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Determination and Estimation of Risk Aversion Coefficients

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Abstract

In the paper we consider two types of utility functions often used in portfolio allocation problems, i.e. the exponential and quadratic utilities. We link the resulting optimal portfolios obtained by maximizing these utility functions to the corresponding optimal portfolios based on the minimum Value-at-Risk (VaR) approach. This allows us to provide analytic expressions for the risk aversion coefficients as functions of the VaR level. The results are initially derived under the assumption that the vector of asset returns is multivariate normally distributed and they are generalized to the class of elliptically contoured distributions thereafter. We find that the choice of the coefficients of risk aversion depends on the stochastic model used for the data generating process. Finally, we take the parameter uncertainty into account and present confidence intervals for the risk aversion coefficients of the considered utility functions. The theoretical results are validated in an empirical study. We conclude that investors fix their risk attitude in a two-step procedure: first, they choose an appropriate light- or heavy-tailed distribution for the asset returns; second, to hedge the remaining risk they choose higher or lower risk aversions respectively.

JEL Classification: G32, G11, G15, C13, C18, C44, C54, C58
Keywords: risk aversion, exponential utility, quadratic utility, elliptically contoured distributions, Laplace distribution, parameter uncertainty.

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1 Introduction

The von Neumann-Morgenstern expected utility theory provides a standard solution to modern asset allocation problems. The aim of the investor is to maximize the expected utility of future wealth with respect to the portfolio weights which denote the fractions of the individual assets in the portfolio. The future wealth or portfolio return is random implying that the expected utility depends on the parameters of the distribution used to model the data generating process of the underlying assets. An issue, which has to be addressed by the investor, is the choice of the functional form of the utility function. Typically, the analysis is constrained to the quadratic or exponential utilities, which allow for an explicit solutions of the portfolio problem. Other functions, like the power utility, require numerical techniques. Usually, most of the utility functions depend on an additional parameter referred to as a risk aversion coefficient. The value of this parameter is subjective and can be hardly justified by economic reasoning. In this paper, we argue that the risk aversion coefficient can be linked to the distribution used as a model for the data generating process and to the level of the Value-at-Risk (VaR) in which the investor is interested in or is required to report.

The quadratic utility function is commonly applied in portfolio theory because of its nice mathematical properties. First, an analytic solution is easy to obtain for the quadratic utility function. Second, Tobin (1958) showed that the Bernoulli principle is satisfied for the mean-variance solution only if one of the following two conditions is valid: the asset returns are normally distributed or the utility function is quadratic. Moreover, the quadratic utility presents a good approximation of other utility functions (see, e.g., Kroll et al. (1984), Brandt et al. (2006)). Levy and Markowitz (1979) considered the expected utility function in terms of the portfolio return and showed that it can be very well approximated by a function of the mean and the variance of the portfolio return. For a portfolio with the weights $\mathbf{w} = (w_1, ..., w_k)'$ such that $\mathbf{w}'\mathbf{1} = 1$ where $\mathbf{1}$ denotes the k-dimensional vector of ones, the quadratic utility is given by

$$U_{quad}(R_{\mathbf{w}}) = R_{\mathbf{w}} - \frac{\gamma_{quad}}{2} R_{\mathbf{w}}^2 = \mathbf{X}'\mathbf{w} - \frac{\gamma_{quad}}{2} (\mathbf{X}'\mathbf{w})^2 \,, \tag{1}$$

where $\mathbf{X} = (X_1, ..., X_k)'$ is the k-dimensional vector of asset returns. The symbol $\gamma_{quad} > 0$ stands for the risk aversion coefficient.

The second utility function considered in the paper is the exponential utility function given by

$$U_{exp}(R_{\mathbf{w}}) = 1 - e^{-\gamma_{exp} R_{\mathbf{w}}} \,, \tag{2}$$

where $\gamma_{exp} > 0$ is the corresponding risk aversion coefficient. If the asset returns are multivariate normally distributed then the maximization of $E(U_{exp}(R_{\mathbf{w}}))$ is equivalent to the so-called mean-variance utility function expressed as (cf. Ingersoll (1987), Okhrin and Schmid (2006, 2008))

$$\boldsymbol{\mu}'\mathbf{w} - \frac{\gamma_{mv}}{2} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \longrightarrow \max \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1 \,, \tag{3}$$

where $\boldsymbol{\mu} = E(\mathbf{X})$ and $\boldsymbol{\Sigma} = Var(\mathbf{X})$. Bodnar et al. (2013) compared the solutions resulting by maximizing $E(U_{quad}(R_{\mathbf{w}}))$, $E(U_{exp}(R_{\mathbf{w}}))$, and (3) and found that although they are mathematically equivalent, this statement does not longer hold from the stochastic point of view.

Note that the choice of the values for both γ_{quad} and γ_{exp} in practice is unclear. There are a few papers dealing with the estimation of the risk aversion coefficient from market data. For instance, Jackwerth (2000) derives the implied absolute risk-aversion coefficient by estimating the risk-neutral and historical probabilities from option prices, while estimators relying on the realized volatility were suggested by Bollerslev et al. (2011). It is important to note, that the corresponding risk aversions characterize an aggregate and not an individual investor. The usual values of γ's considered in empirical applications lie between 1 and 50 (for the quadratic utility) and the choice of the risk aversion coefficient is, usually, performed heuristically. Opposite to these studies, our approach is motivated by the financial interpretations of the solution of the expected utility maximization problems based on the utilities (1) and (2).

In the paper, we link the optimal portfolio obtained from a particular utility function to the minimum VaR optimal portfolio. The latter portfolio is determined by the significance level α, which can be fixed relying on the regulatory recommendations. This allows us to derive analytic expressions for $\gamma_{quad}(.)$ and $\gamma_{exp}(.)$ as functions of the VaR level. Thus, we quantify the investor's attitude towards risk and justify the values of the risk aversion coefficients which are usually used in practice. Furthermore, we argue that modelling the risk attitude is a step-two procedure. First, the investor selects a model for the asset returns. If he opts for a light-tailed distribution, for example the normal distribution, then in order to take into account potentially high losses from large returns, he has to choose a higher risk aversion in the second step. If, however, he chooses a heavy-tailed distribution, then the risk aversion can be smaller. This effect is documented in the empirical study.

The rest of the paper is organized as follows. In Section 2, we present the main results of the paper. Here, the analytical expressions for γ_{quad} and γ_{exp} are presented under the assumption that the asset returns are multivariate normally distributed. In Section 3, these findings are extended to non-normal distributions. The influence of the parameter uncertainty is analyzed in Section 4, while the results of the empirical study are shown in Section 5. Concluding remarks are presented in Section 6. In the appendix (Section 7) all proofs are given.

2 Risk aversion for Gaussian returns

The results of this section are derived assuming that the asset returns are multivariate normally distributed, i.e. $\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, whereas the findings under a more general class of distributions are presented in the next section.

