# Solutions for the MM3001 exam 13/03/2024

## Exercise 1

$f(x) = x \cdot \ln(1 + (x-1)^2) \qquad x_0 = 1$

$f(1) = 1 \cdot \ln(1+0) = 1 \cdot \ln(1) = 0$ 0.5

$f'(x) = \ln(1+(x-1)^2) + x \cdot 2(x-1) \cdot \frac{1}{1+(x-1)^2}$ 0.5 $\quad f'(1) = 0$ 0.5

$$f''(x) = 2(x-1) \cdot \frac{1}{1+(x-1)^2} + 2(x-1)\frac{1}{1+(x-1)^2} + 2x\frac{1}{1+(x-1)^2} - x\,2(x-1)\,2(x-1)\left(\frac{1}{1+(x-1)^2}\right)^2$$

$$= 4(x-1)\frac{1}{1+(x-1)^2} + 2x\frac{1}{1+(x-1)^2} - 4x(x-1)^2\left(\frac{1}{1+(x-1)^2}\right)^2$$ 0.5

$$= 2\frac{1}{1+(x-1)^2}\left(2(x-1) + x - 2x(x-1)^2\frac{1}{1+(x-1)^2}\right)$$

$f''(1) = 2 \cdot \frac{1}{1}(0 + 1 - 0) = 2$ 0.5

$$f'''(x) = -4(x-1)\left(\frac{1}{1+(x-1)^2}\right)^2\left(2(x-1) + x - 2x(x-1)^2\frac{1}{1+(x-1)^2}\right)$$ 0.5

$$+ 2\frac{1}{1+(x-1)^2}\left(3 - 2(x-1)^2\frac{1}{1+(x-1)^2} - 4x(x-1)\frac{1}{1+(x-1)^2} + 4x(x-1)^3\left(\frac{1}{1+(x-1)^2}\right)^2\right)$$

$f'''(1) = 0 + 2 \cdot \frac{1}{1}(3 - 0 - 0 + 0) = 6$ 0.5

$P(x) = 0 + 0(x-1) + \frac{2}{2}(x-1)^2 + \frac{6}{6}(x-1)^3 = (x-1)^2 + (x-1)^3$ 1

$p(1.1) = 0.1^2 + 0.1^3 = \frac{1}{100} + \frac{1}{1000} = \frac{11}{1000} = 0.011$ 0.5

Exercise 2

$$y^2 e^{2x} + C = 3y + x^2$$

(a) we set $x=0$ $y=3$ we get 0.5

$$9 \cdot 1 + C = 9 \implies \boxed{C=0}$$ 0.5

(b) We derive implicitly considering $y=y(x)$

$$y(x)^2 \cdot e^{2x} = 3y(x) + x^2$$

$$2y(x)\, y'(x) \cdot e^{2x} + y(x)^2 \cdot 2 \cdot e^{2x} = 3y'(x) + 2x$$ 1

now we set $x=0$
$y(0)=3$

$$2 \cdot 3 \cdot y'(x) \cdot e^0 + 3^2 \cdot 2 \cdot e^0 = 3y'(x) + 0$$ 0.5

$$3y'(x) + 18 = 0 \qquad \boxed{y'(x) = \frac{-18}{3} = -6}$$ 0.5

(c) We know from (b) that the tangent line has slope $-6$ and passes through $(0,3)$. Thus its equation is of the form

$$y = -6x + m$$

we set in $x=0$ $y=3$ and get

0.5

$$m = 3$$

So the equation is

$$\boxed{y = -6x + 3}$$ 0.5

Exercise 3

$$f(x) = \exp(-x^3 + 30000x)$$

(a) $f'(x) = (-3x^2 + 30000)\exp(-x^3 + 30000x)$ 0.5

$$= -\exp(-x^3 + 30000x)(x^2 - 10000) =$$

$$= \underbrace{-\exp(-x^3 + 30000x)}_{\hat{0}}(x-100)(x+100)$$

$f'(x) \geq 0$ iff $(x-100)(x+100) \leq 0$

iff $-100 \leq x \leq 100$

Thus $f(x)$ is increasing for $-100 \leq x \leq 100$ and decreasing for $x < -100$ 0.5
and $x > 100$

There are two critical points $x = 100$ and $x = -100$ 0.5

| | | -100 | | 100 | |
|---|---|---|---|---|---|
| $f'(x)$ | $-$ | 0 | $+$ | 0 | $-$ |
| $f(x)$ | $\searrow$ | | $\nearrow$ | | $\searrow$ |
| | | MIN | | MAX | |

0.5

we have that $x = -100$ is a local min point and
$x = 100$ is a local max point

(c) $\lim_{x \to +\infty} \exp(-x^3 + 30000x) = \exp\left(\lim_{x \to +\infty} -x^3 + 30000x\right) =$

