Sequences

Let (X, d) be a metric space.

Def A function $x \in X^{\mathbb{N}}$ i.e. $x : \mathbb{N} \to X$, $n \mapsto x_n$ is called a **sequence**

The range of a function determines the function so we will simply write

$$x = (x_n)_{n \geq 1} = (x_1, x_2, x_3, \dots) \qquad (x = (x(n))_{n \geq 1})$$

Note : I will make a distinction between

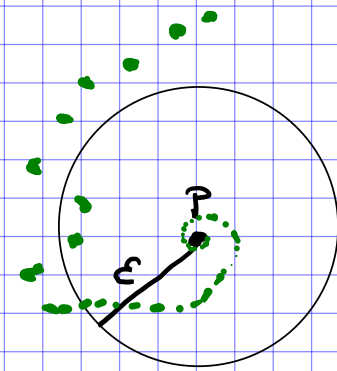
$$(x_n)_{n \geq 1} \quad \text{and} \quad \{x_n\}_{n \geq 1}.$$

If $x_n = (-1)^n$, $\quad (x_n)_{n \geq 1} = (-1, 1, -1, 1, \dots)$

but $\quad \{x_n\}_{n \geq 1} = \{x_1, x_2, \dots\} = \{-1, 1\}$

Def. A sequence $(p_n)_n$ **converges** if

$$(\exists p \in X \quad \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \forall n \geq n_0 \ [\underbrace{d(p_n, p) < \varepsilon}_{p_n \in B_\varepsilon(p)}]$$



In this case we say that $(p_n)_n$ converges to p (or p is the limit of $(p_n)_n$) and we write

$$p = \lim_{n \to \infty} p_n \quad \text{or} \quad p_n \to p \text{ as } n \to \infty \qquad (p = d\text{-}\lim_{n \to \infty} p_n)$$

$$p \overset{d}{=} \lim_{n \to \infty} p_n$$

$\exists \overset{?}{\to} \forall$

$\forall \to \exists$

We say that $(p_n)_n$ diverges if it does not converge

$$\forall p \in X \quad \exists \varepsilon > 0 \quad \forall n_0 \in \mathbb{N} \quad \exists n \geq n_0 \quad p_n \notin B_\varepsilon(p)$$

We say that $(p_n)_n$ is bounded$^{(*)}$ if its range is a bounded in X

Example: $x_n = (-1)^n$ $\quad \{x_n\}_n = \{-1, 1\}$. So $(x_n)_n$ is a bounded sequence in $\mathbb{R}$.

$(*)$ $\exists q \in X : \exists r > 0 \quad \forall n \geq 1 \quad p_n \in B_r(q)$

Thm Let $(p_n)_n$ be a sequence in X and $E \subset X$

i) The limit, if it exists is unique

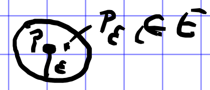
ii) If $(p_n)_n$ converges, it's also bounded

iii) $\forall p \in E' \quad \exists (p_n)_n \ ; \quad \{p_n\}_n \subset E$ such that $\lim_n p_n = p$

Proof (Complete the empty space)

i)

ii)



iii) Recall: $p \in E' \iff \forall \varepsilon > 0 \ \ (B_\varepsilon(p) \setminus \{p\}) \cap E \neq \emptyset$

$\iff \forall \varepsilon > 0 \ \exists p_\varepsilon \in E \ ; p_\varepsilon \neq p$ s.t. $d(p_\varepsilon, p) < \varepsilon$

For all $n \geq 1$, let $p_n \in E$; such that $d(p_n, p) < 1/n$ $(*)$

(It exists as $p \in E'$) & we have constructed $(p_n)_n$; $p_n \in E$

Claim: $\left[\lim_{n\to\infty} p_n = p.\right]$ $(\forall \varepsilon>0\ \exists n_0\in\mathbb{N}:\ \forall n\geq n_0\ d(p_n,p)<\varepsilon)$

For an arbitrary $\varepsilon>0$, let $n_0\in\mathbb{N}$ such that $n_0>\frac{1}{\varepsilon}$ (**) (Arquimedian property of $\mathbb{R}$)

Then $\forall n\geq n_0$

$$d(p_n,p)\overset{(*)}{<}\frac{1}{n}\overset{n\geq n_0}{\leq}\frac{1}{n_0}\overset{(**)}{<}\varepsilon.$$

#

Question:

Is it true that if $\exists (p_n)_n;\ p_n\in E$ such that $\lim_{n\to\infty} p_n=p$ then p is a limit point of E?