We consider two investors who aim to maximize the expected quadratic utility function given by

$$E(U_{quad}(R_{\mathbf{w}})) \to \max \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1 \tag{4}$$

and the expected exponential utility function expressed as

$$E(U_{exp}(R_{\mathbf{w}})) \to \max \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1\,, \tag{5}$$

respectively.

Merton (1969) proved that if the asset returns are multivariate normally distributed then the maximization of the expected exponential utility function is equivalent to the maximization of the mean-variance utility (3). The solution of (3) coincides with the Markowitz efficient frontier which is a parabola in the mean-variance space (Merton (1972)) uniquely determined by the three parameters (cf. Bodnar and Schmid (2008b, 2009))

$$R_{GMV} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,, \quad V_{GMV} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,, \quad \text{and} \quad s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} \ \text{ with } \ \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{6}$$

The quantities R_{GMV} and V_{GMV} are the expected return and the variance of the global minimum variance (GMV) portfolio that determine the location of the parabola's vertex in the mean-variance space, while s is the slope parameter of the parabola. It measures the overall market profitability, i.e. it is equal to the squared excess return of a portfolio on the efficient frontier in comparison to the GMV portfolio for a unit increase in the variance.

The maximization of the exponential utility function, i.e. (5), leads to the optimal portfolio with the weights

$$\mathbf{w}_{EU} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + \gamma_{exp}^{-1}\mathbf{R}\boldsymbol{\mu}\,. \tag{7}$$

Similarly, the solution of (4) is given by

$$\mathbf{w}_{QU} = \frac{\mathbf{A}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{A}^{-1}\mathbf{1}} + \gamma_{quad}^{-1}\mathbf{R}_{\mathbf{A}}\boldsymbol{\mu}\,, \tag{8}$$

where $\mathbf{A} = E(\mathbf{X}\mathbf{X}')$ and $\mathbf{R}_{\mathbf{A}} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{1}\mathbf{1}'\mathbf{A}^{-1}}{\mathbf{1}'\mathbf{A}^{-1}\mathbf{1}}$. Bodnar et al. (2013) derived another expression for the weights of the optimal portfolio in the sense of maximizing the expected quadratic utility function expressed as

$$\mathbf{w}_{QU} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + \tilde{\gamma}_{quad}^{-1}\mathbf{R}\boldsymbol{\mu}\,, \tag{9}$$

with (see, Bodnar et al. (2013, Theorem 1))

$$\tilde{\gamma}_{quad} = \frac{1+s}{\gamma_{quad}^{-1} - 1 - R_{GMV}} \tag{10}$$

In the next step we use another way of constructing an optimal portfolio on the efficient frontier which characterizes the investor's attitude towards risk in a more natural way. A suitable candidate is the minimum value-at-risk (mean-VaR) portfolio suggested by Alexander and Baptista (2002, 2004). The VaR at the confidence level $\alpha \in (0.5, 1)$ (VaR_α) is defined as a portfolio loss satisfying

$$P\{\mathbf{X}'\mathbf{w} < -VaR_\alpha\} = 1 - \alpha\,.$$

If $\mathbf{X}$ is multivariate normally distributed then VaR_α is calculated implicitly and it is given by

$$VaR_\alpha = -\mathbf{w}'\boldsymbol{\mu} - z_{1-\alpha}\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}\,,$$

where $z_\beta = \Phi^{-1}(\beta)$ is the β-quantile of the standard normal distribution. In practice, the values of α are usually taken from the interval $[0.95, 1)$. The VaR is a standard method of risk monitoring suggested by the Basel Committee on Banking Supervision. Alexander and Baptista (2002) went beyond taking of VaR for monitoring purposes, but use the VaR as a risk proxy in portfolio management. The optimization problem is given by

$$VaR_\alpha = -\mathbf{w}'\boldsymbol{\mu} - z_{1-\alpha}\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}} \to \min, \quad \text{subject to } \mathbf{1}'\mathbf{w} = 1. \tag{11}$$

Alexander and Baptista (2002) proved that the solution of (11) lies on the efficient frontier and presented the expression for the weights of this portfolio. Bodnar et al. (2012) rewrote the formula for the weights of the minimum VaR portfolio in the form of (7) expressed as

$$\mathbf{w}_{VaR;\alpha} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{z_{1-\alpha}^2 - s}}\mathbf{R}\boldsymbol{\mu}\,. \tag{12}$$

The above results can be summarized as follows:

- It is important to note that all three solutions of the maximization problems in (4), (5) and (11) (cf. Bodnar et al. (2013)) lie on the Markowitz efficient frontier.

- Equations (7), (9) and (12) show that all optimal portfolios have the same structure. Moreover, from (10) we conclude that the maximization of the expected quadratic utility function with the risk aversion coefficient γ_{quad} leads to the same portfolio as the maximization of the expected exponential utility function with the risk aversion coefficient $\gamma_{exp} = \tilde{\gamma}_{quad}$ where the latter is given in (10).

- The risk aversion coefficients γ_{quad}, γ_{exp} and the α value of VaR are related. The explicit relation depends on the parameters of the efficient frontier, R_{GMV}, s and V_{GMV} only.

Using (12) we are able to specify the closed-form expressions for risk aversion coefficients γ_{quad} and γ_{exp} used in (1) and (2), respectively, such that the corresponding portfolios coincide with the mean-VaR portfolio. The results is summarized in the following theorem.

Theorem 1. *Let* $\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Then*

$$\gamma_{exp} = \frac{\sqrt{z_{1-\alpha}^2 - s}}{\sqrt{V_{GMV}}}, \tag{13}$$

$$\gamma_{quad} = \left(1 + R_{GMV} + \frac{1+s}{\gamma_{exp}}\right)^{-1} = \left(1 + R_{GMV} + \frac{(1+s)\sqrt{V_{GMV}}}{\sqrt{z_{1-\alpha}^2 - s}}\right)^{-1}. \tag{14}$$

The proof of Theorem 1 follows directly from the expressions for the weights given in (7), (9), and (12). For a given level of α the investor attempts to minimize the losses, which corresponds to the upper boundary of the most negative $\alpha{\cdot}\%$ of losses. Thus, α reflects the risk attitude of the investor. If α is small, then (s)he minimizes the extreme losses and, hence, the investor is very risk averse. If α is large or close to 0.5, then the investor cares about losses in general without paying particular attention to large losses. This implies that his risk aversion is moderate. In general it holds that both risk aversion are monotonously increasing in α.

On the first sight, the dependence of the risk aversion on the portfolio characteristics appears to be surprising. However, this evidence is natural since the investor is averse to a particular amount of loss or of risk, implying non-constant γ's. For a very risky portfolio the investor is more risk averse, than for a portfolio with a moderate risk. Moreover, the maximization of the expected quadratic utility function as well as of the expected exponential utility function leads to the portfolios which lie on the efficient frontier. As a results, the investor's attitude towards risk is expected to depend on the situation of the capital market, i.e. on the position of the efficient frontier in the mean-variance space which is fully determined by three parameters R_{GMV}, V_{GMV}, and s.

