Introduction to Real Analysis

Lecture 3: Compact and Connected sets

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Questions?

Lecture Plan

- Compact spaces (Rudin 2.31-2.42)
- Connected sets (Rudin 2.45-2.47)

Section 1
Compact sets continued

Open covering

Given $E \subseteq X$, an open cover of E is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that

$$E \subseteq \bigcup_{\alpha} U_\alpha$$

Definition: Compact set

Given $K \subseteq X$, we say that it is compact if, for every open covering $\{U_\alpha\}_{\alpha \in A}$ there are finitely many $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Compact and closed

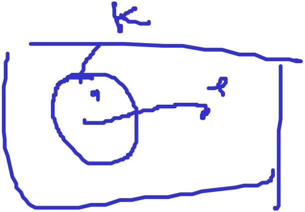
$(X, d) \qquad K \subseteq X$

Theorem

Compact sets are closed.

Proof $\quad \boxed{p \in K^c} \quad$ for all $q \in K$

$d(q) := d(p, q) > 0$



$W_q = N_{\frac{d(q)}{2}}(p)$

$V_q = N_{\frac{d(q)}{2}}(q)$

$V_a \cap W_a = \phi$

$K \subseteq \bigcup_{q \in K} V_q^{\ni q}$ open cover.

$\exists\, q_1 \ldots q_n \in K$ $\quad K \subseteq \bigcup_{1}^{n} V_{q_i}$

$d := \min\{d(q_1) \ldots d(q_n)\} \in \mathbb{R}$

Claim $(N_d(p) \cap K) \setminus \{p\} = \phi$

p is not a limit point

$N_d(p) \cap K$

$N_d(p) \subseteq \bigcap_{i=1}^{n} W_{a_i} \subseteq \bigcap_{1}^{n} (V_{q_i})^c$

$= \left(\bigcup_{1}^{n} V_{a_i} \right)^c$

$= (K)^c$

K^c open

$N_q(p) \cap K = \phi$!!

Compact and closed 2

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Theorem

Let K be a compact set and $C \subseteq K$ a closed set (relatively to X). Then C is compact.

Corollary

If C is closed and K is compact then $C \cap K$ is compact.

Intersection of compact

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Theorem

Let $\{K_\alpha\}$ a collection of compact sets such as any finite intersection is non-empty. Then,

$$\bigcap_\alpha K_\alpha \neq \emptyset$$

Corollary

If $\{K_n\}$ is a sequence of compact sets such that $K_n \supset K_{n+1}$ then we have that

$$\bigcap_n K_n \neq \emptyset$$

Limits in compact sets

Theorem

Any infinite subset of a compact set K has a limit point in K.

Compact subsets of $\mathbb{R}^n$

Let (X, d) be $\mathbb{R}^k$ with the Euclidean distance. We want to prove the following theorem

Theorem: Heine–Borel

For $E \subseteq \mathbb{R}^k$ the following are equivalent

1. E is closed and bounded
2. E is compact
3. every infinte subset of E has a limit point in E.

k-cells

A (closed) k-cell I is a subset of $\mathbb{R}^k$ which is a product of (closed) intervals. That is there are $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$, with $b_i \geq a_i$ such that

$$I = \{(x_1, \ldots, x_k) \in \mathbb{R} \mid a_i \leq x_i \leq b_i\}$$

If we set

$$\delta(I) := \left(\sum_{i=1}^{k} (b_i - a_i)^2 \right)^{\frac{1}{2}}$$

we have that

$$d(x, y) < \delta(I)$$

for all x and y in I.

Proposition

We have that $E \subseteq \mathbb{R}$ is bounded if, and only if, it is contained in a k-cell.

Two Lemmas

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Lemma

The intersection of a sequence of nested interval in $\mathbb{R}$ is not empty.

Lemma

The intersection of a sequence of nested k-cells in $\mathbb{R}^k$ is not empty.

The main technical step

Theorem

Every k-cell is a compact subset of $\mathbb{R}^k$.

Weierstrass Theorem

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Theorem

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Questions?

Section 2
Connected sets

Connected sets

Definiton (usual connected set)

A subset E of X is said to be connected if it cannot be written as union as a union of two disjoint, nonempty sets open with respect to the restricted metric

Connected sets

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In other words, if there are two open sets U_1 and U_2 in X such that

- $E \subseteq U_1 \cup U_2$
- $E \cap U_1 \cap U_2 = \emptyset$

Then $E \cap U_1 = \emptyset$ or $E \cap U_2 = \emptyset$

Connected sets - A la Rudin

Two subsets A and B of a metric space X are said to be separated if

$$\overline{A} \cap B = \emptyset \quad \text{and} \overline{B} \cap A = \emptyset$$

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The two definitions are equivalent. This is a nice exercise to do :).

Connected subset of the real line

A subset $E \subseteq \mathbb{R}$ is connected if, and only if, for all $x < y$ in E we have that $[x, y] \subseteq E$

Questions?

Thank you for your attention!