k) = p(1-p)^{k-1} \qquad \text{for } k \in \mathbb{N} = \{1, 2, \cdots\},$$

where $p \in (0, 1)$. Number the passengers in order of arrival, where the order of passengers on the same bus is arbitrary.

a) Provide a way to simulate from B. (4pt)

Solution: One way is to simulate independent and identically distributed uniformly distributed random variables on $(0, 1)$, $V_1, V_2, \cdots$ one by one. If $V_1 < p$ set $B = 1$. If $V_k > p$ for all $k < j$ (which happens with probability $(1-p)^{j-1}$ and $V_j < p$ (which happens independently with probability p) then set $B = j$. It follows immediately that B has the right distribution.

b) Argue that one can simulate the arrival times of the first n customers (say $T_1, T_2, \cdots, T_n$) using only a sequence of independent uniform random variables, $U_1, U_2, \cdots, U_n$ taking values in the interval $(0, 1)$ as follows:

- Simulate U_1 and set time of first arriving customer as $T_1 = |\log U_1|/\lambda$
- For $\ell \in \{1, 2, \cdots, n-1\}$, simulate $U_{\ell+1}$. If $U_{\ell+1} > p$ set $T_{\ell+1} = T_\ell$, while if $U_{\ell+1} < p$, set $T_{\ell+1} = T_\ell + |\log(U_{\ell+1}/p)|/\lambda$.

(8pt)

Solution: We know from page 650 of Ross that $|\log U_1|/\lambda$ is exponentially distributed with parameter λ as desired. If we follow the algorithm above, the number of arrivals at the same time as the first arrival is equal to $A = \min j \in \{2, 3, \cdots\}; U_j < p - 1$, and we thus have that $\mathbb{P}(A = k-1) = (1-p)^{k-2}p = \mathbb{P}(B = k-1)$. Similarly, if for $\ell \in \mathbb{N}$, we have $T_{\ell+1} > T_\ell$ the probability that $T_{\ell+k} = T_\ell$ is the probability that $U_{\ell+2}, U_{\ell+3}, \cdot, U_{\ell+k}$ are all larger than p which is $(1-p)^{k-1} = \mathbb{P}(B \geq k)$ as desired.

Conditioned on $U_j < p$, U_j/p is a uniform $(0, 1)$ random variable. And $|\log(U_j/p)|/\lambda$ is an exponentially distributed random variable with parameter λ as desired for the inter-arrival times in a Poisson Process.