3 Determination of the Risk Aversion Coefficients for Elliptically Contoured Distributed Asset Returns

In this section we extend the results of Section 2 to elliptically contoured (EC) distributions. This is a large class of multivariate distributions which includes the multivariate normal and

t distributions as special cases. This class has been already discussed in financial literature. For instance, Owen and Rabinovitch (1983) extend Tobin's separation theorem, Bawa's rules of ordering certain prospects to EC distributions. While Chamberlain (1983) showed that elliptical distributions imply the mean-variance utility functions, Berk (1997) argued that one of the necessary conditions for the validity of the capital asset pricing model (CAPM) is an elliptical distribution for the asset returns. Furthermore, Zhou (1993) extended findings of Gibbons et al. (1989) by applying their test to EC distributed returns. A further test for the CAPM under elliptical assumptions is proposed by Hodgson et al. (2002). The application of matrix variate elliptically contoured distributions in portfolio theory is initiated by Bodnar and Schmid (2008a) and Bodnar and Gupta (2009).

In this section, we restrict ourselves to the class of EC distributions for which the density function exists. The vector of asset returns $\mathbf{X}$ is said to be elliptically contoured distributed if its density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = c_k g((\mathbf{x}-\boldsymbol{\mu})\mathbf{D}^{-1}(\mathbf{x}-\boldsymbol{\mu}))\,, \tag{15}$$

where $c_k > 0$ is a constant which depends on the specific type of elliptically contoured distribution, i.e. on the function $g(.)$, and the dimension of the vector $\mathbf{X}$ only. This assertion we denote by $\mathbf{X} \sim E_k(\boldsymbol{\mu}, \mathbf{D}, g)$. The symbol $\boldsymbol{\mu}$ is the location vector, while $\mathbf{D}$ denotes the dispersion matrix. If the second moment of $\mathbf{X}$ exists then

$$\boldsymbol{\mu} = E(\mathbf{X}) \qquad \text{and} \qquad \boldsymbol{\Sigma} = Cov(\mathbf{X}) = \omega\mathbf{D}\,,$$

i.e. $\boldsymbol{\mu}$ is the mean vector and the covariance matrix is proportional to $\mathbf{D}$ with $\omega = E(r^2)$ (see 16). The stochastic representation of the random vector $\mathbf{X}$ is a convenient tool for simulation purposes. If the density function exists for all $k \geq 1$ then the stochastic representation of $\mathbf{X}$ is given by (cf. Fang and Zhang (1990))

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + r\mathbf{D}^{1/2}\mathbf{Z}\,, \tag{16}$$

where $\mathbf{Z} \sim \mathcal{N}_k(\mathbf{0}_k, \mathbf{I}_k)$ is independent of the scalar nonnegative random variable r. The symbol $\mathbf{0}_k$ denotes the k-dimensional zero vector, while $\mathbf{I}_k$ stands for the identity matrix of order k. Moreover, from (16) it holds that r fully determines the type of elliptical contoured distribution.

Using (16) we can calculate the expected quadratic utility and the expected exponential utility under the assumption that the asset returns follow an EC distribution. For the quadratic utility function it holds that

$$\begin{aligned} E(U_{quad}(R_{\mathbf{w}})) &= E(\mathbf{X})'\mathbf{w} - \frac{\gamma_{quad}}{2}E\left((\mathbf{X}'\mathbf{w})^2\right) \\ &= \boldsymbol{\mu}'\mathbf{w} - \frac{\gamma_{quad}}{2}\mathbf{w}'\mathbf{A}\mathbf{w}\,. \end{aligned} \tag{17}$$

Consequently, the optimization problem based on (17) subject to $\mathbf{1}'\mathbf{w} = 1$ is the same as the one in the case of the multivariate normally distributed asset returns. As a result, its solution is given by (9).

Similarly, using that $R_{\mathbf{w}}|r \sim N(\boldsymbol{\mu}'\mathbf{w}, r^2\mathbf{w}'\mathbf{D}\mathbf{w})$ for the exponential utility function we get

$$\begin{aligned} E(U_{exp}(R_{\mathbf{w}})) &= 1 - E\left(e^{-\gamma_{exp}R_{\mathbf{w}}}\right) \\ &= 1 - E\left(E\left(e^{-\gamma_{exp}R_{\mathbf{w}}}|r\right)\right) \\ &= 1 - E\left(e^{-\gamma_{exp}\boldsymbol{\mu}'\mathbf{w} + \frac{\gamma_{exp}^2\mathbf{w}'\mathbf{D}\mathbf{w}}{2}r^2}\right)\,, \end{aligned}$$

where the last identity is obtained observing that the conditional expectation is the moment generating function of the univariate normal distribution at point $-\gamma_{exp}$. Hence,

$$
\begin{aligned}
E(U_{exp}(R_{\mathbf{w}})) &= 1 - e^{-\gamma_{exp}\boldsymbol{\mu}'\mathbf{w}} E\left(e^{\frac{\gamma_{exp}^2 \mathbf{w}'\mathbf{D}\mathbf{w}}{2} r^2}\right) \\
&= 1 - e^{-\gamma_{exp}\boldsymbol{\mu}'\mathbf{w}} m_{r^2}\left(\frac{\gamma_{exp}^2 \mathbf{w}'\mathbf{D}\mathbf{w}}{2}\right) \\
&= 1 - e^{-\gamma_{exp}\boldsymbol{\mu}'\mathbf{w}} m_{r^2}\left(\frac{\gamma_{exp}^2 \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}{2E(r^2)}\right), \qquad (18)
\end{aligned}
$$

where $m_{r^2}(t) = E\left(e^{tr^2}\right)$ is the moment generating function of r^2. As a result, the maximization of the expected exponential utility function is equivalent to

$$
\boldsymbol{\mu}'\mathbf{w} - \frac{1}{\gamma_{exp}} \log\left(m_{r^2}\left(\frac{\gamma_{exp}^2 \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}{2E(r^2)}\right)\right) \to \max \quad \text{subject to} \quad \mathbf{1}'\mathbf{w} = 1\,. \qquad (19)
$$

Lemma 1. *Let $\mathbf{X} \sim E_k(\boldsymbol{\mu}, \mathbf{D}, g)$ with the moment generating function of r^2 given by $m_{r^2}(.)$. Then the solution of the optimization problem (19) is given by*

$$
\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \tilde{\gamma}_{exp}^{-1}\mathbf{R}\boldsymbol{\mu}\,, \qquad (20)
$$

where $\tilde{\gamma}_{exp}$ is the solution of (with respect to κ)

$$
\kappa\psi'\left(\frac{\gamma_{exp}^2(V_{GMV} + \kappa^2 s)}{2E(r^2)}\right) = \frac{E(r^2)}{\gamma_{exp}} \qquad (21)
$$

with $\psi(x) = \log(m_{r^2}(x))$.

The proof of Lemma 1 is given in the appendix. The results of Lemma 1 are very interesting from the practical point of view. Although the maximization of the expected exponential utility function results in a challenging non-linear multivariate optimization problem, it can be simplified to an univariate one whose solution is determined by solving (21) with respect to κ. This result continues to be true independently how large is the dimension of the constructed portfolio. Finally, we note that if the asset returns are multivariate normally distributed then $r = 1$ and, consequently, $E(r^2) = 1$, $m_{r^2}(x) = e^x$, and $\psi'(x) = 1$. Putting these results together, from (21) we get $\kappa = \gamma_{exp}^{-1}$.

