Solutions of the second exam: Stochastic Processes and Simulation II August 10th 2022

Exercise 1: Poisson processes

(i) Give the definition of a homogeneous Poisson process.

Solution: A non-decreasing non-negative integer valued process (counting process) $\{N(t), t \geqslant 0\}$ is said to be a (homogeneous) Poisson process with rate $\lambda > 0$ if the following holds:

(i) $N(0) = 0$;

(ii) $N(t)$ has independent increments;

(iii) $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h)$;

(iv) $\mathbb{P}(N(t+h) - N(t) \geqslant 2) = o(h)$.

A vegan foodtruck is parked every day in front of the university and sells three types of dishes to students: tofu poke bowls (P), falafel wraps (W) and plant-based burgers (B). Let $\{N_P(t), t \geqslant 0\}$ be a Possion process on $(0, \infty)$ with rate λ_P describing the arrivals of students that order a tofu poke bowl, let $\{N_W(t), t \geqslant 0\}$ be a Possion process on $(0, \infty)$ with rate λ_W describing the arrivals of students that order a falafel wrap, and let $\{N_B(t), t \geqslant 0\}$ be a Possion process on $(0, \infty)$ with rate λ_B describing the arrivals of students that order a plant-based burger. Assume that the three Poisson processes are independent of each other. Let $S_1^i, S_2^i, S_3^i, \ldots,$ indicate the arrival times of the Poisson process $\{N_i(t), t \geqslant 0\}$, for $i = P, W, B$, respectively.

(ii) Provide the distribution of the time of the first student that arrives to the foodtruck, i.e., of $\min\{S_1^P, S_1^W, S_1^B\}$, and compute the probability that he/she orders a plant-based burger.

Solution: The arrivals of the three independent homogeneous Poisson processes can be described by a homogenous Poisson process with rate $\lambda_P + \lambda_W + \lambda_B$. The time of the first arrival is then exponentially distributed with mean $\frac{1}{\lambda_P+\lambda_W+\lambda_B}$ and the probability that the first student orders a plant-based burger is

$$\mathbb{P}\left(\min\{S_1^P, S_1^W, S_1^B\} = S_1^B\right) = \frac{\lambda_B}{\lambda_P + \lambda_W + \lambda_B}.$$

(iii) Assume that at the end of the day, at time $T > 0$, we have $N_P(T) = p$, $N_W(T) = w$ and $N_B(T) = b$, with $p, w, b > 0$. Provide the distribution of the time at which the last student of the day arrives, and compute the probability that he/she orders a falafel wrap.

Solution: Using the order statistic property for the combined processes, we obtain that the time of the last arrival in the interval $(0, T)$ is distributed as the largest of $p+w+b$ independent uniform random variables in $(0,T)$. Denote this time by X and note that, for $x \in (0,T)$, we have

$$\mathbb{P}(X \leqslant x) = \mathbb{P}\left(\text{all } p+w+b \text{ arrivals occur before time } x\right) = \left(\frac{x}{T}\right)^{p+w+b}.$$

Consider $p+w+b$ arrivals independently and uniformly distributed on $(0,T)$. Choose w of them uniformly at random (without replacement) and label them as W. The probability that the last arrival is labelled as W is $\frac{w}{p+w+b}$, and this equals the probability the the last student of the day orders a falafel wrap.

Exercise 2: Renewal theory

(i) Give the definition of a continuous-time Markov process/chain and explain the difference from semi-Markov processes.

Solution: A stochastic process $\{X(t), t \geqslant 0\}$ is a continuous-time Markov chain if for all $s,t \geqslant 0$ and nonnegative integers $i, j, x(u), 0 \leqslant u < s$,

$$\mathbb{P}\left(X(t+s) = j \,|\, X(s) = i, X(u) = x(u), 0 \leqslant u < s\right) = \mathbb{P}\left(X(t+s) = j \,|\, X(s) = i\right).$$

In other words, the process has the Markovian property that the conditional distribution of the future given the present and the past depends only on the present and its independent of the past. Alternatively, a stochastic process $\{X(t), t \geqslant 0\}$ is a continuous-time Markov chain on the state space $\mathcal{S}$ if the process moves from state to state in accordance with a discrete-time Markov chain, but the amount of time it spends in each state, before jumping to the next state, is exponentially distributed.
A semi-Markov process is a process on $\mathcal{S}$ that evolves as a continuous-time Markov chain, with the difference that, for all $i \in \mathcal{S}$, the amount of time it spends in state i before jumping into a different state is a random variable (not exponential) with mean μ_i.

