Solutions to Linear Algebra and Learning from Data October 26, 2023

1. The Hessian matrix is $H = \frac{1}{2}(A + A^\top)$. Using then the Jacobi method to check the positivity of the principal minors of H which shows $a \neq 2$. When $a = 2$, H is positive semi-definite so its smallest eigenvalue is 0.

2. (i) See Solutions to the exam 2022-10-25 1(i).

   (ii) See Solutions to the exam 2022-10-25 2(iv).

   (iii) Since $\|A\|_2$ is the largest singular value of A, which is the largest eigenvalue of A in absolut value if A is symmetrc, the statement follows.

   (iv) $\|uu^\top - vv^\top\|_F^2 = \mathrm{tr}((uu^\top - vv^\top)(uu^\top - vv^\top)^\top) = 2 - 2(u^\top v)^2 = 2\sin^2\alpha$

3. Note that

$$\begin{pmatrix} I & 0 \\ -C^\top S^{-1} & I \end{pmatrix} H \begin{pmatrix} I & -S^{-1}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & -C^\top S^{-1} C \end{pmatrix} =: \hat{H},$$

showing that H and $\hat{H}$ have the same number of positive, negative and zero eigenvalues. Since $S > 0$ and C is invertible the matrix $\hat{H}$ has exactly n positive and n negative eigenvalues and so does H., also proving H is indefinite.

4. (i) Clearly $AM = MC$. Then M is invertible we get the desired result.

   (ii) By insepection, we can choose $u = (-p_0 - 1, -p_1, ..., -p_{n-1})^\top$, $v = e_n$.

   (iii) Note that $Q = C_{e_2}$ is an orthogonal matrix so

$$C = C_{e_2} + uv^\top = Q(\underbrace{I + Q^\top uv^\top}_{R}),$$

where apparently $R := I + Q^\top uv^\top$ is an upper triangular matrix. Thus we have $Q_1 = Q, R_1 = R$. Notice that C has rank one update.

   (iv) Now $C_1 = R_1 Q_1 = Q + (\underbrace{Q^\top u}_{u_1})(\underbrace{v^\top Q}_{v_1^\top})$, which is again a rank one update. By more inspection we see that $v_1 = e_{n-1}$. Furthermore we rewrite $C_1 = Q(I + (Q^\top u_1)e_{n-1}^\top)$. Now the matrix $I + (Q^\top u_1)e_{n-1}^\top$ is an upper Hessenberg matrix we can perform an QR (for example via a Givens rotation G_1). This is a well-know method, the QR algorithm, for eigenvalue computation. Due to the special structure of the matrix C we would have a much cheaper QR procedure.

   (v) It is a straightforward check. Do it carefully.

5.(i-iii) This is a Toeplitz matrix but its inverse is not, see Lecture notes Day 11 at the course page.

(iv)

$$\|T_n^{-1}(\rho)\|_\infty = \frac{1+2|\rho|+\rho^2}{|1-\rho^2|}$$

(v) By the Gershgorin circle theorem the eigenvalues must lie in the union of the discs centered at 1 with the radii

$$|\rho|+|\rho|^2+\cdots+|\rho|^{n-1}, 2|\rho|+|\rho|^2+\cdots+|\rho|^{n-2}, ..., |\rho|^k+2(|\rho|+\cdots+|\rho|^k),$$

where $k = \lfloor n/2 \rfloor$. Hence, an estimate for the spectral radius of $T_n(\rho)$ is the maximum of these numbers, which is bounded above by $|\rho|+|\rho|^2+\cdots+|\rho|^{n-1} = \frac{|\rho|(1-|\rho|^{n-1})}{1-|\rho|}$, if $|\rho|<1$ and by $|\rho|^k+2(|\rho|+\cdots+|\rho|^k) = |\rho|^k + \frac{2|\rho|(1-|\rho|^{k-1})}{1-|\rho|}$, if $|\rho|>1$. Thus an estimate for the spectral radius is $1+\frac{|\rho|(1-|\rho|^{n-1})}{1-|\rho|}$, if $|\rho|<1$ and by $1+|\rho|^k+\frac{2|\rho|(1-|\rho|^{k-1})}{1-|\rho|}$, if $|\rho|>1$ If $|\rho|=1$ then the sepctral radius is 1.