Solutions of the second exam:
Stochastic Processes and Simulation II
August 15th 2023

Exercise 1: Poisson processes

(i) Give the definition of a compound Poisson process $\{X(t), t \geqslant 0\}$ with parameter λ and distribution F. Write an expression for its mean $\mathbb{E}[X(t)]$ and its variance $\mathrm{Var}(X(t))$.

Solution: A stochastic process $\{X(t), t \geqslant 0\}$ is said to be a compound Poisson process with parameter λ and distribution F if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \qquad t \geqslant 0,$$

where $\{N(t), t \geqslant 0\}$ is a Poisson process with rate λ, and $\{Y_i, i \geqslant 1\}$ is a family of independent random variables with distribution F that is also independent of $\{N(t), t \geqslant 0\}$. We have that $\mathbb{E}[X(t)] = \lambda t\, \mathbb{E}[Y_1]$ and $\mathrm{Var}(X(t)) = \lambda t\, \mathbb{E}[Y_1^2]$.

One of the many consequences of the climate crisis is the increasing number of wildfires occurring over the summer in Southern European countries such as Portugal, Spain, Italy and Greece. As global temperatures rise, the hot and dry conditions help these fires catch and spread, and it is expected that their size, frequency and severity will increase in the coming years, with extreme wildfires devastating communities and ecosystems.
Assume that in Spain wildfires occur over the summer according to a Poisson process with a rate of $\lambda_S = 2$ per week. Suppose also that each time a wildfire occurs, nearby villages are evacuated and people are relocated, and that the number of relocated people N_S has distribution F_S given by

$$P(N_S = 100) = 1/4, \qquad P(N_S = 300) = 1/2, \qquad P(N_S = 500) = 1/4,$$

independently for each wildfire.

(ii) Let $S(t)$ be the total number of relocated people due to wildfires in Spain over the summer. What type of process is $\{S(t), t \geqslant 0\}$? Calculate the expected value and variance in a summer month, i.e., calculate $\mathbb{E}[S(4)]$ and $\mathrm{Var}(S(4))$, where we approximate 1 month $\approx$ 4 weeks.

Solution: The process $\{S(t), t \geqslant 0\}$ is a compound Poisson process with parameter $\lambda_S = 2$ and distribution F_S. We have that $\mathbb{E}[N_S] = 300$ and $\mathbb{E}[N_S^2] = 100^2 \frac{1}{4} + 300^2 \frac{1}{2} + 500^2 \frac{1}{4} = 110000$. Hence, using point (i) above, $\mathbb{E}[S(4)] = 2 \cdot 4 \cdot 300 = 2400$ and $\mathrm{Var}(S(4)) = 2 \cdot 4 \cdot 110000 = 880000$.

(iii) Assume that in Portugal wildfires occur over the summer according to a Poisson process with rate λ_P per week and that the number of relocated people N_P has distribution F_P, independently for each wildfire. Assume also that wildfires in Portugal occur independently from wildfires in Spain. Let $P(t)$ be the total number of relocated people due to wildfires in Portugal over the summer. What type of process is $\{S(t)+P(t), t \geqslant 0\}$? What is its parameter and its distribution, in terms of $\lambda_S, \lambda_P, F_S, F_P$?

Solution: Note that the process $\{P(t), t \geqslant 0\}$ is a compound Poisson process with parameter λ_P and distribution F_P. The sum $\{S(t)+P(t), t \geqslant 0\}$ of the two independent compound Poisson processes is a compound Poisson process with paramenter $\lambda_S + \lambda_P$ and distribution $F = \frac{\lambda_S}{\lambda_S+\lambda_P} F_S + \frac{\lambda_P}{\lambda_S+\lambda_P} F_P$.

Exercise 2: Renewal theory

(i) Let $\{N(t), t \geqslant 0\}$ be a renewal process with i.i.d. interarrival times X_n, $n \geqslant 1$. Let $\mu = \mathbb{E}[X_n]$ and let $m(t) = \mathbb{E}[N(t)]$ be the renewal function. The elementary renewal theorem states that $\frac{m(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$. Prove the lower bound, i.e., that $\lim_{t\to\infty} \frac{m(t)}{t} \geqslant \frac{1}{\mu}$.

