Introduction to Analysis

SU — 2024 Spring

Assignment 3

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§ Homework 3 §

This homework is worth 10 points that is 1 bonus point. You are welcome to collaborate with your classmates to find a solution, but write and submit your own solution.

Problem 1: Stieltjes Integral

(1) (1 pt.) Show that the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Is not in $\mathcal{R}(\text{id})$.

(2) (2 pts) Let α_+ and α_- the following functions on $[0, 1]$.

$$\alpha_+(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}, \quad \text{and,} \quad \alpha_-(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

Show that a bounded function f is in $\mathcal{R}(\alpha_+) \cap \mathcal{R}(\alpha_-)$ if and only if f is continuous at $\frac{1}{2}$.

(3) (2 pts.) Let α the function on $[0, 1]$ defined by

$$\alpha(x) = \begin{cases} e^x & \text{if } x < \frac{1}{2} \\ 3 + e^x & \text{if } x \geq \frac{1}{2} \end{cases}$$

Let now

$$f(x) = \begin{cases} x & \text{if} x < \frac{1}{3} \\ 2x & \text{if} x \geq \frac{1}{3} \end{cases}$$

Show that $f \in \mathcal{R}(\alpha)$ on $[0, 1]$ and compute

$$\int_0^1 f d\alpha$$

Problem 2: Sequences of functions

(1) (1 pt.) Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly over the rational numbers, show that it converges uniformly over the real numbers.

(2) (1 pt.) Verify that the sequence of function $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ converges uniformly for every closed and bounded interval in $\mathbb{R}$ (with the Euclidean distance $d_E) = (x, y) = |x - y|$.

(3) (1 pt.) Study the convergence (both pointwise and uniform) of the following sequence of function

$$f_n(x) = x(1 - x)^n$$

for $x \in [0, 1]$

Problem 3: Series of functions

(1) (1 pt.) Study the convergence (both pointwise and uniform) of the following series of function

$$\sum_{n=1}^{+\infty} \frac{1}{n(1+nx^2)}$$

for $x \in [0,1]$

(2) (1 pt.) Let $\sum_{n=0}^{+\infty} a_n x^n$ a power series with positive radius of convergence R. Show that for every $0 < r < R$ the series converges uniformly in the interval $[-r, r]$.