MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Topologi MM7052
7,5 hp
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(1) (a) [1 pt] Show that the collection $\mathcal{B} = \{[a, b) \subset \mathbb{R} : a < b\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$.
(b) [1 pt] What are the limit points of the set $(0, 1)$ with respect to $\mathcal{T}$?
(c) [1 pt] What are the connected components of $\mathbb{R}$ with respect to $\mathcal{T}$?
(d) [1 pt] Show that $\mathbb{R}$ with respect to $\mathcal{T}$ is not compact.
(e) [1 pt] Show that $\mathbb{R}$ with respect to $\mathcal{T}$ is not second countable.

Solution: (a) First note that for all $x \in \mathbb{R}$ holds $x \in [x, x + 1)$, so that $\mathcal{B}$ covers $\mathbb{R}$. Moreover, for any $a < b$ and $a' < b'$ in $\mathbb{R}$, we have $[a, b) \cap [a', b') = [\max(a, a'), \min(b, b'))$. This is an element of $\mathcal{B}$ if it is non-empty. Hence, by Proposition 2.44, the set $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

(b) We claim that the set L of limit points of $(0, 1)$ in $\mathcal{T}$ is equal to $[0, 1)$.
Let $x \geq 1$. Then $[1, x+1)$ is an open set disjoint from $(0, 1)$ that contains x; hence $x \notin L$.
Now let $x \in [0, 1)$ and U a neighborhood of x. Then there is a basis element $B = [a, b) \in \mathcal{B}$ such that $x \in B \subseteq U$. In particular $\max(a, 0) \leq x < \min(b, 1)$. Then, for any $y \in \mathbb{R}$ such that $x < y < \min(b, 1)$, we have that $x \neq y \in [0, 1) \cap B \subseteq [0, 1) \cap U$. Hence $x \in L$.
Lastly, let $x < 0$. Then $[x, 0)$ is an open set disjoint from $(0, 1)$ that contains x; hence $x \notin L$.
Together, this proves the claim.

(c) We claim that every point of $\mathbb{R}$ is its own connected component with respect to $\mathcal{T}$ (i.e. the space is totally disconnected). Assume, to the contrary, that there exists a connected component $C \subseteq \mathbb{R}$ with more than one point, and let $x \neq y$ be two points of C. Furthermore, choose some $t \in \mathbb{R}$ such that $x < t < y$. Note that $A = (-\infty, t) = \bigcup_{s<t}[s, t)$ and $B = [t, \infty) = \bigcup_{t<s}[t, s)$ are open with respect to $\mathcal{T}$. Hence $A \cap C$ and $B \cap C$ are two disjoint open subsets of C that together cover C. Both of them are non-empty since $x \in A \cap C$ and $y \in B \cap C$. Hence C is disconnected, which is a contradiction.

(d) Consider the set $\mathcal{C} = \{[n, n + 1) \mid n \in \mathbb{Z}\}$. This is an open cover of $\mathbb{R}$ with respect to $\mathcal{T}$. Since each two of the elements of $\mathcal{C}$ are disjoint, this cover does not have any finite subcover. Hence the space is not compact.

(e) Let $\mathcal{A}$ be any basis for the topology $\mathcal{T}$. For all $x \in \mathbb{R}$ there exists an element $A_x \in \mathcal{A}$ such that $x \in A_x \subseteq [x, x + 1)$; in particular $\inf A_x = x$. Hence, if $x \neq y \in \mathbb{R}$, then $A_x \neq A_y$. Thus $|\mathcal{A}| \geq |\mathbb{R}|$ and so $\mathcal{T}$ does not have a countable basis.

(2) Let X be any non-empty topological space, and define $CX = (X \times I)/(X \times \{0\})$ which is called the cone on X.
(a) [3 pts] Show that CX is contractible.
(b) [2 pts] For any $n \geq 0$, show that $C\mathbb{S}^n$ is homeomorphic to $\overline{\mathbb{B}}^{n+1}$.

Solution: (a) We will prove that the identity of CX is homotopic to the constant map with image $[(x_0, 0)]$, where x_0 is any point of X. Denote by $q\colon X \times I \to CX$ the quotient map. First note that

$$q \times \mathrm{id}_I \colon (X \times I) \times I \longrightarrow CX \times I$$

is a quotient map by Lemma 4.72.