Assume that the owner of the foodtruck has collected data on the orders received during peak hours. He can only prepare one dish at the time and, during these hours, he is always busy preparing some dishes, i.e., as soon as he finishes with one dish and serves it to the student, he immediately takes the next order and starts preparing the following dish. The data collected shows that: if a student orders a tofu poke bowl (P), the following student orders either a falafel wrap or a plant-based burger with equal probability; if a student orders a falafel wrap (W), the following student will order a plant-based burger with probability 2/3 and a tofu poke bowl with probability 1/3; if a student orders a plant-based burger (B), the following students is equally likely to order any of the three dishes. Assume that the time it takes to prepare dish i is on average μ_i, for $i = P, W, B$.

(ii) What proportion of time P_i the owner spends preparing each dish $i = P, W, B$ during peak hours?

Solution: Let π_i, $i = P, W, B$ be the stationary distribution of the embedded discrete-time Markov chain describing the jump between states. We then have that

$$\begin{aligned} \pi_P + \pi_W + \pi_B &= 1, \\ \pi_P &= \frac{1}{3}\pi_W + \frac{1}{3}\pi_B, \\ \pi_W &= \frac{1}{2}\pi_P + \frac{1}{3}\pi_B, \\ \pi_B &= \frac{1}{2}\pi_P + \frac{2}{3}\pi_W + \frac{1}{3}\pi_B, \end{aligned} \tag{0.1}$$

which gives the solution $\pi_P = \frac{1}{4} = \frac{8}{32}$, $\pi_W = \frac{9}{32}$ and $\pi_B = \frac{15}{32}$. Hence the proportion of time the owner spends preparing each dish during peak hours is

$$\begin{aligned} P_P &= \frac{8\mu_P}{8\mu_P + 9\mu_W + 15\mu_B}, \\ P_W &= \frac{9\mu_W}{8\mu_P + 9\mu_W + 15\mu_B}, \\ P_B &= \frac{15\mu_B}{8\mu_P + 9\mu_W + 15\mu_B}. \end{aligned} \tag{0.2}$$

(iii) Give a sufficient condition for the P_i, $i = P, W, B$ to represent the limiting probabilities that a tofu boke bowl, a falalef wrap or a plant-based burger are being prepared during peak hours.

Solution: They represent the limiting probabilities if the distributions of the amount of time it takes to prepare each dish are continuous.

Exercise 3: Queueing theory

Due to the high success of the vegan foodtruck business, the following year the owner decides to hire an assistant that can help when there are lots of students in the queue. Assume that students arrive now at a Poisson rate λ independently of each other and of what dish they will order. Assume also that the time it takes to prepare each dish is exponentially distributed with mean $1/\mu$, both for the owner and the assistant independently. Initially, the owner prepares the dishes alone, while the assistant takes care of other tasks. However, as soon as there are 5 or more students at the foodtruck, the assistant starts preparing the dishes as well. When the assistant finishes with an order, if the number of students at the foodtruck is less than 5, then he stops serving the students and goes back to his other tasks until the number of students at the foodtruck is again 5 or more.

(i) What is the relation that λ and μ must satisfy in order for the number of students not to grow beyond all bounds?

Solution: The necessary condition in order for the number of students not to grow beyond all bounds is that $\lambda < 2\mu$.

(ii) Provide the balance equations describing the above queueing system.

Solution: Let π_i be the asymptotic probability that there are i students at the foodtruck. The balance equations are

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 \\ (\lambda+\mu)\pi_i &= \lambda\pi_{i-1} + \mu\pi_{i+1}, \qquad 1 \leqslant i \leqslant 3, \\ (\lambda+\mu)\pi_i &= \lambda\pi_{i-1} + 2\mu\pi_{i+1}, \qquad i = 4, \\ (\lambda+2\mu)\pi_i &= \lambda\pi_{i-1} + 2\mu\pi_{i+1}, \qquad i \geqslant 5. \end{aligned} \tag{0.3}$$

(iii) Provide the asymptotic time a student spends on average in the queue in terms of the model parameters and of μ_F, the average number of students at the foodtruck.

