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Foundations of Analysis ST24
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Solutions to Homework 3

Problem 1

(1) Let P be a partition of $[0,1]$ given by $0 = x_1 \leq x_2 \leq ... \leq x_n = 1$ and let $\alpha = \mathrm{id}$. We have that

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) = 1$$

and

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x) = 0$$

since every interval contains both rational and irrational numbers. We can then compute

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$$

and

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})) = 0.$$

Note that the above equalities hold for any partition P, thus

$$\overline{\int_0^1} f d\alpha = 1 \neq 0 = \underline{\int_0^1} f d\alpha$$

meaning that f is not Riemann-integrable according to definition 6.2.

(2) First assume that f is continuous at $\frac{1}{2}$. Let P be a partition $0 = x_1 \leq ... \leq x_{k-1} \leq \frac{1}{2} \leq x_k \leq ... \leq x_n = 1$, note that $\alpha_+(x_i) - \alpha_+(x_{i-1})$ is 0 when $i \neq k$ and 1 when $i = k$. From this we can compute

$$U(P, f, \alpha_+) = M_k$$

and

$$L(P, f, \alpha_+) = m_k.$$

Since f is continuous there are points $x_{\max}$ and $x_{\min}$ in $[x_{k-1}, x_k]$ such that $M_k = f(x_{\max})$ and $m_k = f(x_{\min})$.

For an arbitrary $\varepsilon > 0$ we can by continuity find a partition P such that $\left|f(x) - f(\frac{1}{2})\right| < \frac{\varepsilon}{2}$ for all $x \in [x_{k-1}, x_k]$, then we have in particular that

$$\begin{aligned} U(P,f,\alpha_+) - L(P,f,\alpha_+) &= \tfrac{1}{2}|f(x_{\max}) - f(x_{\min})| \\ &\leq \tfrac{1}{2}\left(\left|f(x_{\max}) - f\left(\tfrac{1}{2}\right)\right| + \left|f\left(\tfrac{1}{2}\right) - f(x_{\min})\right|\right) \\ &< \tfrac{1}{2}(\tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}) = \varepsilon \end{aligned}$$

so $f \in \mathcal{R}(\alpha_+)$ by theorem 6.6. In the same way we can show that $f \in \mathcal{R}(\alpha_-)$, which concludes this direction of the proof.

For the other direction, assume that $f \in \mathcal{R}(\alpha_+)$, and consider a partition P of the form $0 = x_1 \leq ... \leq x_{k-1} \leq x_k = \frac{1}{2} \leq x_{k+1} \leq ... \leq x_n = 1$, then $\alpha(x_i) - \alpha(x_{i-1}) = 0$ when $i \neq k - i$ and $\alpha(x_k) - \alpha(x_{k-1}) = 1$ so we have

$$U(P,f,\alpha_+) = M_k = \sup_{x_{k-1} \leq x \leq \frac{1}{2}} f(x)$$

and

$$L(P,f,\alpha_+) = m_k = \inf_{x_{k-1} \leq x \leq \frac{1}{2}} f(x).$$

Then for any $\varepsilon > 0$ we can choose a partition P such that

$$M_k - m_k < \varepsilon,$$

which means that we can find a $\delta = \frac{1}{2} - x_{k-1}$ such that

$$\left|f(x) - f\left(\frac{1}{2}\right)\right| \leq M_k - m_k < \varepsilon$$

whenever $\frac{1}{2} - \delta \leq x \leq \frac{1}{2}$ which means that f is left-continuous at $\frac{1}{2}$. Similarly we can show that $f \in \mathcal{R}(\alpha_-)$ implies that f is right-continuous at $\frac{1}{2}$, which proves the other direction.

(3) Since f is bounded and has finitely many discontinuities and only at points where α is continuous, it follows from theorem 6.10 that $f \in \mathcal{R}(\alpha)$.

To compute the integral we first note that $\alpha(x) = e^x + 3I(x - \frac{1}{2})$ where I is the unit step function. We then use theorem 6.12 to get that

$$\int_0^1 f d\alpha = \int_0^1 f d(e^x) + 3\int_0^1 f d(I(x - \frac{1}{2})).$$

To compute the first integral we use theorem 6.17 to get

$$\int_0^1 f d(e^x) = \int_0^1 f(x) e^x dx = \int_0^{\frac{1}{3}} x e^x + 2\int_{\frac{1}{3}}^1 x e^x$$

after which we use integration by parts to compute both integrals and get

$$\int_0^1 f d(e^x) = \left(1 - \frac{2}{3}e^{\frac{1}{3}}\right) + 2\left(\frac{2}{3}e^{\frac{1}{3}}\right) = 1 + \frac{2}{3}e^{\frac{1}{3}}.$$

We now use theorem 6.15 to get that

$$\int_0^1 fd(I(x-\frac{1}{2})=f(\frac{1}{2})=1$$

and now we can finally conclude that

$$\int_0^1 fd\alpha = 1+\frac{2}{3}e^{\frac{1}{3}}+3=4+\frac{2}{3}e^{\frac{1}{3}}.$$

Problem 2

(1) Let $f(x)=\lim_{n\to\infty} f_n(x)$ for rational x, and for irrational x we choose a sequence $q_n \to x$ with $q_n \in \mathbb{Q}$ and define $f(x)=\lim_{k\to\infty} f(q_k)$. Then by theorem 7.12 $f|_{\mathbb{Q}}$ is continuous and by the definition of $f(x)$ for irrational x we see that f is continuous on all of $\mathbb{R}$.