Now let $H\colon CX \times I \to CX$ be the function given by $H([(x,s)],t) = [(x,st)]$. This is well-defined since $H([(x,0)],t) = [(x,0)] = [(y,0)] = H([(y,0)],t)$ for any $x,y \in X$ and $t \in I$. The composite

$$G = H \circ (q \times \mathrm{id}_I)\colon X \times I \times I \longrightarrow CX$$

is given by $G(x,s,t) = [(x,st)]$. This is equal to $q \circ (\mathrm{id}_X \times \mu)$, where $\mu\colon I \times I \to I$ is given by $\mu(s,t) = st$. Since both q and μ are continuous, so is G. As $q \times \mathrm{id}_I$ is a quotient map, this implies that H is continuous as well.

Lastly note that $H([(x,s)],1) = [(x,s)]$ and $H([(x,s)],0) = [(x,0)] = [(x_0,0)]$. Hence H is a homotopy between id_{CX} and the constant map with image $[(x_0,0)]$.

(b) Let $f\colon C\mathbb{S}^n \to \overline{\mathbb{B}}^{n+1}$ be the function given by $f([(x,s)]) = sx$, where we consider $\mathbb{S}^n$ to be a subset of $\mathbb{R}^{n+1}$. Note that $|sx| = s|x| \le |x| = 1$ since $0 \le s \le 1$ and $x \in \mathbb{S}^n$, so that indeed $sx \in \overline{\mathbb{B}}^{n+1}$. The composite $f \circ q\colon \mathbb{S}^n \times I \to \overline{\mathbb{B}}^{n+1}$ is given by $(x,s) \mapsto sx$, which is continuous. Hence f is continuous as well. Moreover, note that f is bijective, and that $C\mathbb{S}^n$ is compact since it is a quotient of the compact space $\mathbb{S}^n \times I$. Since $\overline{\mathbb{B}}^{n+1}$ is Hausdorff, this implies, by Lemma 4.50, that f is a homeomorphism.

(3) We define a group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by putting $(m,n)\cdot(x,y) = (x+m, y+n)$ for any $(m,n) \in \mathbb{Z}^2$ and $(x,y) \in \mathbb{R}^2$.

(a) [2 pt] Show that this is a covering space action (also called a properly discontinuous action).

(b) [3 pt] Show that $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

Solution: (a) This is a special case of Proposition 12.15 since $\mathbb{R}^2$ is a connected and locally path-connected topological group and $\mathbb{Z}^2$ is a discrete subgroup such that the action is precisely given by translation.

(b) Denote by $\varepsilon\colon \mathbb{R}^1 \to \mathbb{S}^1$ the exponential map $\varepsilon(x) = e^{2\pi i x}$. We consider $\mathbb{S}^1$ to be a subgroup of the multiplicative group $\mathbb{C}^*$. Then ε is a surjective group homomorphism with kernel $\mathbb{Z}$. Hence $\varepsilon \times \varepsilon$ is a surjective group homomorphism with kernel $\mathbb{Z} \times \mathbb{Z}$, and thus it induces an isomorphism of groups $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$; in particular f is a bijection. Since $\varepsilon \times \varepsilon$ is continuous and $q\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a quotient map, the induced map f is also continuous. Note that $\mathbb{R}^2/\mathbb{Z}^2$ is equal to $q([0,1]\times[0,1])$ and hence compact. Since $\mathbb{S}^1 \times \mathbb{S}^1$ is Hausdorff, this implies, by Lemma 4.50, that f is a homeomorphism.

(4) Let $q : E \to X$ be a covering map.

(a) [3 pt] Show that if X is Hausdorff then E is Hausdorff.

(b) [2 pt] Show that if q is proper then $q^{-1}(x)$ consists of finitely many points for all $x \in X$.

Solution: (a) Let $e \neq e'$ be two points of E. If $q(e) \neq q(e')$, then these images have disjoint open neighborhoods U and V in X and $q^{-1}(U)$ and $q^{-1}(V)$ are disjoint open neighborhoods of e and e'. If $q(e) = q(e')$, then this image x has an evenly covered neighborhood $U \subseteq X$. In particular $q^{-1}(U)$ is a disjoint union of open subsets of E, each of which contains exactly one preimage of x. Hence e and e' are contained in two different of these open subsets; this yields disjoint open neighborhoods of e and e'.

