Introduction to Analysis

SU — 2024 Summer

Assignment 1

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Problem 1: (2 pts) LUP

Show that $\{x \in \mathbb{Q} | x \geq 0,\ x^2 > 2\}$ has no greatest lower bound in $\mathbb{Q}$. Use this to deduce that $\mathbb{Q}$ does not sarisfy the least upper bound property.

Solution: Let $S \coloneqq \{x \in \mathbb{Q} | x \geq 0,\ x^2 > 2\}$. First note that 1 is clearly a lower bound of S since for any $q \in S$ we have $q^2 > 2 > 1^2$ and $f(x) = x^2$ is an increasing function on the positive rational numbers. Thus we have reduced the problem to showing that S has rational greatest lower bound $l \geq 1$.

For this, I first claim that any positive lower bound l of S has the property $l^2 \leq 2$. To see this note that for any l with $l^2 > 2$ we can define

$$m \coloneqq l + \frac{1}{2l}(2 - l^2) < l$$

but

$$m^2 = l^2 + (2 - l^2) + \left(\frac{1}{2l}(2 - l^2)\right)^2 = 2 + \left(\frac{1}{2l}(2 - l^2)\right)^2 > 2$$

since the expression in the parenthesis is not 0 by assumption. It is also clear that $m > 0$ since

$$l + \frac{1}{2l}(2 - l^2) = \frac{l}{2} + \frac{1}{l} > \frac{l}{2} > 0.$$

Hence $m \in S$ and $m < l$ which contradicts l being a lower bound of S.

Next, I claim that if l is a positive lower bound of S with $l^2 < 2$ then l is not a greatest lower bound. To show this, let $1 \leq l$ be a lower bound of S with $l^2 < 2$. Then define

$$m \coloneqq l + \frac{2 - l^2}{4l}.$$

Clearly we have $m > l$. Furthermore, we have

$$2 - m^2 = 2 - l^2 - \frac{(2 - l^2)}{2} - \frac{(2 - l^2)^2}{16l^2} = \frac{2 - l^2}{16l^2}\left(9l^2 - 2\right).$$

Since $l > 1$ (meaning that $9l^2 - 2 > 7$) and $2 - l^2 > 0$ both factors in the above expression are positive and thus $2 - m^2 > 0$. This means that for any $q \in S$ we have

$$q^2 - m^2 > 2 - m^2 > 0.$$

Since m, q are positive and $f(x) = x^2$ is an increasing function on the positive rational numbers this implies that $q > m$ and thus m is a lower bound of S. Since $m > l$ this means that l is not a greatest lower bound of S.

This means that if l is a greatest lower bound for S we must have $l^2 = 2$. I claim that no $l \in \mathbb{Q}$ has this property. If this is true then no $l \in \mathbb{Q}$ can be a greatest lower bound of S and so we are done. To prove this claim let $\frac{a}{b} \in \mathbb{Q}$ be reduced as far as possible (i.e. $\mathrm{GCD}(a, b) = 1$). Then $\left(\frac{a}{b}\right)^2 = 2$ implies that $a^2 = a \cdot a = 2b^2$. Since 2 is prime this means that 2 divides one of the factors on the left hand side, both of which are a, i.e. 2 divides a. Thus we can write $a = 2c$ where c is some integer. This gives $2b^2 = (2c)^2 = 4c^2$. Dividing both sides by 2 gives $b^2 = 2c^2$. By the same argument as before 2 divides b. However, this is a contradiction since $\mathrm{GCD}(a, b) = 1$ and thus no $l \in \mathbb{Q}$ has the property $l^2 = 2$. This completes the proof.

For the second part note that by Rudin, theorem 1.11, if an ordered set has the least upper bound property it has it has the greatest lower bound property. By the proof above there is a subset of $\mathbb{Q}$ with a lower bound but not a greatest lower bound. Therefore $\mathbb{Q}$ does not have the greatest lower bound property and thus it does not have the least upper bound property.

