Consider the function $f:[0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We show that f is not continuous at $a = \frac{1}{2}$

To this aim we need to find an $\varepsilon > 0$ such that, for every $\delta > 0$ there is a real number p with $|p - 1/2| < \delta$ and $|f(p) - f(1/2)| > \varepsilon$.

Let ε be any number $< \frac{1}{2}$ for example $\varepsilon = \frac{1}{3}$ then for any $\delta > 0$ there is an irrational number $p \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$

Thus we have that

$|p-\frac{1}{2}|<\delta$ and

$|f(p)-f(1/2)| = |0-\frac{1}{2}| = \frac{1}{2} > \varepsilon$

Thus the function is not continuous at $\frac{1}{2}$.

This function is going to be continuous a $x=0$

Fix $\varepsilon>0$. Let $\delta=\varepsilon$

for any $x\in(-\delta,\delta)$ we have that

$$|f(x)-f(0)| = |f(x)| = \begin{cases} 0 \text{ if } x\notin\mathbb{Q} \\ |x| \text{ if } x\in\mathbb{Q} \end{cases}$$

$$<\delta=\varepsilon$$