(b) Since the subspace $\{x\} \subseteq X$ is compact and q is proper, the preimage $q^{-1}(x)$ is compact. Since q is a covering map, the fiber $q^{-1}(x)$ is also discrete. But compact discrete spaces are finite.

(5) [5 pts] Let $\overline{\mathbb{B}}^2$ denote the closed disc with boundary $\mathbb{S}^1$. Fix a point $p \in \mathbb{S}^1$. Compute the fundamental group of the union $(\mathbb{S}^1 \times \mathbb{S}^1) \cup (\overline{\mathbb{B}}^2 \times \{p\})$ inside $\overline{\mathbb{B}}^2 \times \mathbb{S}^1$.

Solution: Let $X = (\mathbb{S}^1 \times \mathbb{S}^1) \cup (\overline{\mathbb{B}}^2 \times \{p\})$ and let Y be the space obtained by attaching a disk $\overline{\mathbb{B}}^2$ to $\mathbb{S}^1 \times \mathbb{S}^1$ along the map

$$\varphi\colon \partial\overline{\mathbb{B}}^2 = \mathbb{S}^1 \cong \mathbb{S}^1 \times \{p\} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

given by the inclusion. We claim that $X \cong Y$. Let $f\colon (\mathbb{S}^1 \times \mathbb{S}^1) \amalg \overline{\mathbb{B}}^2 \to X$ be the continuous map given by the identity on $\mathbb{S}^1 \times \mathbb{S}^1$ and by $f(b) = (b, p)$ on $\overline{\mathbb{B}}^2$. Since $f(\varphi(b)) = f(b)$ for all $b \in \partial\overline{\mathbb{B}}^2$, it induces a continuous map $g\colon Y \to X$. Note that g is a bijection. Moreover $X \subseteq \overline{\mathbb{B}}^2 \times \mathbb{S}^1$ is Hausdorff and Y is compact since it is a quotient of $(\mathbb{S}^1 \times \mathbb{S}^1) \amalg \overline{\mathbb{B}}^2$. Hence g is a homeomorphism by Lemma 4.50. In particular the fundamental group of X is isomorphic to the one of Y.

Now, by Proposition 10.13, the fundamental group of Y (based at a point represented by $(x_0, p) \in \mathbb{S}^1 \times \mathbb{S}^1$ for some $x_0 \in \mathbb{S}^1$) is isomorphic to the quotient of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ by the normal closure of $\varphi_*(\alpha)$, where α corresponds to 1 under the isomorphism $\pi_1(\partial\overline{\mathbb{B}}^2) = \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Note that the map $\Psi\colon \pi_1(S^1 \times S^1) \to \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1)$ given by $\Psi(\gamma) = ((\mathrm{pr}_1)_*(\gamma), (\mathrm{pr}_2)_*(\gamma))$ is an isomorphism by Proposition 7.34. Hence $\pi_1(Y)$ is isomorphic to the quotient of $\pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1)$ by the normal closure of $\Psi(\varphi_*(\alpha))$. But $\mathrm{pr}_1 \circ \varphi = \mathrm{id}_{\mathbb{S}^1}$ and $\mathrm{pr}_2 \circ \varphi$ is constant with value p. Hence $\Psi(\varphi_*(\alpha)) = (\alpha, e)$, where e is the neutral element of $\pi_1(\mathbb{S}^1)$. Using the isomorphism $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, this implies that $\pi_1(Y)$ is isomorphic to the quotient of $\mathbb{Z} \times \mathbb{Z}$ by the normal closure of $(1, 0)$. This normal closure is the subgroup $\mathbb{Z} \times \{0\}$, and hence $\pi_1(Y) \cong (\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times \{0\}) \cong \mathbb{Z}$.

(6) [5 pts] Prove the following theorem:

Theorem 1. *(Homotopy invariance of π_1.) If $\varphi : X \to Y$ is a homotopy equivalence, then for any point $p \in X$, $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is an isomorphism.*

Solution: This is Theorem 7.40 from the book, and its proof can be found there.