Next, we compare the solution of the maximization problems (4) and (5) under the assumption of elliptically distributed asset returns with the one obtained by minimizing the VaR at the confidence level α. In the case of elliptically contoured distribution, the VaR of the portfolio with weights $\mathbf{w}$ is given by

$$
VaR_\alpha = -\mathbf{w}'\boldsymbol{\mu} - d_{1-\alpha}\sqrt{\mathbf{w}'\mathbf{D}\mathbf{w}} = -\mathbf{w}'\boldsymbol{\mu} - \frac{d_{1-\alpha}}{\sqrt{E(r^2)}}\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}\,,
$$

where $d_{1-\alpha}$ depends on k and $g(.)$ only and it is independent of $\mathbf{w}$. Minimizing VaR_α with respect to $\mathbf{w}$ leads to the following expression for the weights

$$
\mathbf{w}_{VaR;\alpha} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{d_{1-\alpha}^2/E(r^2) - s}}\mathbf{R}\boldsymbol{\mu}\,. \qquad (22)
$$

Using (22) we are able to specify the closed-form expressions for the risk aversion coefficients γ_{quad} and γ_{exp}. It holds that

Theorem 2. *Let* $\mathbf{X} \sim E_k(\boldsymbol{\mu}, \mathbf{D}, g)$ *with the moment generating function of* r^2 *given by* $m_{r^2}(.)$. *Then*

$$\gamma_{quad} = \left(1 + R_{GMV} + \frac{(1+s)\sqrt{V_{GMV}}}{\sqrt{d^2_{1-\alpha}/E(r^2) - s}}\right)^{-1} .$$

Let $\psi(x) = \log(m_{r^2}(x))$. *Then* γ_{exp} *is the solution of*

$$\gamma_{exp}\psi'\left(\frac{\gamma^2_{exp}(V_{GMV} + \kappa^2 s)}{2E(r^2)}\right) = \frac{E(r^2)}{\kappa} ,$$

with $\kappa = \frac{\sqrt{V_{GMV}}}{\sqrt{d^2_{1-\alpha}/E(r^2)-s}}$.

The proof of Theorem 2 is given in the appendix.

4 Estimation and Inference Procedure

In this section we deal with the problem of parameter uncertainty. The parameters of asset returns, i.e. $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are unknown and have to be estimated from a sample. Replacing the population parameters with their sample counterparts in the equations for the risk aversion coefficients we obtain the estimated risk aversion coefficients. In order to access their statistical properties we derive useful stochastic representations of the estimated risk aversion coefficients.

Let $\mathbf{X}_1, ..., \mathbf{X}_n$ be a sample of asset returns used to estimate the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{X}_j - \hat{\boldsymbol{\mu}})(\mathbf{X}_j - \hat{\boldsymbol{\mu}})' . \tag{23}$$

Substituting $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ from (23) in (6), the estimators for the three parameters of the efficient frontier R_{GMV}, V_{GMV}, and s are obtained, namely,

$$\hat{R}_{GMV} = \frac{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}},\ \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}},\ \hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}} \ \text{ with } \ \hat{\mathbf{R}} = \hat{\boldsymbol{\Sigma}}^{-1} - \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} . \tag{24}$$

Because $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are random quantities, the estimated characteristics in (24) are random too. Assuming that the asset returns are iid and normal, Bodnar and Schmid (2008b, 2009) derived the exact distributions of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$. Let $\phi(\cdot)$ be the density function of the standard normal distribution. By $f_{\chi^2_n}(\cdot)$ we denote the density of the χ^2-distribution with n degrees of freedom, while $f_{F_{n_1,n_2,\lambda}}(\cdot)$ stands for the density of the non-central F-distribution with n_1 and n_2 degrees of freedom and the non-centrality parameter λ. The symbol $f_{N(\mu,\sigma^2)}(.)$ is used for the density function of the normal distribution with mean μ and variance σ^2.

In the following lemma we summarize some results of Bodnar and Schmid (2008b, 2009). Particularly, we provide the exact joint and marginal distributions of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$.

Lemma 2. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample of independent vectors such that $\mathbf{X}_i \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ for $i = 1, \ldots, n$ and $n > k$. Let $\mathbf{\Sigma}$ be positive definite. Then it holds that*

a) $\hat{V}_{GMV}$ *is independent of* $(\hat{R}_{GMV}, \hat{s})$.

b) $(n-1)\hat{V}_{GMV}/V_{GMV} \sim \chi^2_{n-k}$.

c) $\frac{n(n-k+1)}{(n-1)(k-1)}\hat{s} \sim F_{k-1,n-k+1,n\,s}$.

d) $\hat{R}_{GMV}|\hat{s} = y \sim \mathcal{N}\left(R_{GMV}, \frac{1+\frac{n}{n-1}y}{n}V_{GMV}\right)$.

e) The joint density function of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ is given by

$$f_{\hat{R}_{GMV},\hat{V}_{GMV},\hat{s}}(x,y,z) = \frac{n(n-k+1)}{(k-1)V_{GMV}} f_{\chi^2_{n-k}}\left(\frac{n-1}{V_{GMV}}z\right)$$
$$\times \quad f_{N(R_{GMV},\frac{1+\frac{n}{n-1}y}{n}V_{GMV})}(x) f_{F_{k-1,n-k+1,n\,s}}\left(\frac{n(n-k+1)}{(n-1)(k-1)}y\right).$$

The application of the closed-form expressions for the risk aversion coefficients derived in Theorem 1 leads to the following estimators of these quantities given by

$$\hat{\gamma}_{exp} = \frac{\sqrt{z^2_{1-\alpha} - \hat{s}}}{\sqrt{\hat{V}_{GMV}}}, \tag{25}$$

$$\hat{\gamma}_{quad} = \left(1 + \hat{R}_{GMV} + \frac{(1+\hat{s})\sqrt{\hat{V}_{GMV}}}{\sqrt{z^2_{1-\alpha} - \hat{s}}}\right)^{-1}. \tag{26}$$

The formulas (25) and (26) show that the risk aversion coefficients can be estimated only if $\hat{s} < z^2_{1-\alpha}$. Moreover, the corresponding population quantities can be interpreted if $s < z^2_{1-\alpha}$ only. These two observations show that we are not able to derive the unconditional distributions of $\hat{\gamma}_{exp}$ and $\hat{\gamma}_{exp}$ but only the corresponding conditional distributions provided that $\hat{s} < d^2_{1-\alpha}$. Following the approach of Bodnar et al. (2012) we first establish the conditional distributions of $\hat{\gamma}_{quad}$ and $\hat{\gamma}_{exp}$ given $\hat{s} = s^*$ and generalize thereafter.

