let $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x - \frac{1}{2} & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

this is continuos in $\frac{1}{2}$.

$\alpha_+ : [0,1] \longrightarrow \mathbb{R}$

$$\alpha_+ = \begin{cases} 0 & \text{if } x \le \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

Let $P = \{0, \frac{1}{2}, x_2, 1\}$ with $x_2 \in \mathbb{Q}$

$\Delta\alpha_1 = 0 \quad \Delta\alpha_2 = 1 \quad \Delta\alpha_3 = 0$

$$L(P, f, \alpha_+) = \inf_{[\frac{1}{2}\, x_2]} f \cdot 1$$

now on $[\frac{1}{2}, x_2]$ $f$ is either $0$ (in $\frac{1}{2}$ and in irrational numbers)

or $x - \frac{1}{2} > 0$ on rational $\neq \frac{1}{2}$

Thus $\inf_{[\frac{1}{2}, x_2]} f = 0$

and $L(P, f, \alpha_+) = 0$

At the same way

$$\sup_{[\frac{1}{2}, x_2]} f = x_2 - \frac{1}{2}$$

So $U(P, f, \alpha_+) = \left(x_2 - \frac{1}{2}\right).$

Now fix $\varepsilon > 0$ and let

$x_2 = \frac{1}{2} + \frac{1}{n}$ with $\frac{1}{n} < \varepsilon$

then

$$U(P, f, \alpha_+) - L(P, f, \alpha_+) = \frac{1}{n} < \varepsilon$$

So $f$ is in $\mathcal{R}(\alpha_+)$

Similarly we can prove that $f$ is in $\mathcal{R}(\alpha_-)$ with

$$\alpha_-(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}$$

$P = \{0, x_1, \frac{1}{2}, 1\}$ with $x_1 \in \mathbb{Q}$

$\Delta\alpha_1 = 0 \qquad \Delta\alpha_2 = 1 \qquad \Delta\alpha_3 = 0$

$$\inf_{[x_1, \frac{1}{2}]} f = x_1 - \frac{1}{2} < 0$$

$$\sup_{[x_1, \frac{1}{2}]} f = 0$$

$U(P, f, \alpha_-) - L(P f \alpha_-) =$

$$0 - (x_1 - \tfrac{1}{2}) = \tfrac{1}{2} - x_1$$

Fix $\varepsilon > 0$ and let $\frac{1}{n} < \varepsilon$

and $x_1 = \frac{1}{2} - \frac{1}{n}$

$$U(P, f, \alpha_-) - L(P, f, \alpha_-) < \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad f \in \mathcal{R}(\alpha_-)$$