STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5012
TENTAMEN
May 31, 2021

Second exam Stochastic Processes and Simulation II

May 31, 2021 kl. 9–15

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Preliminaries

This exam is a home-exam. The candidate may use books, mathematical software, material available on the internet etc. If you use results from the 11th edition of "Introduction to Probability Models" by Sheldon Ross, you do not have to copy the argument but a proper reference (including page number) is sufficient. Otherwise proper referencing is also needed, but arguments need to be copied or paraphrased.

The exam should be handed in May 31 no later than 15:00 (deadline). This should be done using the function given on the course website (near bottom of website). If problems arise, send solution by email to ptrapman@math.su.se.

The solutions to the exam should be submitted on the course homepage as a single PDF file. There are no restrictions regarding what your PDF should contain. For example, the PDF may be based on a Word document, a Latex document, or scanned nicely handwritten solutions. If you plan on "scanning" handwritten solutions using your mobile phone, I suggest downloading and using a "scanning app". If you scan and thereby obtain several PDF files, then there are many programs that can be used to merge several PDF files into one PDF file.

The home exam will follow the same style as old exams. Hence, your solution should be of the same type as for usual exams (i.e. not of "thesis type").

If something is unclear or if you experience problems during the exam, please notify me as soon as possible by sending an e-mail to ptrapman@math.su.se

If I need to get in touch with you during the exam I will use the news forum on the course home page, so please check this regularly

If you want to ask questions, that are hard to formulate by email, concerning the exam you may send an e-mail to ptrapman@math.su.se with the subject "Exam MT5012" together with a Zoom meeting ID. I will contact you as soon as possible.

Grading

The exam consists of 5 problems each divided in subproblems. Maximum of 60 points.

In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

	A	B	C	D	E
Needed points	50	45	40	35	30

Partial answers (such as not evaluating integrals or sums) might be worth points, sometimes even a maximal score! Answers “out of the blue” will not be rewarded.

The problems differ in difficulty. Do not be alarmed if you think the question is too easy, and only require a one line answer. It might be just that.

Statement by student

Write out in full on the exam that you promise that you (1) have written the exam without the assistance of any other person and (2) have not discussed the exam with anyone while writing it (apart from possibly the lecturer).

Problem 1: Poisson Processes

Consider a homogeneous Poisson process $\{N(t); t \geq 0\}$ with intensity $\lambda > 0$. Let $c > 0$ be a positive constant. Now color the point in the Poisson process independently red or blue where for $t > 0$ a point at position t is red with probability e^{-ct}, otherwise the point is blue.

a) What is the probability that the first point is red? (4pt)

b) What is the distribution of the number of red points in $(0, \infty)$? (4pt)

Denote the positions of the red points by $R_1 < R_2 < \cdots$, where $R_k = \infty$ if the total number of red points is less than k.

c) Let $s < t < \infty$. Compute $\mathbb{P}(R_1 < s | R_3 = t)$. (4pt)

Problem 2: Renewal Theory

A quiz show candidate is asked a series of questions. Each answer can be evaluated to be either C (for correct) or F (for false). The sequence of answers can be described by a Markov chain. The first answer is correct with probability p_1. A correct answer is directly followed by another correct answer with probability p_C, while a false answer is directly followed by a correct answer with probability p_F. Let $C(n)$ be the number of correct answers among the first n answers.

a) What is the expected number of questions needed to obtain the first correct answer? (4pt)

b) What is the (almost sure) long-run fraction of answers that is correct? That is, what is the almost sure limit of $C(n)/n$? (4pt)

Hint: Note that p_1, p_C and p_F are not necessarily equal.

c) Let $X(n)$ be the number of subsequent correct answers given at time n. So, if the sequence starts

$$C, C, F, C, F, \cdots,$$

then

$$X(1) = 1, \quad X(2) = 2, \quad X(3) = 0, \quad X(4) = 1 \quad \text{and} \quad X(5) = 0.$$

For all $n \in \{1, 2, \cdots\}$, assume that at time n the candidate receives a reward of $X(n)$. What is the expected long run income of the candidate per time unit? That is what is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X(j) \quad ?$$

(4pt)

Problem 3: Queueing Theory

Consider the following $M/M/\infty$ queue with catastrophes: Customers arrive in the system according to a homogeneous Poisson Process with intensity $\lambda > 0$. Upon entering the system, a customer immediately receives service from one of the infinitely many servers. Workloads are independent and exponentially distributed with rate $\mu > 0$ (So, with expectation $1/\mu$). A customer who has been in the system for the duration of his or her workload, immediately leaves the system. In addition catastrophes occur according to a Poisson process with intensity δ. This Poisson process is independent of the arrival process of customers. At the moment of a catastrophe, all customers present leave the system immediately. Workloads are independent of both the arrival process and the times of catastrophes.

