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Matematiska institutionen
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Exam / Tentamen
MM7022 Logik II, 7,5 hp
HT 2023
Fri, 2023-12-15

This exam consists of 6 questions, worth a total of 40 points. Not all questions are equally difficult. You may submit your answers in either English or Swedish. Write clearly and justify your answers appropriately.

Good luck! — Lycka till!

1. (4p) Recall that ordinal addition $\alpha + \beta$, multiplication $\alpha \cdot \beta$, and exponentiation α^β are each defined by induction on the second argument β. Which of the following identities are valid for all ordinals α, β, γ? For each, prove or give a counterexample.
   (a) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
   (b) $\alpha^{\beta\cdot\gamma} = (\alpha^\beta)^\gamma$
   (c) $(\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$

2. (6p) For each of the following sets, determine whether its cardinality is equal to $\|\mathbb{R}\|$, or strictly greater, or strictly less.
   (a) $\mathbb{R}^{<\omega}$, the set of all finite sequences of reals;
   (b) $\mathrm{Bij}(\mathbb{R}, \mathbb{R})$, the set of all bijections $\mathbb{R} \to \mathbb{R}$;
   (c) $\mathrm{Mon}(\mathbb{N}, \mathbb{N})$, the set of all monotone functions $(\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$.

3. (9p) Equivalents of the axiom of choice.
   (a) State the three main equivalent forms of the axiom choice: AC itself, Zorn's lemma, and the well-ordering principle (WOP).

   The *principle of cardinal comparability* (PCC) states that for any sets X and Y, either there exists some injection from X to Y, or some injection from Y to X. In class, we proved PCC from WOP.

   (b) Prove PCC directly from Zorn's lemma.
   (c) Prove that PCC implies one of AC, Zorn, or WOP (and so is equivalent to AC). *(Hint: There are many possible approaches here. Hartogs' lemma may be helpful.)*

4. (7p) Let $\mathcal{L}_G$ be the language consisting of a single binary predicate symbol, $\sim$. An $\mathcal{L}_G$-structure (i.e. a set G together with a binary relation $\sim$) is called a *directed graph*, or *digraph*.

   A *cycle of length n* in a digraph $(G, \sim)$ (for $n \geq 1$) is a sequence $x_1, \ldots, x_n \in G$, such that $x_i \sim x_{i+1}$ for each $1 \leq i < n$, and $x_n \sim x_1$. A digraph is *cycle-free* if it contains no cycles.

   (a) Show that the class of cycle-free digraphs is axiomatisable.
   (b) Show that the class of digraphs containing a cycle is not axiomatisable.
   (c) Show that the class of cycle-free digraphs is not finitely axiomatisable.

5. (5p) We can generalise the μ-operator of recursive functions as follows: For any (possibly partial) function $f : \mathbb{N}^{p+1} \rightharpoonup \mathbb{N}$, and any $k \geq 1$, take $\mu^k y.\, [f(\vec{x}, y) = 0]$ to be defined as follows:

   - $\left(\mu^k y.\, [f(\vec{x}, y) = 0]\right) = n$ just if $f(\vec{x}, y)$ is defined for all $y \leq n$, $f(\vec{x}, n) = 0$, and there are precisely k values of y with $0 \leq y < n$ such that $f(\vec{x}, y) = 0$;
   - $\left(\mu^k y.\, [f(\vec{x}, y) = 0]\right)$ is undefined if there is no such n.

   Show that if $f : \mathbb{N}^{p+1} \rightharpoonup \mathbb{N}$ is recursive, then the function $g : \mathbb{N}^{1+p} \rightharpoonup \mathbb{N}$ defined by $g(k, \vec{x}) = \mu^k y.\, [f(\vec{x}, y) = 0]$ is again recursive.

6. (9p) Gödel's first incompleteness theorem tells us that no recursively enumerable theory extending Robinson arithmetic P_0 can be complete and consistent. Show, however, that any three of these four properties can coexist: give theories of arithmetic which are…

   (a) …complete, recursively enumerable, and extending P_0, but not consistent;
   (b) …consistent, recursively enumerable, and extending P_0, but not complete;
   (c) …complete, consistent, and extending P_0, but not recursively enumerable;
   (d) …complete, consistent, and recursively enumerable, but not an extension of P_0.

   In each case, justify why the three claimed properties hold. (Failure of the fourth property then follows by the incompleteness theorem.)

——— End of exam — Slut på provet ———