Solution: By Little's formula, $\mu_F = L = \lambda W$, where W is the average time a student spends at the foodtruck. Furthermore, the average time a student spends being served is $\frac{1}{\mu}$. So the asymptotic time a student spends on average in the queue is $\frac{\mu_F}{\lambda} - \frac{1}{\mu}$.

(iv) Bonus (2 points): compute the asymptotic probability π_0 that there is no student at the foodtruck when $\lambda = 2$ and $\mu = 4$.

Solution: The balance equations become

$$\begin{aligned} \pi_0 &= 2\pi_1 \\ 3\pi_i &= \pi_{i-1} + 2\pi_{i+1}, \qquad 1 \leqslant i \leqslant 3, \\ 3\pi_i &= \pi_{i-1} + 4\pi_{i+1}, \qquad i = 4, \\ 5\pi_i &= \pi_{i-1} + 4\pi_{i+1}, \qquad i \geqslant 5. \end{aligned}$$

We have that $\pi_i = \left(\frac{1}{2}\right)^i \pi_0$ for $i = 0, 1, 2, 3, 4$, while $\pi_i = \left(\frac{1}{2}\right)^4 \left(\frac{1}{4}\right)^{i-4} \pi_0$ for $i \geqslant 5$. Hence, we have

$$\begin{aligned} 1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \left(\sum_{i=0}^{3} \left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^4 \sum_{i=4}^{\infty} \left(\frac{1}{4}\right)^{i-4} \right) = \pi_0 \left(\frac{1 - \left(\frac{1}{2}\right)^4}{\frac{1}{2}} + \frac{\left(\frac{1}{2}\right)^4}{1 - \frac{1}{4}} \right) \\ = \pi_0 \left(\frac{15}{8} + \frac{1}{12} \right) = \pi_0 \frac{47}{24}, \end{aligned}$$

which implies that $\pi_0 = \frac{24}{47}$.

Exercise 4: Simulation

Assume now that students arrive at the vegan foodtruck according to a nonhomogeneous Poisson process with intensity function $\lambda(t), t \geqslant 0$.

(i) If a student arrives at time x, compute the density function $f_x(t)$ of the time at which the next student arrives.

Solution: We have that

$$\begin{aligned} F_x(t) &= \mathbb{P}\left(\text{next arrival in } (x, x+t) \,|\, \text{arrival at } x\right) \\ &= 1 - \mathbb{P}\left(\text{no arrivals in } (x, x+t) \,|\, \text{arrival at } x\right) \\ &= 1 - \mathbb{P}\left(\text{no arrivals in } (x, x+t)\right) \\ &= 1 - e^{-\int_0^t \lambda(x+y)\, dy}. \end{aligned}$$

Hence by differentiating we get

$$f_x(t) = \lambda(x+t) e^{-\int_0^t \lambda(x+y)\, dy}.$$

(ii) For $T > 0$, provide a method for simulating the student arrival process when $\lambda(t) = 1 + \frac{1}{t+2}$ in $(0, T)$, by simulating only standard uniform random variables.

Solution: One way of simulating this process is to first simulate a homogeneous Poisson process with rate 1 and then simulate a nonhomogeneous Poisson process with intensity function $\frac{1}{t+2}$. The superposition of these two processes in $(0, T)$ is the desired Poisson process.
We can simulate the homogeneous Poisson process with rate 1 by simulating the sequence of exponentially distributed arrival times $X_1, X_2, \ldots$. We can simulate standard uniform random variables $U_1, U_2, \ldots$, and use the inverse transformation method setting $X_i = -\log(U_i)$.
In a similar way, we can simulate the inhomogeneous Poisson process with intensity function $\frac{1}{t+2}$ by simulating the event times in the order in which they occur. Using point (i), we can simulate the time of the first event Y_1 from the distribution F_0. If $Y_1 = y_1$, then we simulate Y_2 by adding y_1 to a value simulated from F_{y_1}. If $Y_2 = y_2$, then we simulate Y_3 by adding y_2 to a value simulated from F_{y_2}, and so on. In particular,

$$F_x(t) = 1 - e^{-\int_0^t \lambda(x+y)\, dy} = 1 - e^{-\int_0^t \frac{1}{x+y+2}\, dy} = 1 - e^{-\log\left(\frac{x+t+2}{x+2}\right)} = 1 - \frac{x+2}{x+t+2} = \frac{t}{x+t+2}.$$