a) Provide the expected time a customer is in the system? (2pt)

b) What is the long run average number of customers in the system? (3pt)

c) For $k \in \{1, 2, \cdots\}$ Let N_k be the number of customers that leaves the system at the k-th catastrophe. What is $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} N_k$? (3pt)

d) Let $\mathcal{S} = \{0, 1, 2, \cdots\}$ be the "state space" of the process describing the number of customers in the system. Let $\{P_s\}_{s\in\mathcal{S}}$ be the stationary distribution of the Queueing system described in this problem. Provide the "balance equations" characterizing this stationary distribution. That is, provide relations beween the P_s ($s \in \mathcal{S}$), which are in theory enough to compute them all. (4pt)

Hint: Note you do not have to solve the balance equations.

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t); t \geq 0\}$ be a standard Brownian motion (So we assume $B(0) = 0$ and $Var(B(1)) = 1$). Let

$$m(t) = \min_{0 \leq s \leq t} B(t).$$

a) Argue that for $x, y > 0$ we have

$$\mathbb{P}(m(t) < -x, B(t) > y - x) = \mathbb{P}(B(t) > y + x).$$

(4pt)

b) Again for $x, y > 0$, compute $\mathbb{P}(B(t) > y - x | m(t) > -x)$. (4pt)

Hint: You may use a) to obtain $\mathbb{P}(m(t) > -x, B(t) > y - x)$.

c) Show that

$$\lim_{\epsilon \searrow 0} \mathbb{P}(B(t) > y | m(t) > -\epsilon) = e^{-y^2/(2t)}$$

and compute

$$\lim_{\epsilon \searrow 0} \mathbb{E}[B(t) | m(t) > -\epsilon].$$

Here $\lim_{\epsilon \searrow 0}$ means the limit as ϵ decreases to 0. (4pt)

Problem 5: Simulation

Let X and Y be random variables with density functions

$$f_X(x) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-x^2/2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad f_Y(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Assume that it is easy to simulate a realisation of the random variable Y.

a) If you use the rejection method using $f_Y(x)$ to simulate a random variable with density function $f_X(x)$, what value of $\lambda \in (0, \infty)$ gives the lowest possible probability of rejection of a proposed realisation of X? (6pt)

Remark: The rejection method above can be used (after an extra step) to simulate a realisation of a standard normal random variable.

Let $\mathbb{N}$ be the set of strictly positive integers and For every $k \in \mathbb{N}$, define $[k] = \mathbb{N} \cup (0, k]$ as the set of positive integers not exceeding k.

Consider the following construction of the random process $\{W(t); t \in [0,1]\}$. **Only Steps 1, 2 and equation (1) are needed for solving part b). You can safely ignore the other steps!**

1. For all $n \in \mathbb{N}$ and $j \in [2^{n-1}]$, let N_0 and $N_{n,j}$ be independent standard normal distributed random variables.
2. Set $W(0) = 0$ and $W(1) = N_0$.
3. Assume that you know $W(i2^{-(n-1)})$, for all $i \in [2^{n-1}]$ (which you do for $n = 1$). Then for all $j \in [2^{n-1}]$ define
$$W((2j-1)2^{-n}) = \frac{W((j-1)2^{-(n-1)}) + W(j2^{-(n-1)})}{2} + 2^{-(n+1)/2} N_{n,j}.$$
In this way you can find $W(x)$ for all $x \in [0,1]$ with finite binary representation. In particular, $n = j = 1$ gives
$$W(1/2) = \frac{W(0) + W(1)}{2} + \frac{1}{2} N_{1,1}. \tag{1}$$
4. For $n \in \mathbb{N}$, Define $\{W_n(t); t \in [0,1]\}$ by connecting for $i \in [2^n]$, the points $((i-1)2^{-n}, W((i-1)2^{-n}))$ and $(i2^{-n}, W(i2^{-n}))$ by straight line segments. That is, for $i \in [2^n]$ and $t \in [0, 2^{-n}]$ define
$$W_n((i-1)2^{-n} + t) = W((i-1)2^{-n}) + 2^n t [W(i2^{-n}) - W((i-1)2^{-n})].$$
5. $\{W(t); t \in [0,1]\}$ is the pointwise limit of $\{W_n(t); t \in [0,1]\}$ as $n \to \infty$.

b) Deduce from the definition of $W(0), W(1)$ and $W(1/2)$ that $W(1/2)$ has a normal distribution with expectation 0 and variance 1/2. Show further that $W(1/2)$ and $W(1) - W(1/2)$ are independent. (6pt)

Remark: The above construction gives a way to construct and approximate a Brownian Motion on $[0,1]$.