From the results of Lemma 2 we obtain the following stochastic representations of $\hat{R}_{GMV}$ and $\hat{V}_{GMV}$ given $\hat{s} = s^*$, denoted by $\hat{R}^*_{GMV}$ and $\hat{V}^*_{GMV}$. They are expressed as

$$\hat{R}^*_{GMV} \stackrel{d}{=} R_{GMV} + \sqrt{\frac{1}{n} + \frac{s^*}{n-1}}\sqrt{V_{GMV}}\xi_1, \qquad \hat{V}^*_{GMV} \stackrel{d}{=} \frac{V_{GMV}}{n-1}\xi_2, \tag{27}$$

where $\xi_1 \sim N(0,1)$ and $\xi_2 \sim \chi^2_{n-k}$ are independently distributed. Then the stochastic representations of $\hat{\gamma}_{exp}$ and $\hat{\gamma}_{quad}$ under the condition $s^* = \hat{s}$, denoted by $\hat{\gamma}^*_{exp}$ and $\hat{\gamma}^*_{quad}$ respectively, are obtained and they are presented in Theorem 4.

Theorem 3. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample of independent vectors such that $\mathbf{X}_i \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ for $i = 1, \ldots, n$ and $n > k$. Let $\mathbf{\Sigma}$ be positive definite. Then it holds that*

$$\hat{\gamma}^*_{exp} \stackrel{d}{=} \frac{\sqrt{z^2_{1-\alpha} - s^*}\sqrt{n-1}}{\sqrt{V_{GMV}}\sqrt{\xi_2}}, \tag{28}$$

$$\hat{\gamma}^*_{quad} \stackrel{d}{=} \left(1 + R_{GMV} + \sqrt{\frac{1}{n} + \frac{s^*}{n-1}}\sqrt{V_{GMV}}\xi_1 + \frac{(1+s^*)\sqrt{V_{GMV}}\sqrt{\xi_2}}{\sqrt{n-1}\sqrt{z^2_{1-\alpha} - s^*}}\right)^{-1}. \tag{29}$$

The results of Theorem 3 possess several interesting applications. First, for simulating $\hat{\gamma}^*_{exp}$ and $\hat{\gamma}^*_{quad}$ it is not necessary to generate n independent k-variate normally distributed random vectors. It is enough to simulate two independent random variables ξ_1 and ξ_2 from the well-known univariate distributions and then to apply the expressions of Theorem 3. Second, Theorem 4 is very useful for the derivation of the densities of $\hat{\gamma}^*_{exp}$ and $\hat{\gamma}^*_{quad}$.

5 Extension to Robust Portfolio Selection

The obtained results can be further extended to a general portfolio selection problem, i.e. to the case of the investor who aims to maximize the expected utility function expressed as

$$\max E(u(\mathbf{X}'\mathbf{w})) \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1\,, \tag{30}$$

where $u(.)$ is a utility function. If the distribution of asset returns is partially known then the portfolio selection problem (30) can be reformulated with robust optimization technique as follows (cf. Fabozzi et al. (2010))

$$\max \min_{\mathbf{X}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma})} E(u(\mathbf{X}'\mathbf{w})) \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1\,, \tag{31}$$

where the notation $\mathbf{X} \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ indicates that the distribution of $\mathbf{X}$ belongs to the class of k-dimensional distributions with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Let

$$V(\mathbf{w}) = \min_{\mathbf{X}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma})} E(u(\mathbf{X}'\mathbf{w})) \tag{32}$$

denote the value of the inner minimization problem for given weights $\mathbf{w}$. Popescu (2007) proved that (32) is equivalent to an optimization problem with univariate distributions with a given mean and variance, i.e.

$$V(\mathbf{w}) = \min_{R_{\mathbf{w}}\sim(\mu_{\mathbf{w}},\sigma^2_{\mathbf{w}})} E(u(R_{\mathbf{w}}))\,. \tag{33}$$

Moreover, if $V(\mathbf{w})$ is continuous, non-decreasing in $\mu_{\mathbf{w}}$, non-increasing in $\sigma^2_{\mathbf{w}}$, and quasi-concave, then (31) is equivalent to the following quadratic optimization problem (cf. Popescu (2007))

$$\max\ \gamma\boldsymbol{\mu}'\mathbf{w} - (1-\gamma)\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1\,, \tag{34}$$

where $\gamma \in [0,1]$. Moreover, if $\mathbf{w}(\gamma)$ is the solution of (34) then $V(\mathbf{w}(\gamma))$ is continuous and unimodal in γ.

For a portfolio with the weights $\mathbf{w}$ and for a confidence level $\alpha \in (1/2, 1]$, Fabozzi et al. (2010) considered the robust version of the VaR, the so-called RVaR. Within our notations, the RVaR at the confidence level α is defined by

$$RVaR_\alpha = \max_{R_{\mathbf{w}}\sim(\mu_{\mathbf{w}},\sigma^2_{\mathbf{w}})} VaR_\alpha\,. \tag{35}$$

The application of Chebyshev's inequality leads to (see Alexander and Baptista (2002, Section 3.2))

$$RVaR_\alpha = -\mathbf{w}'\boldsymbol{\mu} + d_{1-\alpha}(\mathbf{w})\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}, \tag{36}$$

where $d_{1-\alpha} = 1/\sqrt{1-\alpha}$. The solutions of the optimization problem (34) and

$$RVaR_\alpha \to \min \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1$$

provide us the probabilistic interpretation of γ. Following the proof of Theorem 1 we get

$$\gamma = \left(1 + \frac{\sqrt{d_{1-\alpha}^2 - s}}{2\sqrt{V_{GMV}}}\right)^{-1}. \tag{37}$$

6 Empirical illustration

For illustration purposes we use monthly data for country indices from MSCI Developed Markets Indexes and from MSCI Emerging Markets Indexes. The markets cover 23 and 21 countries respectively and the time span from June 2004 till March 2014, resulting in 117 observations. To assess the impact of the dimension we consider portfolios consisting of k = 2, 5, 10, 15 assets. For simplicity we select the assets for the first part of the study in the alphabetic order. The characteristics of the frontier are summarized in Table 1. As expected the variance of the GMV portfolio V_{GMV} decreases with increasing k, the return R_{GMV} and the slope s increase. Furthermore, the return and variance of the global minimum variance portfolio are lower for the developed markets, compared to the emerging markets, which is consistent with our expectations.