Solution: Consider the time $S_{N(t)+1}$ of the first renewal after t. Note that $N(t)+1$ is a stopping time, since $N(t)+1 = n \Leftrightarrow N(t) = n-1 \Leftrightarrow X_1 + \cdots X_{n-1} \leqslant t, X_1 + \cdots X_n > t$. Then, $\mathbb{E}[S_{N(t)+1}] = \mathbb{E}[X_1 + \cdots + X_{N(t)+1}] = \mathbb{E}[X]\mathbb{E}[N(t)+1] = \mu(m(t)+1)$. Define the excess time as $Y(t) = S_{N(t)+1} - t$. Taking expectations and rearranging the terms, we get $\mu(m(t)+1) = t + \mathbb{E}[Y(t)]$, which implies

$$\frac{m(t)}{t} = \frac{1}{\mu} + \frac{\mathbb{E}[Y(t)]}{t\mu} - \frac{1}{t} \geqslant \frac{1}{\mu} - \frac{1}{t} \to \frac{1}{\mu}.$$

(ii) Let $\{N(t), t \geqslant 0\}$ be as above and consider a renewal reward process $\{R(t) = \sum_{n=1}^{N(t)} R_n, t \geqslant 0\}$ where R_n, $n \geqslant 1$ are i.i.d. and represent the rewards earned each time a renewal occurs. State and prove the reward theorem for $\frac{R(t)}{t}$.

Solution: The renewal reward theorem says that, if the expected reward in a cycle $\mathbb{E}[R]$ and the expected cycle length $\mathbb{E}[X]$ are finite, then $\frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$ almost surely as $t \to \infty$. To prove it, write $\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t}$. Then, by the strong low of large numbers, $\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \to \mathbb{E}[R]$ a.s., and the rate of the renewal process $\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[X]}$ a.s..

Suppose that each time a wildfire occurs a relocation center is installed nearby in order to help people evacuate and move to the nearest city. Assume that people arrive at the relocation center according to a Poisson process with a rate of 10 per hour, and that, as soon as there are 30 people, a bus picks them all up and departs. The bus service association incurs a cost at a rate of $4k$ euros per unit time whenever there are k people waiting at the relocation center.

(iii) Describe the problem in terms of a renewal reward process. What is the expected length of a cycle? What is the expected cost in a cycle? Use the renewal reward theorem to compute the long-run average cost.

Solution: We can model the problem as a renewal reward process where the renewals/cycles are described by the arrivals of buses and the reward is a cost which is paid gradually through a cycle at a rate of $4n$ per unit time. The expected time between two arrivals is $\frac{1}{10}$, hence the length of a cycle is $\mathbb{E}[X] = \frac{30}{10} = 3$ hours. The expected cost up to the arrival of the first person is 0. The expected cost between the first and the second arrival is $\frac{4}{10}$, and in general the expected cost between the k-th and the $(k+1)$-st arrival is $\frac{4k}{10}$. Hence we have that $\mathbb{E}[R] = \sum_{k=0}^{29} \frac{4k}{10} = \frac{4}{10}\frac{29 \cdot 30}{2} = 174$ euros. By the renewal reward theorem, the long-run average cost is given by $\frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{174}{3} = 58$ euros.

Exercise 3: Queueing theory

Assume that when people arrive at the relocation center they first have to go to a registration room to identify themselves. Assume they arrive at a Poisson rate λ independently of each other and they form a single queue for two registration desks: one is always active, while the other one is active if and only if there are at least 3 people in the registration room. Assume that the time it takes at each desk is exponentially distributed with mean $1/\mu$, independently of everything else.

(i) Specify what type of queueing model best describes the registration process. What condition must λ and μ satisfy in order for the number of people not to grow beyond all bounds?

Solution: The registration process can be described as a variation of an $M/M/2$ queueing model, where the second server is operating only when there are at least 3 people in the system. The arrival times are i.i.d. Exp(λ) and the service times are i.i.d. Exp(μ). The necessary condition in order for the number of people not to grow beyond all bounds is that $\lambda < 2\mu$.

(ii) Write down the balance equations for the above queueing system.