Remarks: Some additional comments are in order here to make it easier to see how I came up with these steps and hopefully make it clearer to you how one can go about solving a similar exercise. To me, the most difficult part in this proof is how to construct m from l both in the first and second step. The idea I use in both cases is to start with l and try to find a rational number somewhere between l and $\sqrt{2}$. I will tell you how I did this for the first part, i.e. $l^2 > 2$, and then hint at how it can be done for the second part. The idea is to first define $f(x) = x^2 - 2$. The derivative of this function, $f'(x) = 2x$ is strictly increasing on the interval $x > 0$ and in particular when $x^2 > 2$. Thus, the tangent line $T(l)$ through any point $(l, f(l))$ on the graph of f will lie strictly below the graph of f and meet the graph only at the point $(l, f(l))$. This means that if $(x, y) \in T(l)$ is a point on the tangent line then $y \leq f(x)$ with equality if and only if $x = l$. In particular if m is the x-coordinate at which $T(l)$ meets the x-axis (i.e. $(m, 0) \in T(l)$ then we have $m < l$ and $f(m) = m^2 - 2 > 0$. This m is precisely the point m I used for the solution For the second part the function $f(x)$ does not work. Do you see why? If not try using $f(x) = 2 - x^2$ to find a point m from l where $l^2 < 2$ and you will hopefully see why $m^2 > 2$. Plotting a graph might be helpful. Instead we do the same thing but with the function $g(x) = (x^2 - 2)^2$. The derivative of this function is increasing on the interval $1 < l < \sqrt{2}$ but not on the entire interval $l < \sqrt{2}$. This is why I first had to mention that $l = 1$ is a lower bound so we may assume $l \geq 1$.

Finally I also want to mention that this exercise is much easier to solve if you decide to use real numbers. For this first note that $\sqrt{2}$ is clearly a lower bound for S in $\mathbb{R}$. Next note that for any real number $r > \sqrt{2}$ there exists a rational number q such that $\sqrt{2} < q < r$ which implies that $2 < q^2$, i.e. $q \in S$. Hence, r is not a lower bound for S whenever $r > \sqrt{2}$ and thus $\sqrt{2}$ is the greatest lower bound of S. By the last part of my solution, $\sqrt{2}$ is not a rational number. Therefore, for any rational lower bound l of S we have $l < \sqrt{2}$ which means that there exists some m such that $l < m < \sqrt{2}$. This is also a lower bound for S and thus l is not a greatest lower bound. This argument holds in more generality. If $S \subseteq \mathbb{Q}$ is a set such that the greatest lower bound $l \in \mathbb{R}$ of S is not a rational number then S has no greatest lower bound in $\mathbb{Q}$.

Problem 2: (1 pt) Cardinality

Show that the set of sequences with values in $\{\clubsuit, \heartsuit\}$ is not countable.

Solution: Let S be the set of sequences with values in $\{\clubsuit, \heartsuit\}$ and let T be the set of sequences with values in $\{0, 1\}$. I claim that there is a bijection $F\colon S \to T$. To construct this first note that there is a bijection $f\colon \{\clubsuit, \heartsuit\} \to \{0, 1\}$ defined by $f(\clubsuit) = 0$ and $f(\heartsuit) = 1$ with inverse g defined by $g(0) = \clubsuit$ and $g(1) = \heartsuit$. Using this we can define F be the function that sends

$$F\colon \{a_n\}_{n\in\mathbb{N}} \mapsto \{f(a_n)\}_{n\in\mathbb{N}}.$$

Now I claim that

$$G\colon T \to S, \{b_n\}_{n\in\mathbb{N}} \mapsto \{g(b_n)\}_{n\in\mathbb{N}}$$

is an inverse of F. To see this note that

$$G(F(\{a_n\}_{n\in\mathbb{N}})) = G(\{f(a_n)\}_{n\in\mathbb{N}}) = \{g(f(a_n))\}_{n\in\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}},$$

and similarly

$$F(G(\{b_n\}_{n\in\mathbb{N}})) = F(\{g(b_n)\}_{n\in\mathbb{N}}) = \{f(g(b_n))\}_{n\in\mathbb{N}} = \{b_n\}_{n\in\mathbb{N}}.$$

Thus, $G \circ F = \mathrm{id}_S$ and $F \circ G = \mathrm{id}_T$ which means that G is the inverse of F and thus F is a bijection. Since $F\colon S \to T$ is a bijection and we know from Rudin, theorem 2.14, that T is uncountable it follows that S is uncountable.