	Developed markets			Emerging markets		
k	V_{GMV}	R_{GMV}	s	V_{GMV}	R_{GMV}	s
2	0.0049632	0.0088234	0.0225480	0.0044765	0.0072103	0.0019479
5	0.0026577	0.0136901	0.0597653	0.0036172	0.0071194	0.0292601
10	0.0021298	0.0090921	0.1335025	0.0030592	0.0103470	0.0956051
15	0.0010852	0.0029841	0.2105829	0.0014732	0.0099718	0.1044366
full	0.0006360	0.0054814	0.2927886	0.0012206	0.0075236	0.1441346

Table 1: Characteristics of the mean-variance frontier for the first k developed markets (left) and emerging markets (right) for the period from June 2004 to March 2014

Gaussian returns

The risk aversion coefficients γ_{quad} and γ_{exp} for Gaussian returns as functions of α are shown in Figure 1. We conclude that they are monotonously increasing in α. Thus if an investor is more concerned with extreme losses, then his implied risk aversion is higher. On the other hand the risk aversions are higher for less risky, i.e. larger and better diversified, portfolios. This seems to be counterintuitive, but in fact this evidence supports the idea that looking at high level VaR for less risky portfolios artificially inflates the aversion to risk. The same is observed by comparing the results for developed and emerging markets, where for the latter the implied risk aversion is lower despite of higher volatility. In general, for $\alpha > 0.95$ the risk aversion is high and the portfolio attains the GMV portfolio. Even for small values of α the risk aversion is much higher than the frequently used values from 1 to 10 for γ_{exp}. This implies that if the investors are interested in the commonly used 99%-Value-at-Risk, then they are much more risk averse, than imposed in the empirical studies.

In order to make the study robust to the choice of the indices we fix α at 99% and sample 200 portfolios of different sizes from both pools of indexes. For each portfolio we compute the

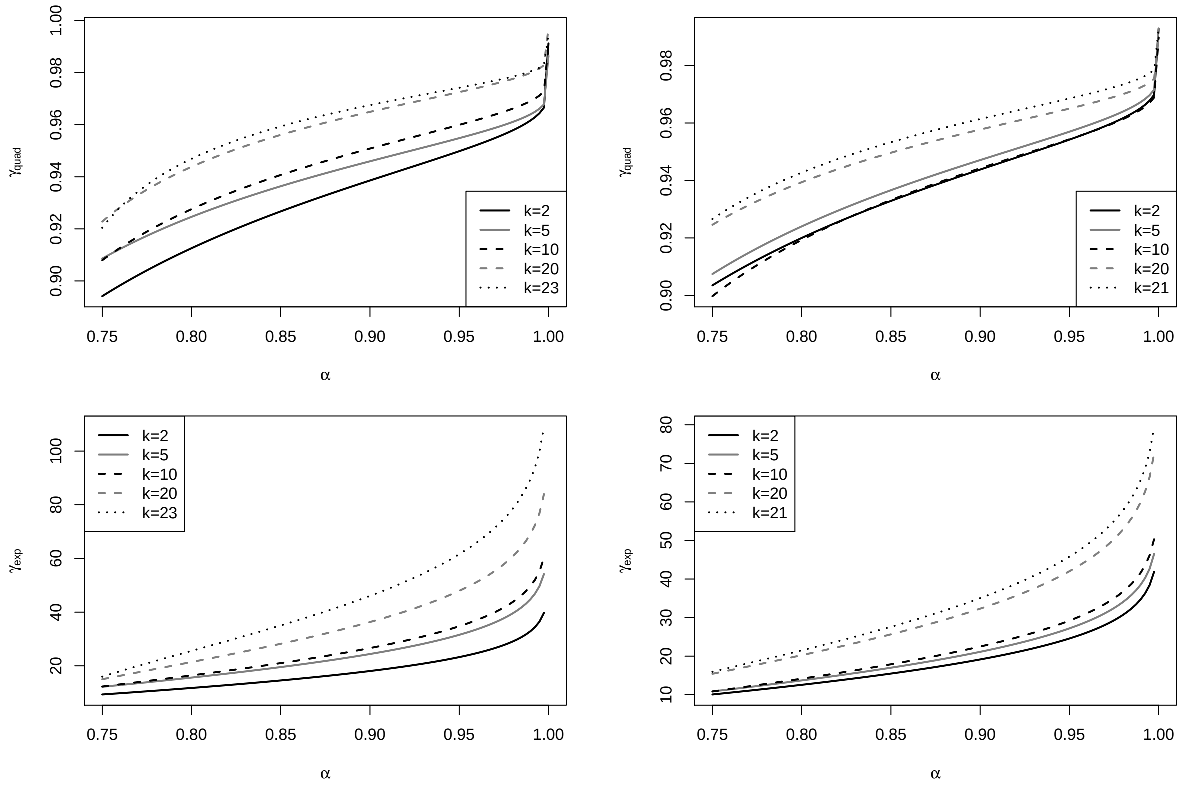

Figure 1: The risk aversion coefficients γ_{quad} (top) and γ_{exp} (bottom) as functions of α for portfolios consisting of the first k developed markets (left) and emerging markets (right).

corresponding risk aversion coefficients and plot their histograms in Figure 2. We conclude that the above conclusions are robust to the choice of markets. Small portfolios lead to systematically lower risk aversions compared to larger portfolios. Additionally we observe bimodal histograms, implying two classes of markets leading to different risk aversions.

Elliptical returns

To illustrate the theoretical results for elliptical distributions we concentrate on the multivariate Laplace distribution. It is obtained by assuming that r^2 follows exponential distribution with intensity equal to 1. In a univariate framework the resulting distribution of X_i is the Laplace distribution with the density $\frac{1}{2}\sqrt{\frac{2}{\lambda}}exp\left\{-\sqrt{\frac{2}{\lambda}}|x_i-\mu_i|\right\}$. The $1-\alpha$ quantile of this distribution is used as the $d_{1-\alpha}$-quantile in Theorem 2. Technical details on the multivariate Laplace distribution can be found in Eltoft *et al.* (2006), whereas Kotz et al. (2001) discuss the application of the multivariate Laplace distribution in portfolio theory.

The Laplace distribution has heavier tails compared to the normal and thus is a reasonable alternative in financial applications. Explicit application of this distribution is technically demanding due to complex expressions for the density (see Eltoft *et al.*, 2006). In our case, however, the stochastic representation allows us to work with r only. The corresponding risk aversion coefficients γ_{quad} and γ_{exp} as functions of α are shown in Figure 4. Similarly to the normal distribution, the risk aversion attains very high values even for modest levels of α. However, it is important to note that the values of the coefficients are lower than for the normal distribution. This implies that the investor reflects his aversion to risk in two steps: he