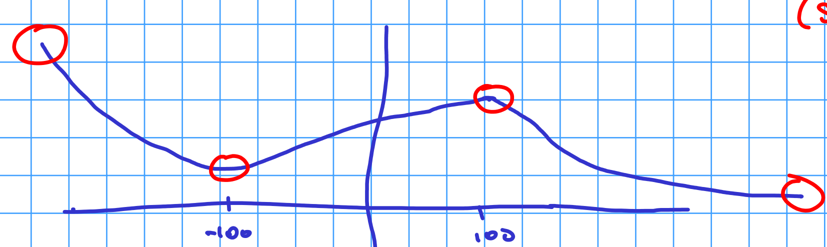
$$= \exp\left(\lim_{x \to +\infty} -x^3\left(1 - \cancel{\frac{30000}{x^2}}^{\to 0}\right)\right) = 0 \quad 0.5$$

0.5

$$\lim_{x \to -\infty} \exp(-x^3 + 3000x) = \exp\left(\lim_{x \to -\infty} -x^3\right) = +\infty$$

(show the limits & the critical points)

1



(b) From (a) we have that there is just a critical pt $x=-100$ in the interval & the function is always derivable in the interval.

Thus we need to test the value that $f(x)$ takes in

$x = \quad -101 \quad\quad -100 \quad\quad -99$

0.5

$f(x) = \exp(-1999699) \quad \exp(-200000) \quad \exp(-1999701)$

largest value MAX

smallest value MIN

0.5

$\Rightarrow$ The function has min value $\exp(-200000)$ and max value $\exp(-1999699)$ in the interval

Exercise 4

(a) $$\int \frac{\ln(t^5+1)}{t^5+1} t^4 + 4t^{7/5} \, dt =$$

$$\int \frac{\ln(t^5+1)}{t^5+1} t^4 \, dt + \int 4 \cdot t^{7/5} \, dt =$$

0.5

$u = \ln(t^5+1)$

$du = \frac{5t^4}{t^5+1} dt$

$$= \frac{1}{5} \int u \, du + 4 \frac{1}{\frac{7}{5}+1} t^{7/5+1} + C_1 =$$

$$= \frac{1}{5} \cdot \frac{1}{2} u^2 + 4 \frac{1}{\frac{12}{5}} t^{12/5} + C$$

0.5 0.5

$$= \frac{1}{10} \left( \ln(t^5+1) \right)^2 + \cancel{4} \frac{5}{\cancel{12}_{3}} t^{12/5} + C$$ 0.5

(b) $$\int_0^{+\infty} t \cdot e^{-t} \, dt = \lim_{b \to +\infty} \left( \int_0^b t e^{-t} \, dt \right)$$ 0.5

$$= \lim_{b \to +\infty} \left( \left[ -t e^{-t} \right]_0^b - \int_0^b -e^{-t} \cdot 1 \, dt \right)$$

0.5

$$= \lim_{b \to +\infty} \left( -b e^{-b} - 0 + \int_0^b e^{-t} \, dt \right)$$

$$= \lim_{b \to +\infty} \left( -b e^{-b} + \left[ -e^{-t} \right]_0^b \right)$$

0.5

0.5 0.5

$$= \lim_{b \to +\infty} \left( \cancel{-b e^{-b}}^{\to 0} - \cancel{e^{-b}}^{\to 0} + 1 \right) = 1$$ 0.5

$$\lim_{b \to +\infty} -b \cdot e^{-b} = \lim_{b \to +\infty} -\frac{b}{e^b} \overset{H}{=} \lim_{b \to +\infty} -\frac{1}{e^b} = 0$$

## Exercise 5

1,3

(a) $\det(A) = \begin{vmatrix} 0 & 0 & 1 \\ 2 & c & 2 \\ c-3 & 6 & 1 \end{vmatrix} = (-1)^{1+3}\left(2\cdot 6 - c(c-3)\right)$ 0.5 for correct formula

$= \boxed{12 - c^2 + 3c}$ 0.5 for correct answer

(b) A is invertible $\Leftrightarrow \det(A) \neq 0$

$-c^2+3c+12=0$ 0.5 $\qquad c_\pm = \dfrac{-3 \pm \sqrt{9+4\cdot 12}}{-2}$ 0.5 $= \dfrac{-3\pm\sqrt{57}}{-2}$

So A is invertible whenever $c \neq +3 \pm \sqrt{57}/2$

(c) $\left(\begin{array}{ccc|c} 2 & 0 & \boxed{1} & 1 \\ 8 & 1 & 2 & 3 \\ 1 & 6 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 4 & \boxed{1} & 0 & 1 \\ -1 & 6 & 0 & 2 \end{array}\right)$ $R_3 \to R_3 - R_1$