Remarks: Most of you used the same argument as Rudin does in the proof of theorem 2.14 to solve this problem. This is of course also a valid solution.

Problem 3: Closure

Let $\{E_k\}_{k\in\mathbb{N}}$ be subsets of a metric space (X, d). Show the following statements

(1) (1 pts) If $B = \cup_{k=1}^{n} E_k$ then $\overline{B} = \cup_{k=1}^{n} \overline{E}_k$

(2) (1 pts) If $B = \cup_{k\in\mathbb{N}} E_k$ then $\overline{B} \supseteq \cup_{k\in\mathbb{N}} \overline{E}_k$

Solution: I will start with part 2. This statement is true for $k \in \mathcal{I}$ where $\mathcal{I}$ is any index set, not just $\mathcal{I} = \mathbb{N}$. That is, if $B = \cup_{k\in\mathcal{I}} E_k$ then $\overline{B} \supseteq \cup_{k\in\mathcal{I}} \overline{E}_k$. To see this first note that by definition, $\overline{B} \supseteq B \supseteq E_k$ for every $k \in \mathcal{I}$. Hence $\overline{B}$ is a closed subset containing E_k which implies that $\overline{B} \supseteq \overline{E_k}$ by definition of the closure. This is true for every k and thus $\overline{B} \supseteq \bigcup_{k\in\mathcal{I}} \overline{E_k}$.

For part 1, first note that this proves that if $B = \bigcup_{k=1}^{n} E_k$ then $\overline{B} \supseteq \bigcup_{k=1}^{n} \overline{E_k}$. To show the other inclusion note that $\bigcup_{k=1}^{n} \overline{E_k}$ is a finite union of closed sets and hence it is closed. Since $\bigcup_{k=1}^{n} \overline{E_k} \supseteq \bigcup_{k=1}^{n} E_k = B$ this means that $\bigcup_{k=1}^{n} \overline{E_k} \supseteq \overline{B}$ by definition of the closure. This completes the proof.

Remarks: To me the definition of the closure of a set A is the intersection of all closed sets containing A and so, the statement that if $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$ follows immediately from the definition. However, it seems this is not the definition Rudin uses and instead this result is a theorem which I would reference here but sadly I do not have access to the book at the time of writing this. Regardless, this is true.

Problem 4: Pre-images

Let $f : X \to Y$ be a function and let A_α is a collection of subsets of Y, with α in an index set $\mathcal{I}$. Recall that, if $A \subseteq Y$, the pre-image of A through f is

$$f^{-1}(A) := \{x \in X \mid f(x) \in A\}.$$

(1) (1 pt) Show that $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) = \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$

(2) (1 pts) Show that $f^{-1}(\bigcap_{\alpha\in\mathcal{I}} A_\alpha) = \bigcap_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$

Solution: For part 1, let $x \in f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha)$. Then $f(x) \in \bigcup_{\alpha\in\mathcal{I}} A_\alpha$ which means that $f(x) \in A_\beta$ for some index $\beta \in \mathcal{I}$ by definition of a union. Since $f(x) \in A_\beta$ we have, by definition, $x \in f^{-1}(A_\beta) \subseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$, i.e. $x \in \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$ and thus, $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) \subseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$. Similarly, if $x \in \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$ then $x \in f^{-1}(A_\beta)$ for some $\beta \in \mathcal{I}$. This implies that $f(x) \in A_\beta \subseteq \bigcup_{\alpha\in\mathcal{I}} A_\alpha$. Thus, $x \in f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha)$ which implies that $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) \supseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$. Since we have shown $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) \subseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$ and $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) \supseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$ we must have $f^{-1}(\bigcup_{\alpha\in\mathcal{I}} A_\alpha) = \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$. This completes the proof.

The second part is analogous.

Problem 5: Compact spaces

(1) (1 pt) Show that $0 \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is compact in $\mathbb{R}$.