Solution: Let π_n be the asymptotic probability that there are n people in the registration room. The balance equations are

$$\begin{aligned}
\lambda\pi_0 &= \mu\pi_1 \\
(\lambda+\mu)\pi_1 &= \lambda\pi_0 + \mu\pi_2, \\
(\lambda+\mu)\pi_2 &= \lambda\pi_1 + 2\mu\pi_3, \\
(\lambda+2\mu)\pi_n &= \lambda\pi_{n-1} + 2\mu\pi_{n+1}, \qquad n \geqslant 3.
\end{aligned}$$

(iii) Let μ_R be the average number of people in the registration room. What is the asymptotic average time that a person spends in the registration room? What is the asymptotic average time that a person spends in queue in the registration room?

Solution: We know that $\mu_R = L = \lambda W$, where W is the average time a person spends in the registration room. Hence, $W = \frac{\mu_R}{\lambda}$. Moreover, the average time a person spends in service is $\frac{1}{\mu}$, so the average time a person spends in the queue is $W_Q = W - \frac{1}{\mu} = \frac{\mu_R}{\lambda} - \frac{1}{\mu}$.

Exercise 4: Simulation

(i) Assume that wildfires in Spain occur over the summer according to a Poisson process $\{W_S(t), t \geqslant 0\}$ with rate $\lambda_S = 2$. Describe how we can simulate this process by simulating only standard uniform random variables.

Solution: We can simulate the standard Poisson process with rate 2 by simulating the sequence of exponentially distributed arrival times $X_1, X_2, \ldots$. We can simulate standard uniform random variables $U_1, U_2, \ldots$, and use the inverse transformation method setting $X_i = -\frac{\log(U_i)}{2}$.

Assume that in Italy wildfires occur over the summer according to a nonhomogeneous Poisson process $\{W_I(t), t \geqslant 0\}$ with intensity function $\lambda_I(t) = \frac{1}{t+3}$.

(ii) In Italy, given that a wildfire occurs at time x, compute the density function $f_x(t)$ of the time at which the next wildfire occurs.

Solution: We have that

$$\begin{aligned} F_x(t) &= \mathbb{P}\left(\text{next arrival in } (x, x+t) \,|\, \text{arrival at } x\right) \\ &= 1 - \mathbb{P}\left(\text{no arrivals in } (x, x+t) \,|\, \text{arrival at } x\right) \\ &= 1 - \mathbb{P}\left(\text{no arrivals in } (x, x+t)\right) \\ &= 1 - e^{-\int_0^t \lambda_I(x+y)\,dy} \\ &= 1 - e^{-\int_0^t \frac{1}{x+y+3}\,dy} \\ &= 1 - e^{-\log\left(\frac{x+t+3}{x+3}\right)} \\ &= 1 - \frac{x+3}{x+t+3} \\ &= \frac{t}{x+t+3}. \end{aligned}$$

Hence by differentiating we get $f_x(t) = \frac{x+3}{(x+t+3)^2}$.

(iii) Describe how we can simulate the occurrence of wildfires over the summer in Italy by simulating only standard uniform random variables.

Solution: We can simulate the nonhomogeneous Poisson process with intensity function $\frac{1}{t+3}$ by simulating the event times in the order in which they occur. Using point (ii), we can simulate the time of the first event Y_1 from the distribution F_0. If $Y_1 = y_1$, then we simulate Y_2 by adding y_1 to a value simulated from F_{y_1}. If $Y_2 = y_2$, then we simulate Y_3 by adding

y_2 to a value simulated from F_{y_2}, and so on. In particular, $F_x(t) = \frac{t}{x+t+3}$, hence using the inverse transformation method with

$$F_x^{-1}(u) = (x+3)\frac{u}{1-u},$$

we can simulate the successive event times $Y_1, Y_2, \ldots$ by simulating standard uniform random variables U_1', U_2', ... and then setting

$$Y_1 = F_0^{-1}(U_1') = \frac{3U_1'}{1-U_1'},$$
$$Y_2 = Y_1 + F_{Y_1}^{-1}(U_2') = Y_1 + (Y_1+3)\frac{U_2'}{1-U_2'},$$
$$Y_i = Y_{i-1} + F_{Y_{i-1}}^{-1}(U_i') = Y_{i-1} + (Y_{i-1}+3)\frac{U_i'}{1-U_i'}, \qquad i \geqslant 3.$$

(iv) Bonus (2 points): how can we simulate the time of the first wildfire that occurs either in Spain or Italy?