$\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|c} 2 & 0 & \boxed{1} & 1 \\ 16 & \boxed{1} & 0 & 1 \\ \boxed{-25} & 0 & 0 & -4 \end{array}\right) \sim$ 1 pt for correct Gauss elimination

only 1 solution

III $\rightsquigarrow$ $-25x = -10$ $\quad x = \dfrac{4}{25}$ we plug in into II 1 pt for correct answer

II $\rightsquigarrow$ $4x + y = 1$ $\quad y = \dfrac{25-16}{25} \Leftrightarrow y = \dfrac{9}{25}$

I $\rightsquigarrow$ $2x + z = 1$ $\quad z = 1 - 2x = \dfrac{25-8}{25} = \dfrac{17}{25}$

SOLUTION: $(x\ y\ z) = \left(\dfrac{4}{25}, \dfrac{9}{25}, \dfrac{17}{25}\right)$

Exercise 6

$$f(x,y) = 48xy - 32x^3 - 24y^2$$

(a) $\frac{\partial}{\partial x} f(x,y) = 48y - 96x^2 = 0$

0.5

$\frac{\partial}{\partial y} f(x y) = 48x - 48y = 0 \iff x = y$

So $48x - 96x^2 = 0 \iff x(48 - 96x)$

$x = 0$ or $x = \frac{48}{96} = \frac{1}{2}$

Thus we get two critical points $(0,0)$ and $(\frac{1}{2} \frac{1}{2})$ 0.5

$$H(x y) = \begin{vmatrix} -192x & 48 \\ 48 & -48 \end{vmatrix} = (192)(48)x - 48 \cdot 48$$

0.5

$H(0 0) = -48^2 < 0$ SADDLE

0.5

$H(\frac{1}{2} \frac{1}{2}) = 96 \cdot 48 - 48^2 > 0$ so we look at the sign of $-192\frac{1}{2} < 0$ the point is a max

(b) There are 4 sides

1) $x = 0$ $y \in [0, 1]$ $g(y) = f(0 y) = -24y^2$ $g'(y) = -48y$ has a critical point in $y = 0$ which is an extreme

$\boxed{f(0,0) = 0}$ $\boxed{f(0,1) = -24}$ 0.5

2) $x = 1$ $y \in [0,1]$ $g(y) = f(1,y) = 48y - 32 - 24y^2$

$g'(y) = 48 - 48y = 0$ $y = 1$ 0.5

again this is an extreme

$f(1,0) = \boxed{-32}$ $f(1,1) = 48 - 32 - 24 = \boxed{-8}$

3) $y = 0$ $x \in [0,1]$ $g(x) = f(x,0) = -32x^3$ 0.5

$g'(x) = -96x^2$ which has a critical pt in $x = 0$ which is one of the extremes

$f(0,0) = g(0) = \boxed{0}$ $f(1,0) = g(1) = \boxed{-32}$

4) $y = 1$ $x \in [0,1]$ $g(x) = f(x 1) = 48x - 32x^3 - 24$ 0.5

$g'(x) = 48 - 96x^2 = 48(1 - 2x^2) = 48(1 + \sqrt{2}x)(1 - \sqrt{2}x)$

we have two critical points $x = -\frac{1}{\sqrt{2}}$ $x = +\frac{1}{\sqrt{2}}$

of these only $x = \frac{1}{\sqrt{2}}$ lies in $[0,1]$.

$$f(0,1) = \boxed{-24}$$

$$f\left(\tfrac{1}{\sqrt{2}},1\right) = 48\frac{\sqrt{2}}{2} - \frac{32\sqrt{2}\cdot 2}{8} - 24 = 24\sqrt{2} - 8\sqrt{2} - 24$$

$$= 16\sqrt{2} - 24 \approx \boxed{-1.37}$$

$$f(1,1) = \boxed{-8}$$

of all the boxed values the biggest is 0 and the smallest is $-32$ so

MAX VALUE $= 0$

MIN VALUE $= -32$

(c) Of the critical pts that we found only $\frac{1}{2}\,\frac{1}{2}$ lies in the interior of D. We have to compare $f(\frac{1}{2}\,\frac{1}{2})$ with the max & min value gotten from the boundary

$$f\left(\tfrac{1}{2}\,\tfrac{1}{2}\right) = 48\,\frac{1}{2}\cdot\frac{1}{2} - 32\,\frac{1}{8} - 24\,\frac{1}{4}$$

$$= 12 - 4 - 6 = 2 > 0$$ so this is the max on D

Answer the max value on D is 2 taken on $\left(\frac{1}{2}\,\frac{1}{2}\right)$

The min value on D is $-32$ taken in $(1,0)$