For any $\varepsilon>0$ and real number x we can find a $\delta>0$ such that $|f(x)-f(y)|<\frac{\varepsilon}{3}$ and $|f_n(x)-f_n(y)|<\frac{\varepsilon}{3}$ whenever $|y-x|<\delta$ since f and f_n are continuous. Now let y be a rational number such that $|y-x|<\delta$, then by uniform convergence there is some natural number N such that $|f_n(y)-f(y)|\le\frac{\varepsilon}{3}$ whenever $n>N$, so then we have for all $n>N$ that

$$|f_n(x)-f(x)|\le|f_n(x)-f_n(y)|+|f_n(y)-f(y)|+|f(y)-f(x)|<\varepsilon$$

using the triangle inequality together with the above. This shows that $f_n\to f$ uniformly.

(2) We first note that for all $x\in\mathbb{R}$ we have

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$$

so $f_n\to f$ pointwise where $f(x)=e^x$. Note that f_n and f are all continuous.

Now we want to show that the convergence is uniform on an arbitrary interval $[a,b]$. If $n>-a$ we have that $1+\frac{x}{n}>0$, so we can apply the AM-GM inequality to get that

$$1+\frac{x}{n+1}=\frac{1+n\left(1+\frac{x}{n}\right)}{n+1}\ge\left(\left(1+\frac{x}{n}\right)^n\right)^{\frac{1}{n+1}}$$

and then raising both sides to the power of $n+1$ we find that $f_{n+1}(x)\ge f_n(x)$ for all $x\in[a,b]$ whenever $n>-a$. We have thus shown that the sequence is eventually monotone, and then theorem 7.13 proves that the convergence is uniform on $[a,b]$.

(3) First note that $f_n(0)=f_n(1)=0$ for all n, and for $x\in(0,1)$ we get

$$\lim_{n\to\infty}x(1-x)^n=x\lim_{n\to\infty}(1-x)^n=x\cdot 0=0$$

since $0<1-x<1$, so $f_n\to 0$ pointwise.

For every n the function f_n is differentiable for all $x \in [0,1]$, so its maximum value is attained either at an endpoint of the interval (where $f_n(x)$ is zero) or at a point where $f_n'(x) = 0$. We can compute

$$f_n'(x) = (1-x)^n - nx(1-x)^{n-1} = (1-x)^{n-1}(1-(1+n)x)$$

from which we can see that $f_n'(x) = 0$ only for $x = \frac{1}{1+n}$, whence

$$\sup_{x\in[0,1]} f_n(x) = f_n(\frac{1}{1+n}) = \frac{1}{1+n}\left(\frac{n}{1+n}\right)^n = \frac{1}{1+n}\left(1+\frac{1}{n}\right)^{-n}$$

so

$$\lim_{n\to\infty} \sup_{x\in[0,1]} f_n(x) = 0 \cdot \frac{1}{e} = 0$$

which shows that $f_n \to 0$ uniformly by theorem 7.9.

Problem 3

(1) Let

$$f_n(x) = \frac{1}{n(1+nx^2)}.$$

For $x = 0$ we get $f_n(x) = \frac{1}{n}$ so $\sum_{n=1}^{\infty} f_n$ does not even converge at this point. Now consider $x \in (0,1]$. Since

$$0 \le \sum_{n=1}^{\infty} f_n(x) \le \sum_{n=1}^{\infty} \frac{1}{n^2x^2} = \frac{1}{x^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6x^2}$$

we see that the series converges pointwise on all of $(0,1]$.

We also have that the series converges uniformly on $[\varepsilon, 1]$ for any $\varepsilon > 0$, this follows from theorem 7.10 since

$$|f_n(x)| \le \frac{1}{n^2\varepsilon^2}$$

for all $x \in [\varepsilon, 1]$.

The convergence is not uniform on all of $(0,1]$, however. We can see this by noting that

$$\sum_{n=M}^{N} f_n(x)$$

is a continuous function on $[0,1]$ and then computing

$$\sup_{x\in(0,1]} \sum_{n=M}^{N} f_n(x) = \max_{x\in[0,1]} \sum_{n=M}^{N} \frac{1}{n(1+nx^2)} = \sum_{n=M}^{N} \frac{1}{n}$$

and since the harmonic series is divergent it is not Cauchy, and thus by the above our series is not uniformly Cauchy, hence not uniformly convergent.

(2) Recall that the radius of convergence is defined as

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}.$$

For all $x \in [-r, r]$ we have $|a_n x^n| \leq |a_n| r^n$ and

$$\sum_{n=0}^{\infty} |a_n| r^n$$

converges since $r < R$ and since the radius of convergence of

$$\sum_{n=0}^{\infty} |a_n| x^n$$

is also

$$\frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} = R.$$

This shows that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly in $[-r, r]$ by theorem 7.10.