Solutions of the first exam:
Stochastic Processes and Simulation II
June 1st 2022

Exercise 1: Poisson processes

(i) Give the definition of a nonhomogeneous Poisson process and of its mean value function.

Solution: A non-decreasing non-negative integer valued process (counting process) $\{N(t), t \geqslant 0\}$ is said to be a nonhomogeneous Poisson process with intensity function $\lambda(t) > 0$, $t \geqslant 0$ if the following holds:

(i) $N(0) = 0$;

(ii) $N(t)$ has independent increments;

(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$;

(iv) $\mathbb{P}(N(t+h) - N(t) \geqslant 2) = o(h)$.

Its mean value function is defined by $m(t) = \int_0^t \lambda(y)\, dy$.

Greta owns a vegan restaurant which is open every day from 09:00 to 23:00. Customers arrive at a Poisson rate that increases steadily from 6 customers per hour (c/ph) at 09:00 to 30 c/ph at 12:00. In the following two hours, around lunch time, the average rate remains constant at 30 c/ph. After that, it steadily drops until 16:00, at which time it has the value of 10 c/ph, and then it steadily increases again until 18:00 to 24 c/ph. Around dinner time, the arrival rate remains constant at 24 c/ph in the following three hours, after which it steadily drops again until closing time to 6 c/ph. Assume that the numbers of customers arriving during disjoint time periods are independent.

(ii) Define a good probability model that describes the arrival of customers at Greta's restaurant.

Solution: A good model would be assume that the arrivals constitute a nonhomegeneous Poisson process with intensity function $\lambda(t)$ given by

$$\lambda(t) = \begin{cases} 6 + 8t, & 0 \leqslant t \leqslant 3, \\ 30, & 3 \leqslant t \leqslant 5, \\ 30 - 10(t-5), & 5 \leqslant t \leqslant 7, \\ 10 + 7(t-7), & 7 \leqslant t \leqslant 9, \\ 24, & 9 \leqslant t \leqslant 12, \\ 24 - 9(t-12), & 12 \leqslant t \leqslant 14, \\ \lambda(t-14), & t > 14. \end{cases}$$

(iii) What is the probability that no customers arrive between 10:00 and 11:00? What is the expected number of customers that arrive at the restaurants in a day?

Solution: The number of arrivals between 10:00 and 11:00 is a Poisson random variable with mean $m(2) - m(1)$. The probability that no customers arrive between 10:00 and 11:00 and is the probability that this random variable is zero. Hence, it is given by

$$e^{-\int_1^2 \lambda(t)\,dt} = e^{-\int_1^2 (6+8t)\,dt} = e^{-(6+12)} = e^{-18}.$$

The average number of arrivals in a day is given by

$$\begin{aligned}\int_0^{14} \lambda(t)\,dt = &\int_0^3 (6+8t)\,dt + \int_3^5 30\,dt + \int_5^7 (30 - 10(t-5))\,dt \\ &+ \int_7^9 (10 + 7(t-7))\,dt + \int_9^{12} 24\,dt + \int_{12}^{14} (24 - 9(t-12))\,dt \\ = &\, 54 + 60 + 40 + 34 + 72 + 30 = 290.\end{aligned}$$

Exercise 2: Renewal theory

(i) Let $\{N(t), t \geqslant 0\}$ be a renewal process with interarrival times X_n, $n \geqslant 1$. Consider a renewal reward process $\{R(t) = \sum_{n=1}^{N(t)} R_n, t \geqslant 0\}$ where R_n, $n \geqslant 1$ are i.i.d. and represent the rewards earned each time a renewal occurs. State and prove the reneward reward theorem for $\frac{R(t)}{t}$. Note: you can use the elementary renewal theorem without proving it.

Solution: The renewal reward theorem says that, if $\mathbb{E}[R_n] = \mathbb{E}[R] < \infty$ and the mean interarrival time $\mathbb{E}[X_n] = \mu < \infty$, then $\frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mu}$ almost surely as $t \to \infty$. In order to prove it, first write $\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t}$. Then, by the strong law of large numbers, $\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \to \mathbb{E}[R]$ a.s., and, by the elementary renewal theorem, $\frac{N(t)}{t} \to \frac{1}{\mu}$.

In order to be able to cook healthier food, Greta decides to buy an air fryer machine for her restaurant. The machine breaks down according to a renewal process, whose interarrival times are uniformly distributed between 0 and 2 years. Assume that the cost of a new machine is 75000 SEK, while the expected cost of a new repair increases with the number of earlier repairs in such a way that the expected cost of the k-th repair is $600 \cdot k$ SEK. Assume that at the K-th breakdown Greta decides to replace the air fryer and buy a new one instead of repairing it.