No. If $E=\{0\}\subset\mathbb{R}$. And we define $p_n=0$ $\forall n\geq 1$, we have that $\lim_n p_n=0$ but $E'=\emptyset$. $(0\notin E')$

Def. Given a sequence $(p_n)_n$, consider a sequence of natural numbers $(n_k)_k$ with $n_1<n_2<n_3<\dots$

The sequence $(p_{n_k})_{k\geq 1}$ is called a subsequence of $(p_n)_n$

If $(p_{n_k})_k$ converges, its limit is called a subsequential limit.

Example: $x_n=(-1)^n$; let $n_k=2k$ $k\geq 1$ $n_1=2<n_2=4<n_3=6<\dots$

So $x_{n_k}=(-1)^{2k}=1^k=1$ (x_{n_k}) is a subsequence of $(x_n)_n$

$(x_n)_n$ does not converges in $\mathbb{R}$ but $(x_{n_k})_k$ converges to 1

Theorem Let $(p_n)_n$ be a sequence in X, $p \in X$ and $K \subset X$ be a compact set.

i) The sequence $(p_n)_n$ converges to p iff every subsequence converges to p

ii) If $(p_n)_n$ is a sequence in K (i.e. $\forall n \geq 1 \ p_n \in K$) there exists a convergent subsequence

iii) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence

$\forall \varepsilon > 0 \ \exists \ell \in \mathbb{N} : \forall n \geq \ell \quad d(p_n, p) < \varepsilon$)

Proof (Fill the gaps)

i) $\Rightarrow$)

$((A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A))$

$\Leftarrow$) Let $(p_n)_n$ be a sequence that does not converge to p i.e.

$[\exists \varepsilon > 0 \quad \forall \ell \in \mathbb{N} \quad \exists n \geq \ell : \quad d(p_n, p) \geq \varepsilon]$

Let $\ell = 1+1$ $\exists n_1 \geq 1+1 (\geq 1)$: $d(p_{n_1}, p) \geq \varepsilon$.

Let $\ell = n_1 + 1$. $\exists n_2 \geq \ell > n_1 : d(p_{n_2}, p) \geq \varepsilon$ $\left(\ell = n_2 + 1 \quad \exists n_3 \geq \ell > n_2 \quad d(p_{n_3}, p) \geq \varepsilon \right)$

Iterating the argument, we can construct a subsequence of $(p_n)_n$ $(p_{n_k})_{k \geq 1}$ $n_1 < n_2 < \dots$ that does not converge to p.

ii) Let E be the range of $(p_n)_n$ ($E = \{p_n\}_{n\geq 1}$ in my notation); $E \subset K$.

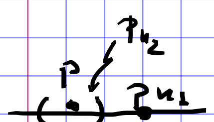
If E is finite, the sequence has to repeat at least one value an infinite number of times i.e. $\exists (n_k)_k$ in $\mathbb{N}$ with $n_1 < n_2 < \dots$ such that

$$p_{n_1} = p_{n_2} = p_{n_3} = \dots$$

Let p be this common value. Then $\lim_k p_{n_k} = p$.

$$p_n = \begin{cases} +1 + \frac{1}{n} & \text{if } n \text{ even} \\ -1 + \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

If E is infinite, since $E \subset K$ & K is compact by thm 2.37, $\exists \underline{p} \in E' \cap K \subset \underline{E'}$ ($E = \underline{\{p_n\}}_{n\geq 1}$; $p \in E'$ [$\forall \varepsilon > 0$ $\exists p_\varepsilon \in E$: $p_\varepsilon \neq p$ & $d(p_\varepsilon, p) \leq \varepsilon$])



So $\exists n_1 \in \mathbb{N}$: $d(p, p_{n_1}) < \frac{1}{2}$ $(p_{n_1} \in E)$

$\exists n_2 \in \mathbb{N}$ [that we can assume $n_2 > n_1$]$^{(*)}$: $d(p, p_{n_2}) < \frac{1}{2^2}$ $(p_{n_2} \in E)$

Iterating the argument we can construct

$$n_1 < n_2 < \dots \quad \text{such that} \quad \forall k \quad d(p_{n_k}, p) < \frac{1}{2^k}$$

Claim (show this using the definition of limit)

$$\left[\lim_{k\to\infty} p_{n_k} = p. \right]$$

(*) If not: $\forall n \geq n_1$
$d(p, p_n) \geq \frac{1}{4}$.
This would contradict that $p \in E'$

iii) If $(p_n)_n$ is bounded in $\mathbb{R}^k$, we can find a k-cell I (compact set) such that

$$\{p_n\}_n \subset \dot{I}$$

By ii), there exists a convergent subsequence. #

Thm Let $(p_n)_n$ be a sequence in X.