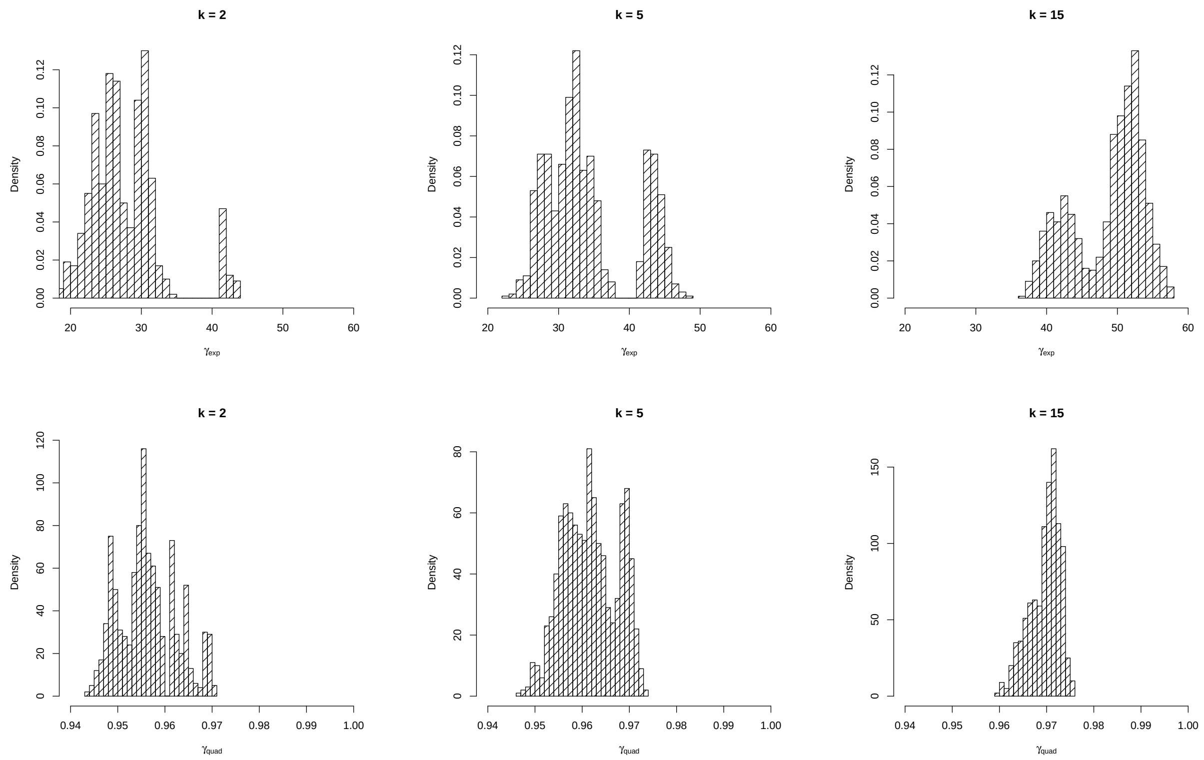

Figure 2: Histograms of risk aversion coefficients γ_{quad} (top) and γ_{exp} (bottom) for $\alpha = 0.99$ and portfolios consisting of $k = 2$ (left), $k = 5$ (middle) and $k = 15$ (right) randomly sampled emerging markets.

selects an appropriate model for the asset returns and the remaining risk aversion is captured by the risk aversion coefficient. If he opts for a light tailed distribution, then he is forced to choose a higher aversion coefficient to hedge against high losses. Alternatively, a heavy-tailed distribution takes high losses into account and allows for lower risk aversions.

Estimation risk

With the next example we illustrate the simulated density functions of the sample risk aversions $\hat{\gamma}^*_{quad}$ and $\hat{\gamma}^*_{exp}$ by relying on Theorem 3. We use the same data as in the above examples and condition on $s^* = \hat{s}$, where $\hat{s}$ is obtained individually for each portfolio. We observe that the precision of the estimators is relatively high. Note that the number of assets has an opposite impact on the precision for the two risk aversions. While for the exponential utility the densities are narrower for small portfolios, they become wider for the quadratic utility.

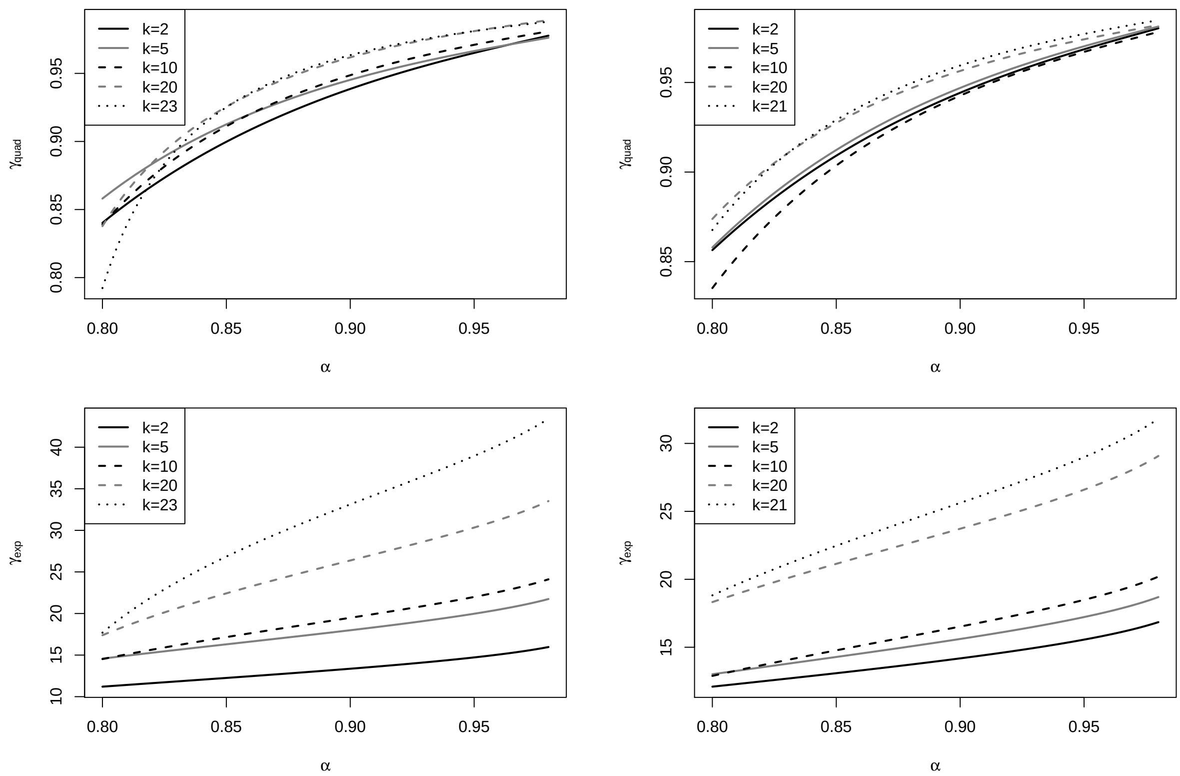

Figure 3: The risk aversion coefficients γ_{quad} (top) and γ_{exp} (bottom) as functions of α for portfolios consisting of the first k developed markets (left) and emerging markets (right) assuming multivariate Laplace distribution for the returns.