(ii) What is the expected age of an air fryer when being replaced?

Solution: Denote by X_1 the time of the first breakdown and by X_i the time between the $(i-1)$-st and i-th breakdown for $i \geqslant 2$. For all $i \geqslant 1$, since $X_i \sim U(0,2)$, we have that

$\mathbb{E}[X_i] = 1$. The expected age of an air fryer when being replaced is then given by (in years)

$$\mathbb{E}\left[\sum_{i=1}^{K} X_i\right] = \sum_{i=1}^{K} \mathbb{E}[X_i] = K \cdot 1 = K.$$

(iii) What is the expected cost per year for Greta's restaurant using an air fryer? For which value of K is this cost the lowest?

Solution: Using renewal reward theory, we know that

$$\mathbb{E}[\text{cost per time unit}] = \frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{length of a cycle}]}.$$

Let a new cycle begin as soon as a new air fryer is bought. The expected length of a cycle is then K, as seen in (i). The expected cost per cycle is

$$75000 + \sum_{i=1}^{K-1} 600 \cdot i = 75000 + 600\frac{K(K-1)}{2} = 75000 + 300K(K-1).$$

The expected cost per year is then

$$\frac{75000 + 300K(K-1)}{K} = \frac{75000}{K} + 300(K-1).$$

This expression attains its minimum when $\frac{d}{dK}\left(\frac{75000}{K} + 300(K-1)\right) = 0$ i.e., when $\frac{75000}{K^2} = 300$, i.e., when $K = \sqrt{\frac{75000}{300}} = \sqrt{250} \approx 15.8$. Since K only takes discrete values, the expected cost per year is the lowest for $K = 16$. Indeed, for $K = 16$ it is 9187.5 SEK, while for $K = 15$ it is 9200 SEK.

Exercise 3: Queueing theory

Right next to her restaurant, Greta also runs a vegan ice cream shop where she works serving customers. Assume that customers arrive at a Poisson rate λ independently of each other and they order one ice cream each as soon as it's their turn to be served. Assume also that the time it takes Greta to prepare an ice cream is exponentially distributed with mean $1/\mu$, independently of everything else.

(i) Specify what type of queueing model best describes the ice-cream shop activity and provide the relation that λ and μ must satisfy in order for the number of customers not to grow beyond all bounds. What is the asymptotic distribution of the number of customers in the shop?

Solution: The ice-cream shop activity can be described by an $M/M/1$ queueing model in which customers arrive according to a Poisson Process with rate λ and their service time S is exponentially distributed with mean $1/\mu$. Customers are served according to a "first come

first served" rule. The necessary condition in order for the number of customers not to grow beyong all bounds is that $\lambda < \mu$.
Solving the balance equations for an $M/M/1$ queueing model, we get that the asymptotic distribution of the number of customers in the shop is given by $P_0 = 1 - \frac{\lambda}{\mu}$ and, for $n \geqslant 1$, $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$.

(ii) What is the average time a newly arrived customer has to wait in the queue before Greta takes his/her order?

Solution: For an $M/M/1$ queueing model, the average number of customers in the shop is $L = \frac{\lambda}{\mu-\lambda}$ and the average time a customer spends in the shop is $W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$. Then the average time a customer has to wait in the queue is given by

$$W_Q = W - \mathbb{E}[S] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

Assume now that, as soon as there are no more customers to be served, Greta leaves the shop and goes to the restaurant to help in the kitchen. She comes back to check on the ice cream shop after an exponentially distributed time with mean $1/\gamma$. If there are still no customers, she returns to the restaurant, again for an exponentially distributed time with mean $1/\gamma$, independently of anything else. If there are customers in the queue when she comes back to the ice-cream shop, then she starts serving until the shop is empty again.

(iii) Describe the number of customers and the location of Greta as a continuous-time Markov chain and derive its balance equations.

Solution: Consider the state space $\{(S,1),(S,2),\dots,(R,0),(R,1),(R,2),\dots\}$, where we are in state (S,k) if Greta is at the shop and there are $k \geqslant 1$ customers in the shop, while we are in state (N,k) if Greta is at the restaurant and there are $k \geqslant 0$ customers in the shop. Denote by $P_{(S,1)}, P_{(S,2)}, \dots, P_{(R,0)}, P_{(R,1)}, P_{(R,2)}, \dots$ the probabilities of being in each of the states, respectively. The transition rates to go from one state to another are the followings: from (S,k) to $(S,k+1)$ the rate is λ, for $k \geqslant 1$; from (R,k) to $(R,k+1)$ the rate is also λ, for $k \geqslant 0$; from (S,k) to $(S,k-1)$ the rate is μ, for $k \geqslant 2$; from $(S,1)$ to $(R,0)$ the rate is also μ; from (R,k) to (S,k) the rate is γ, for $k \geqslant 1$. Hence, we can write the balance equations:

$$\begin{aligned}
\lambda P_{(R,0)} &= \mu P_{(S,1)} \\
(\lambda+\mu) P_{(S,1)} &= \mu P_{(S,2)} + \gamma P_{(R,1)} \\
(\lambda+\gamma) P_{(R,k)} &= \lambda P_{(R,k-1)}, \qquad \text{for } k \geqslant 1 \\
(\lambda+\mu) P_{(S,k)} &= \lambda P_{(S,k-1)} + \mu P_{(S,k+1)} + \gamma P_{(R,k)}, \qquad \text{for } k \geqslant 2.
\end{aligned}$$

(iv) Bonus (2 points): Show that the long-run proportion of time that there are $k \geqslant 1$ customers in the ice-cream shop while Greta is at the ice-cream shop is

$$\frac{\gamma(\mu-\lambda)}{\mu(\mu-\lambda-\gamma)}\left(\left(\frac{\lambda}{\lambda+\gamma}\right)^k - \left(\frac{\lambda}{\mu}\right)^k\right),$$

and the long-run proportion of time that there are $k \geqslant 0$ customers in the ice-cream shop while Greta is at the restaurant is

$$\frac{\gamma(\mu-\lambda)}{\mu(\lambda+\gamma)}\left(\frac{\lambda}{\lambda+\gamma}\right)^k.$$

Solution: Fill in the suggested expressions for $P_{(S,k)}$ and $P_{(R,k)}$ and verify the balance equations.

Exercise 4: Simulation

Consider the Poisson process $\{N(t), t \geqslant 0\}$ with rate λ describing the arrivals of customers at Greta's vegan ice-cream shop. Let $U_1, U_2, \dots$ be i.d.d. random variables uniformly distributed on the interval $[0,1]$ and assume that we can easily simulate them.

(i) Explain in detail why we can simulate the time of the first customer arriving at the ice cream shop by simulating only U_1.

Solution: The time of the first customer arriving at the ice-cream shop is $X \sim \text{Exp}(\lambda)$. Note that

$$\mathbb{P}(X \leqslant x) = 1 - e^{\lambda x} = \mathbb{P}(U_1 \leqslant 1 - e^{-\lambda x}) = \mathbb{P}(\log(1-U_1) \geqslant -\lambda x) = \mathbb{P}(-\frac{\log(1-U_1)}{\lambda} \leqslant x),$$

hence X is distributed as $-\frac{\log(1-U_1)}{\lambda}$, which has the same distribution of $-\frac{\log(U_1)}{\lambda}$. Using the inverse transformation method, we can then simulate $U_1 \sim U(0,1)$ and set $X = -\frac{\log(U_1)}{\lambda}$.

(ii) Recall that if $X \sim \Gamma(n, \lambda)$, then its density $f_X(x) = \frac{\lambda^n}{(n-1)!}x^{n-1}e^{-\lambda x}$. Let Y be Gamma distributed with density $f_Y(x) = 4\lambda^3 x^2 e^{-2\lambda x}$. Show that Y can be simulated by simulating only U_1, U_2, U_3 and setting $Y = -\frac{1}{2\lambda}\left(\log(U_1) + \log(U_2) + \log(U_3)\right)$.

Solution: Note that $Y \sim \Gamma(3, 2\lambda)$ is distributed as the sum of three i.i.d. exponential random variables with paramenter 2λ. The result follows then from (i).

(iii) Show how we can simulate the time of the k-th customer arriving at the ice-cream shop using the rejection method with $g(x) = \frac{\lambda}{k}e^{-\frac{\lambda}{k}x}$ as trial density and compute the average number of iteration in the most efficient case.

Solution: The arrival time of the k-th customer S_k has a Gamma distribution with parameters k and λ, so its density is $f(x) = \frac{\lambda^k}{(k-1)!}x^{k-1}e^{-\lambda x}$. We have that

$$\frac{f(x)}{g(x)} = \frac{k\lambda^{k-1}x^{k-1}}{(k-1)!}e^{-\lambda\left(1-\frac{1}{k}\right)x}$$

and its derivative

$$\frac{d}{dx}\frac{f(x)}{g(x)} = (k-\lambda x)\frac{\lambda^{k-1}x^{k-2}}{(k-2)!}e^{-\lambda\left(1-\frac{1}{k}\right)x}$$

