Introduction to Analysis

SU — 2024 Spring

Assignment 2

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§ Homework 2 §

Problem 1: Equivalent metric

Let X be a set with two distance functions $d_1(x,y)$ and $d_2(x,y)$. We say that d_1 and d_2 are equivalent if there exist positive real constants c and C such that

$$cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y)$$

(1) (1 pt) Show that, if d_1 and d_2 are equivalent, a sequence with value in X is convergent with respect to d_1 if and only if it is convergent with respect of d_2.

(2) (1 pt) Show that, if d_1 and d_2 are equivalent, a sequence with value in X is Cauchy with respect to d_1 if and only if it is Cauchy with respect of d_2.

(3) (1 pt) Deduce that, in the above assumptions, (X, d_1) is complete if, and only if (X, d_2) is complete.

(4) (2 pts) Always assuming that d_1 and d_2 are equivalent, show that neighbourhoods with respect of d_1 are open with respect of d_2, and that neighbourhoods with respect d_2 are open with respect d_1.

(5) (2 pts) Always assuming that d_1 and d_2 are equivalent, show that a function $f : X \to (Y, d)$ is continuous with respect of d_1 if and only if it is continuous with respect d_2.

Problem 2: Power series

Given a power series $\sum a_n x^n$ with real coeffiecients, recall that the radius of convergence is

$$R := \sup\{r \geq 0 \,|\, \sum a_n r^n \text{converges}\}.$$

Compute the radius of convergences of the following power series.

(1) (1 pt)

$$\sum_{n=1}^{+\infty} \frac{\ln n}{n 2^n} x^n$$

(2) (1 pt)

$$\sum_{n=0}^{+\infty} \frac{2^n}{\sqrt{n+3}} x^n$$

(3) (1 pt)

$$\sum_{n=0}^{+\infty} \frac{1}{\sqrt{3^n + 9^n}} x^n$$