MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Topologi MM7052
7,5 hp
2023-12-19

(1) (a) [1 pt] Show that the collection $\mathcal{B} = \{(a, \infty) \subset \mathbb{R} : a \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$.
(b) [1 pt] What are the closed sets in $\mathcal{T}$?
(c) [1 pt] What are the limit points of the set $(0, 1)$ with respect to $\mathcal{T}$?
(d) [1 pt] What are the path connected components of $\mathbb{R}$ with respect to $\mathcal{T}$?
(e) [1 pt] Determine if any of the sets $\{\frac{1}{n} : n \in \mathbb{N}\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ are compact with respect to $\mathcal{T}$.

(2) Let X be a topological space and define $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$.
(a) [2 pts] Show that Δ is continuous and then that it is an embedding.
(b) [3 pts] Show that $\Delta(X)$ is closed if and only if X is Hausdorff.
(Note that this is a standard result, but you need to prove it and cannot just refer to a text book.)

(3) (a) [3 pt] Consider the cylinder $\mathbb{S}^n \times I$ and define $f : \mathbb{S}^n \times I \to \overline{\mathbb{B}}^{n+1}$ by $(x, t) \mapsto tx$. Show that f is a quotient map for all $n \geq 0$.
(b) [2 pt] Show that the quotient map $\mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $x \sim y$ if $x = \lambda y$ for some non-zero $\lambda \in \mathbb{R}$, is not a covering map for any $n \geq 0$.

(4) [5 pts] Let G be a connected topological group with neutral element e and $f : \mathbb{R} \to G$ a continuous group homomorphism. For every $n \in \mathbb{Z}$ let $\alpha_n : I \to G$ be the path $\alpha_n(t) = f(nt)$. Prove that if $\mathbb{Z} \subset f^{-1}(e)$, the map $\mathbb{Z} \to \pi_1(G, e)$ defined by $n \mapsto [\alpha_n]$, is a group homomorphism.

(5) [5 pts] Compute the fundamental group of the complement of n points on $\mathbb{S}^2$ for all $n \geq 1$.

(6) [5 pts] Prove the following theorems:

Theorem 1. *(Homotopy Classification of Loops in $\mathbb{S}^1$) Two loops in $\mathbb{S}^1$ based at the same point are path-homotopic if and only if they have the same winding number.*

Theorem 2. *(Fundamental Group of the Circle) The group $\pi_1(\mathbb{S}^1, 1)$ is an infinite cyclic group generated by the loop $\omega : I \to \mathbb{S}^1$ defined by $\omega(s) = e^{2\pi i s}$.*