Solution: We can simulate the time X_1 of the first wildfire in Spain and the time Y_1 of the first wildfire in Italy, following the methods described in points (i) and (iii) respectively. Then, $\min(X_1, Y_1)$ gives the time of the first event in the superposition of the two Poisson processes.

Exercise 5: Brownian motion

(i) Give the definition of a Brownian motion $\{X(t), t \geqslant 0\}$ with drift coefficient μ and variance parameter σ^2. Show that $\frac{X(t)}{t} \to \mu$ almost surely as $t \to \infty$.

Solution: The process $\{X(t), t \geqslant 0\}$ is a Brownian motion with drift coefficient μ and variance parameter σ^2 if $X(0) = 0$, $\{X(t), t \geqslant 0\}$ has stationary and independent increments, and $X(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$. Alternatively, if $\{B(t), t \geqslant 0\}$ is a standard Brownian motion and $\mu \in \mathbb{R}$, then the process $\{X(t) = \sigma B(t) + \mu t, t \geqslant 0\}$ is a Brownian motion with drift μ and variance parameter σ^2. Since $\frac{\sigma B(t)}{t} \sim \mathcal{N}(0, \sigma^2/t)$, and its variance converges to 0, we have that $\frac{\sigma B(t)}{t} \to 0$ in probability and almost surely. Hence we have that $\frac{X(t)}{t} = \frac{\sigma B(t)}{t} + \mu \to \mu$ almost surely as $t \to \infty$.

Wildfires in Southern Europe are expected to increase in the coming years, due to global warming and temperature rise. Indeed, we know that in 2023 the global temperature has increased by 1.1 degrees Celsius compared with pre-industrial levels, and it is predicted to keep increasing in the next decades.

(ii) Assume that the global temperature evolves according to a Brownian motion with drift and that exactly in 2035 it will be 1.5 degrees Celsius higher compared with pre-industrial levels. What is the value of the drift coefficient μ in the unit Celsius/years? In which year it is expected to reach the 2 degrees Celsius threshold?

Solution: The process is expected to increase by 0.4 degrees Celcius in 12 years from now, so the drift coefficient must be $\mu = \frac{0.4}{12} = \frac{1}{30}$. To reach the 2 degrees Celsius threshold, it needs to further increase by 0.5 degrees Celcius after 2035, which is expected to happen in $\frac{0.5}{\mu} = \frac{0.5}{1/30} = 15$ years, hence in 2050.

Assume that we will be able to find solutions to the climate crisis and manage to half the effect of the drift in the year 2035 and to completely cancel it in the year 2041, so that the global temperature will evolve according to a Brownian motion with drift μ until 2035, then with drift $\mu/2$ until 2041, and then without drift.

(iii) If in 2041 the global temperature will be exactly at its mean value and if $\sigma = 1$, what is the probability that it will not reach the 2 degrees Celsius threshold by 2050?

Solution: We know that in 2035 the global temperature will be 1.5 degrees Celcius warmer than pre-industrial times. Moreover, we know that it will evolve according to a Brownian motion with drift $\mu/2 = 1/60$ for the following 6 years until 2041, so it is expected to increase by $6\frac{1}{60} = 0.1$ degrees Celsius and to reach level 1.6 degrees Celsius warmer than pre-industrial times. It will then evolve according to a standard Brownian motion without drift. Note that the probability of not hitting the threshold of 2 degrees Celsius before 2050 is equivalent to the probability that a standard Brownian motion $\{\bar{X}(t), t \geqslant 0\}$ starting at 0 takes longer than 9 years to increase by 0.4 degrees Celsius. Recall that, if we let $T_a = \inf\{t \geqslant 0 : \bar{X}(t) \geqslant a\}$ be the hitting time of barrier a, then $\mathbb{P}(T_a > t) = 1 - \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy$. Hence, we have that

$$\mathbb{P}(T_{0.4} > 9) = 1 - \frac{2}{\sqrt{2\pi}} \int_{0.4/\sqrt{9}}^{\infty} e^{-y^2/2}\, dy = 1 - \frac{2}{\sqrt{2\pi}} \int_{0.4/3}^{\infty} e^{-y^2/2}\, dy.$$