is 0 when $x = 0$ or $x = \frac{k}{\lambda}$. We get that $\frac{f(0)}{g(0)} = 0$ and $\frac{f\left(\frac{k}{\lambda}\right)}{g\left(\frac{k}{\lambda}\right)} = \frac{k^k}{(k-1)!}e^{-(k-1)}$, so that $\frac{f(x)}{g(x)}$ attains its maximum value $c = \frac{k^k}{(k-1)!}e^{-(k-1)}$ in $x = \frac{k}{\lambda}$. We can now use the rejection method: simulate Y with density $g(x)$ (for example using the method in (i)), simulate $U \sim U(0,1)$, and set $S_k = Y$ if $U \leqslant \frac{f(Y)}{cg(Y)}$, otherwise repeat the procedure.
The most efficient case is when $c = \max_{x\geqslant 0}\frac{f(x)}{g(x)}$. The average number of iterations is exactly $c = \frac{k^k}{(k-1)!}e^{-(k-1)}$.

Exercise 5: Brownian motion

After years of collaboration with Heura Foods to have their plant-based meat products in her restaurant, Greta decides to buy a share of Heura's stock whose price changes according to a standard Brownian motion. Assume that Greta buys the stock at some moment when the price is a, $a > 0$, and decides to sell it either as soon as the price reaches the value b, $b > a$, or as soon as a period of time of length s goes by.

(i) What is the probabily of Greta selling the stock at the price b?

Solution: The probability of Greta selling the stock at the price b is the probability that the price reaches the value b within time s from the moment Greta buys the stock. If we look at the (shifted) Brownian motion $\{B(t), t \geqslant 0\}$ starting from the moment Greta buys the stock, then we are interested in the probability that the process reaches the value b before time s, i.e.,

$$\mathbb{P}(T_b \leqslant s \,|\, B(0) = a) = \mathbb{P}(T_{b-a} \leqslant s \,|\, B(0) = 0) = \mathbb{P}(T_{b-a} \leqslant s) = \frac{2}{\sqrt{2\pi}}\int_{\frac{b-a}{\sqrt{s}}}^{\infty} e^{-\frac{y^2}{2}}\,dy.$$

Starting from the moment Greta buys the stock, let T_{bc} be the first time the price reaches the value c, $c < a$, after reaching the value b.

(ii) State the reflection principle and use it to compute $\mathbb{P}(T_{bc} > t)$.

Solution: The reflection principle says that, if $\{X(t), t \geqslant 0\}$ is a standard Brownian motion and T a stopping time, then the process $\{X_T(t), t \geqslant 0\}$ defined as

$$X_T(t) = \begin{cases} X(t), & 0 \leqslant t \leqslant T, \\ 2X(T) - X(t), & t > T, \end{cases}$$

is also a standard Brownian motion.
Since hitting times are stopping times, by the reflection principle, we know that T_{bc} is distributed as T_{2b-c}. Indeed, from T_b the time until reaching c (from b to a and then from a to

c) is distributed as the time until reaching $b + (b - a) + (a - c) = 2b - c$. Hence,

$$\mathbb{P}(T_{bc} > t) = \mathbb{P}(T_{2b-c} > t) = 1 - \mathbb{P}(T_{2b-c} \leqslant t) = 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{2b-c}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}\, dy.$$

(iii) Compute the probability that the price returns to the value a at least once in the time interval $(1, 2)$ after Greta buys the stock.

Solution: We are interested in the probability of the event $\{\exists\, t \in (1,2) : B(t) = a \,|\, B(0) = a\}$, which is the same probability of the event $A = \{\exists\, t \in (1,2) : B(t) = 0 \,|\, B(0) = 0\}$. Conditioning on the value of $B(1)$, since $B(1) \sim \mathcal{N}(0,1)$, we get

$$\begin{aligned}
\mathbb{P}(A) &= \int_{-\infty}^{\infty} \mathbb{P}(A \,|\, B(1) = x) f_{B(1)}(x)\, dx = \int_{-\infty}^{\infty} \mathbb{P}(A \,|B(1) = x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\
&= \int_{-\infty}^{\infty} \mathbb{P}\left(\exists\, t \in (0,1) : B(t) = -x \,|\, B(0) = 0\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \int_{-\infty}^{\infty} \mathbb{P}(T_{-x} < 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\
&= \int_{-\infty}^{\infty} \mathbb{P}(T_x < 1) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \int_{|x|}^{\infty} e^{-\frac{y^2}{2}}\, dy\, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \frac{2}{\pi} \int_0^{\infty} \int_x^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy dx \\
&= \frac{1}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx dy = 2 \left(\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}}\, dx \right)^2 = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}.
\end{aligned}$$