Hence, using the inverse transformation method with

$$F_x^{-1}(u) = (x+2)\frac{u}{1-u},$$

we can simulate the successive event times $Y_1, Y_2, \ldots$ by simulating standard uniform random variables U_1', U_2', $\ldots$ and then setting

$$\begin{aligned} Y_1 &= F_0^{-1}(U_1') = \frac{2U_1'}{1-U_1'}, \\ Y_2 &= Y_1 + F_{Y_1}^{-1}(U_2') = Y_1 + (Y_1+2)\frac{U_2'}{1-U_2'}, \\ Y_i &= Y_{i-1} + F_{Y_{i-1}}^{-1}(U_i') = Y_{i-1} + (Y_{i-1}+2)\frac{U_i'}{1-U_i'}, \qquad i \geqslant 2. \end{aligned}$$

(iii) How can we simulate the arrival time of the first student?

Solution: To simulate the arrival time of the first student we can simply simulate X_1 and Y_1 as above and take $\min(X_1, Y_1)$, which gives the first event in the superposition of the two Poisson processes.

Exercise 5: Brownian motion

(i) Give the definition of a standard Brownian motion and explain how any Brownian motion can be rescaled to a standard one. Also, give the definition of a standard Brownian brigde and specify its mean and variance at time $t > 0$.

Solution: A stochastic process $\{B(t), t \geqslant 0\}$ is said to be a standard Brownian motion if:

(i) $B(0) = 0$;

(ii) $\{B(t), t \geqslant 0\}$ has independent and stationary increments;

(iii) for every $t > 0$, $B(t) \sim \mathcal{N}(0, t)$.

A general Brownian motion $\{X(t), t \geqslant 0\}$ is defined in the same way, but with (iii) replaced by $X(t) \sim \mathcal{N}(0, \sigma^2 t)$, hence any Brownian motion can be converted to the standard one by letting $B(t) = X(t)/\sigma$.
A standard Brownian bridge is a stochastic process on $[0, 1]$ whose probability distribution is the probability distribution of a standard Brownian motion conditional on $B(1) = 0$, i.e., the process $\{B(t), t \in [0, 1] \,|\, B(1) = 0\}$. We have that $B(t) \,|\, B(1) = 0 \sim \mathcal{N}(0, t(1-t))$, hence it has mean 0 and variance $t(1-t)$.

After months of selling the plant-based Beyond Burger at his foodtruck, the owner decides to buy a share of the stock of Beyond Meat, the company that produces it. Assume that the stock price changes according to a standard Brownian motion. Moreover, assume that he buys the stock at a certain time when the price is x, $x > 0$, and that he decides to sell it when one of the two following events occurs: either when the price reaches the value y, $y > x$, or when a period of time of length t has passed.

(ii) What is the probabily that the owner of the foodtruck does not sell the stock at the price y?

Solution: The probability of the owner of the foodtruck not selling the stock at the price y is the probability that the price does not reach the value y within time t from the moment he buys the stock. If we look at the (shifted) Brownian motion $\{B(s), s \geqslant 0\}$ starting from the moment he buys the stock, then we are interested in the probability that the process does not reach the value y before time t, i.e.,

$$\begin{aligned}\mathbb{P}(T_y > t \,|\, B(0) = x) &= 1 - \mathbb{P}(T_y \leqslant t \,|\, B(0) = x) \\ &= 1 - \mathbb{P}(T_{y-x} \leqslant t \,|\, B(0) = 0) \\ &= 1 - \mathbb{P}(T_{y-x} \leqslant t) \\ &= 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{y-x}{\sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}} \, ds.\end{aligned}$$

(iii) Starting from the moment he buys the stock, let T_{yz} be the first time the price reaches the value z, $z < x$, after it has reached the value y, i.e., the time it takes for the price to go from x to z passing by y. Compute $\mathbb{P}(T_{yz} \leqslant s)$ using the reflection principle.

Solution: Since hitting times are stopping times, by the reflection principle, we know that T_{yz} is distributed as T_{2y-z}. Indeed, from T_y the time until reaching z (from y to x and then from x to z) is distributed as the time until reaching $y + (y - x) + (x - z) = 2y - z$. Hence,

$$\mathbb{P}(T_{yz} \leqslant s) = \mathbb{P}(T_{2y-z} \leqslant s) = \frac{2}{\sqrt{2\pi}} \int_{\frac{2y-z}{\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}} \, du.$$