Let E^* be the set of all subsequential limits of $(p_n)_n$.

Then, E^* is a closed set.

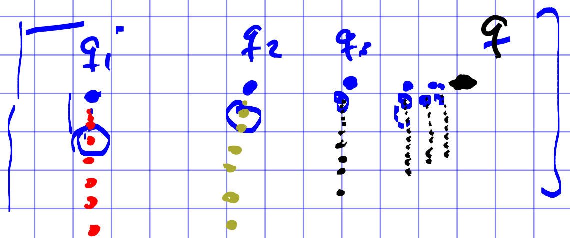
$x_n = (-1)^n$, $(x_n)_n$ $\quad E^* = \{-1, 1\} = \{-1\} \cup \{1\}$

Proof. (A closed $\iff A = \overline{A} (= A \cup A') \iff A' \subset A$)

It suffices to prove that $(E^*)' \subset E^*$

WLOG we can assume that $(E^*)' \neq \emptyset$.

Let $q \in (E^*)'$. Then $\forall n \geq n_0$ $[\exists q_n \in E^*]$: $q_n \neq q$: $d(q_n, q) < \frac{1}{2^n}$.



For $q_1 \in E^*$ there exists a subsequence of $(p_n)_n$ convergent to q_1

Let p_{k_1} an element in that subsequence such that $d(p_{k_1}, q_1) < \frac{1}{2}$.

$[x \in A' \iff \forall \varepsilon > 0 \ \exists y \in A, y \neq x \ d(y,x) < \varepsilon]$

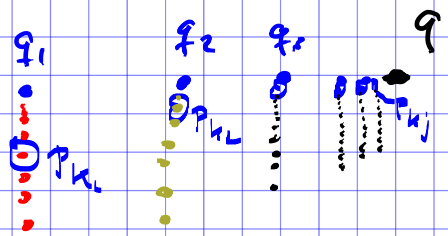
Since $q_2 \in E^*$, there exists a subsequence of $(p_n)_n$ convergent to q_2

So we can choose an element p_{k_2} of that subsequence such that

$$k_1 < k_2 \quad \text{and} \quad d(p_{k_2}, q_2) < \frac{1}{2^2}$$

Iterating the process we can construct $(k_j)_{j\geq 1}$ $\{k_j\} \subset \mathbb{N}$

$$k_1 < k_2 < k_3 < \dots \qquad d(p_{k_j}, q_j) < \frac{1}{2^j}$$



Then $(p_{k_j})_j$ is a subsequence of $(p_n)_n$ such that $\forall j \geq 1$

$$d(p_{k_j}, q) \leq \underbrace{d(q_{k_j}, q_j)}_{<\frac{1}{2^j}} + \underbrace{d(q_j, q)}_{<\frac{1}{2^j}} < \frac{2}{2^j} = \frac{1}{2^{j-1}}$$

Claim: It follows that $(p_{k_j})_j$ converges to q,

which implies that $[q \in E^*]$

#

Cauchy sequences

Def. Given $E \subset X$, one defines its diameter as

$$\mathrm{diam}(E) := \sup_{p,q \in E} d(p,q)$$

Def. A sequence $(x_n)_n$ is said to be a Cauchy sequence iff $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} : \forall n,m \geq n_0 \quad d(x_n, x_m) < \varepsilon$

Exercises: 1) Given a sequence $(x_n)_n$, lets define $E_n := \{x_n, x_{n+1}, x_{n+2}, \dots\}$ $n \geq 1$.

Show that TFAE

i/ $(x_n)_n$ is Cauchy

ii/ $\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \mathrm{diam}(E_n) < \varepsilon$

iii/ $\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \mathrm{diam}(E_{n_0}) < \varepsilon$

iv/ $\lim_{n \to \infty} \mathrm{diam}(E_n) = 0$

2) Study the validity of the following statement

A set E is bounded $\iff \mathrm{diam}(E) < +\infty$.