(2) (1 pt) Determine whether $\{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is compact in $\mathbb{R}$. Justify your answer

Solution: Let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be an open cover of $0 \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$. By definition of an open cover there is some $\alpha \in \mathcal{I}$ we have $0 \in U_\alpha$. I will let U_{α_0} denote this open set. Since U_{α_0} is open and $0 \in U_{\alpha_0}$ there is some open interval $(-b, b) \subseteq U_{\alpha_0}$ where $0 < b$ by the definition of an open set. By Rudin, theorem 1.20, any real number $b > 0$ has the property that there is some $N \in \mathbb{Z}^+$ such that $nb > 1$, or equivalently $b > \frac{1}{n}$, for every $n > N$. Thus, there exists some $n \in \mathbb{Z}^+$ such that $\frac{1}{n} \in (-b, b)$ for every $n > N$. Now, for every integer $1 \le i \le n$ let $U_{\alpha_i} \in \{U_\alpha\}_{\alpha\in\mathcal{I}}$ be any element of the open cover such that $\frac{1}{i} \in U_{\alpha_i}$. Then $V_0 \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$ contains all elements of

$0 \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ and thus $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ has a finite subcover. By definition, this means that $0 \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is compact.

The set $\{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is not compact. To prove all we must do is to find an open cover with no finite open subcover. I claim that $\{U_n\}_{n \in \mathbb{Z}^+}$ where $U_n := (\frac{1}{n+1}, \infty)$ has this property. Clearly this is an open cover since $\frac{1}{n} \in U_n$ for every n. Now, let $U_{i_1}, \ldots, U_{i_N}$ be some finite subset of the cover. Then let $m := \max(i_1, i_2, \ldots, i_N) + 1$. Then we have $\frac{1}{m} < \frac{1}{i_k + 1}$ for each i_k and therefore $\frac{1}{m} \notin U_{i_k}$ for every U_{i_k}. Hence, $U_{i_1}, \ldots, U_{i_N}$ is not a cover of $\{\frac{1}{n} | n \in \mathbb{Z}^+\}$ and therefore $\{U_n\}_{n \in \mathbb{Z}^+}$ does not have a finite subcover

Problem 6: (1 pt) Connected spaces

Show that the union of circles (not disks!) with center $(0, 0)$ and rational radii < 1 is a nonconnected set of $\mathbb{R}^2$ with the Euclidean distance.

Solution: Let X denote the space of all such circles and let ξ be any real but not rational number in the interval $(0, 1)$, i.e. $\xi \in (0, 1) \setminus \mathbb{Q}$. Note that it is true in general that for any real numbers $x < y$ there is some irrational number z such that $x < z < y$ but I could not find any reference for this in Rudin so if you do not think this is obvious then just take $\xi = \frac{\sqrt{2}}{2}$ which we know is irrational from exercise 1. Now, define $A := \{x \in X | d(x, 0) < \xi\}$ and let $B := \{x \in X | d(x, 0) > \xi\}$. First note that since every point $x \in X$ has the property $d(x, 0) \in \mathbb{Q}$ we have in particular that $d(x, 0) \neq \xi$ $A = \{x \in X | d(x, 0) \leq \xi\}$ and $B = \{x \in X | d(x, 0) \geq \xi\}$. Thus, A, B are closed and therefore $A = \overline{A}$, $B = \overline{B}$. By definition $A \cap B = \emptyset$ and thus $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Finally, $A \cup B = \{x \in X | d(x, 0) \neq \xi\} = X$ since ξ is not rational. By definition of connectedness this means that X is not connected.

Remarks: I have based this solution on Rudin's definition of connectedness. However, it is more common to say that a space X is not connected if it has two open subsets A, B with $A \cup B = X$ and $A \cap B = \emptyset$. This is an equivalent definition and using this all we have to do is conclude that A, B are both open.

The claim that for any real numbers $x < y$ there is some irrational number z such that $x < z < y$ is a good exercise.

Hint: Start by showing that there is an irrational number w (e.g. exercise 1). Then construct z as wq where q is some appropriately chosen rational number.