Thm

i) Every Cauchy sequence is bounded

ii) Every convergent sequence is Cauchy

iii) If a Cauchy sequence has a convergent subsequence with subsequential limit p, the sequence is convergent with limit p.

iv) Every Cauchy sequence in a compact metric space converges

v) In $\mathbb{R}^k$, every Cauchy sequence converges

[In $\mathbb{R}^k$, or in compact metric spaces
A sequence converges $\Leftrightarrow$ It is Cauchy.]

Cauchy criterion of convergence.

Proof i) Let $\varepsilon = 1$. $\exists n_0 : \forall n \geq n_0 \quad d(x_n, x_{n_0}) < 1$.

$$\text{Let } M = \max_{j=1,\dots,n_0} d(x_j, x_{n_0})$$

So we have that

$$\forall n \geq 1 \quad d(x_n, x_{n_0}) \leq 1 + M < +\infty.$$

ii) Exercise

iii) Ass. $(p_n)_n$ is Cauchy & $\exists (p_{n_k})_k$ $n_1 < n_2 < n_3 < \dots$ & $\exists p$

$$\lim_{k\to\infty} p_{n_k} = p \quad \left(\forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N} : \ \forall k \geq k_0 \ d(p_{n_k}, p) < \varepsilon \right)$$

Fix $\varepsilon > 0$.

$\exists n_0 \in \mathbb{N} \ \forall m, n \geq n_0 \quad d(p_m, p_n) < \varepsilon/2$. (Cauchy)

$\exists k_1 \in \mathbb{N}$ that we can assume $n_{k_1} \geq n_0$ such that $d(p_{n_{k_1}}, p) < \varepsilon/2$

Then $\forall n \geq n_0 \quad (m = n_{k_1})$

$$d(p_n, p) \leq d(p_n, p_{n_{k_1}}) + d(p_{n_{k_1}}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $p_n \to p$ as $n \to \infty$.

iv/ Exercise

v/ (Exercise)

#

Def. A metric space in which every Cauchy sequence converges is said to be **complete**

Examples

- $(\mathbb{R}^n, \|\cdot\|_2)$ is complete
- All compact metric spaces are complete.
- $(\mathbb{Q}, |\cdot|)$ is not complete

  Let $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad x_0 = 3/2$.

  $n \geq 0$

One can show

- $x_{n+1} \le x_n$ for all $n \ge 0$
- $(x_n)_n$ is bounded.
- $x_n \in \mathbb{Q}$ $\forall n \ge 0$
- $(x_n)_n$ is Cauchy

BUT the limit of $(x_n)_n$ does not exists (in $\mathbb{Q}$)

One can indeed <u>complete</u> $\mathbb{Q}$ (construct a set that contains a copy of $\mathbb{Q}$, for which all Cauchy sequence has a limit): How does the process go?

One says that two Cauchy sequences $(x_n)_n, (y_n)_n$ are equivalent (and one writes $(x_n)_n \sim (y_n)_n$) if $\lim_{n\to\infty}(x_n - y_n) = 0$

$\to$ This defines a relation of equivalence in the set of Cauchy sequences: $\mathcal{C}$

- Forms a Ring
- I set of all $(x_n) \sim (0,0,\dots)$ | Ideal

Define $R = \mathcal{C}/\sim$ so elements of R are equivalence classes $[(x_n)_n] =: x$

One defines $\forall x, y \in \mathbb{R}$.

$x \cdot y = [(x_n \cdot y_n)_n]$

$x + y := [(x_n + y_n)_n]$

$\mathbb{0} := [(0, 0, 0, \dots)]$, $\mathbb{1} := [(1, 1, 1 \dots)]$

$|x| := [(|x_n|)_n]$

- Well defined & indep. of the representative of x and y.
- $(\mathbb{R}, +, \cdot)$ forms a Field.
- $\mathbb{Q} \hookrightarrow \mathbb{R}$ Injective.

  $q \longmapsto [(q, q, q, \dots)]$

After one shows that

i) $(\mathbb{R}, |\cdot|)$ is complete.

ii) We can give an order (compatible with the usual order in $\mathbb{Q}$)

iii) $(\mathbb{R}, |\cdot|)$ is arquimedian

iv) $\mathbb{Q}$ is dense in $(\mathbb{R}, |\cdot|)$

The advantage of this construction is that has the ingredients to prove that [Any metric space can be completed] (1418036 Adv. Real. Anal. I)