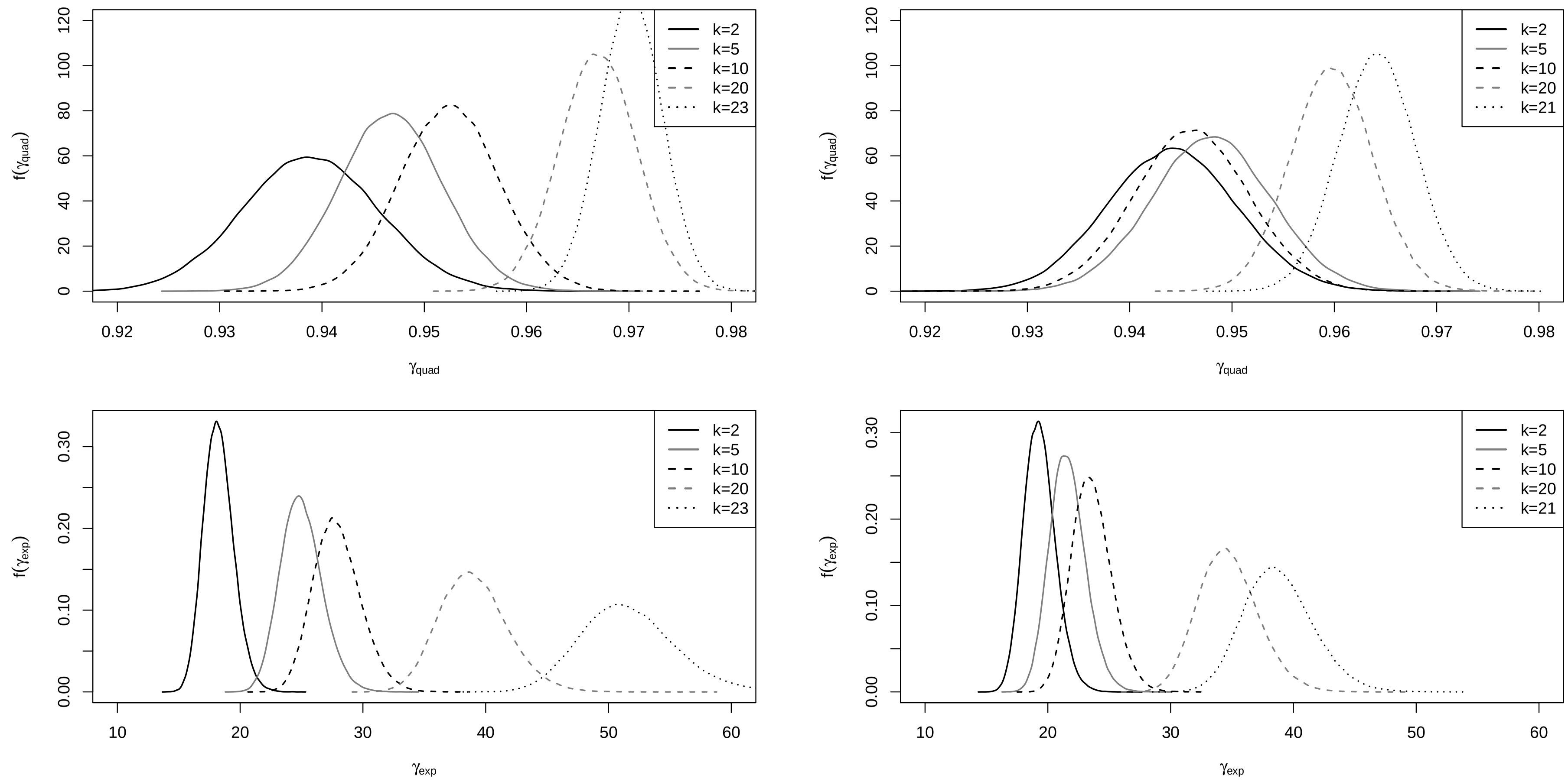

Figure 4: The conditional densities of γ^*_{quad} (top) and γ^*_{exp} (bottom) given s^* for portfolios consisting of the first k developed markets (left) and emerging markets (right) assuming multivariate normal distribution for the returns.

7 Summary

In the paper we consider the exponential and quadratic utility functions which are frequently applied in portfolio management. Since the VaR plays a key role in monitoring risk, many investors follow the minimum VaR portfolio strategies. We link both approaches to obtain a functional relationships between the risk aversions and the level of VaR. The results are obtained assuming that the vector of asset returns is multivariate normally distributed and they are generalized to the class of elliptically contoured distributions. The latter is particularly important due to well known heavy-taildness of asset returns. Finally, we take the parameter uncertainty into account and give conditional stochastic representation of the empirical risk aversion coefficients. The theoretical results are validated in an empirical study. We found evidence that investors model their risk attitude in a two-step procedure. At the first step they choose an appropriate light- or heavy-tailed distribution for the asset returns. In the second they hedge the remaining risk by choosing higher or lower risk aversions respectively.

8 Appendix

Proof of Lemma 1: First, we note that $\log(m_{r^2}(.))$ is an increasing function since $\log(.)$ and $m_{r^2}(.)$ are both increasing. Using this result we show next that the solution of (19) lies in the efficient frontier in the mean-variance space which consists of all portfolios such that it exists no portfolio with a larger expected return and a smaller variance (cf., Alexander and Baptista (2004, Definition 6)).

We prove the last statement by using the method from contradiction. Let $\tilde{\mathbf{w}}$ be a solution of (19) but the portfolio with the weights $\tilde{\mathbf{w}}$ does not lie in the efficient frontier. Then it exists a portfolio from the efficient frontier, say $\tilde{\mathbf{w}}_0$, such that $E(R_{\tilde{\mathbf{w}}_0}) \geq E(R_{\tilde{\mathbf{w}}})$ and $Var(R_{\tilde{\mathbf{w}}_0}) \leq Var(R_{\tilde{\mathbf{w}}})$ where at least one inequality is strict. Then

$$
\begin{aligned}
& \boldsymbol{\mu}'\tilde{\mathbf{w}} - \frac{1}{\gamma_{exp}} \log\left(m_{r^2}\left(\frac{\gamma_{exp}^2 \tilde{\mathbf{w}}'\boldsymbol{\Sigma}\tilde{\mathbf{w}}}{2E(r^2)} \right) \right) \\
= \;& E(R_{\tilde{\mathbf{w}}}) - \frac{1}{\gamma_{exp}} \log\left(m_{r^2}\left(\frac{\gamma_{exp}^2 Var(R_{\tilde{\mathbf{w}}})}{2E(r^2)} \right) \right) \\
< \;& E(R_{\tilde{\mathbf{w}}_0}) - \frac{1}{\gamma_{exp}} \log\left(m_{r^2}\left(\frac{\gamma_{exp}^2 Var(R_{\tilde{\mathbf{w}}_0})}{2E(r^2)} \right) \right)
\end{aligned}
$$

which shows that

$$E(U_{exp}(R_{\tilde{\mathbf{w}}})) < E(U_{exp}(R_{\tilde{\mathbf{w}}_0}))\,.$$

The last equality contradicts to the assumption that the portfolio with the weights $\tilde{\mathbf{w}}$ maximizes the expected exponential utility function.

Hence, the weights of the optimal portfolio in the sense of maximizing the expected exponential utility is given by

$$\mathbf{w}_{EU} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \kappa \mathbf{R}\boldsymbol{\mu}\,.$$

Substituting the last equality in (19) leads to

$$R_{GMV} + \kappa s - \frac{1}{\gamma_{exp}} \log\left(m_{r^2}\left(\frac{\gamma_{exp}^2 (V_{GMV} + \kappa^2 s)}{2E(r^2)} \right) \right) \tag{38}$$

which has to be maximized with respect to κ. Let $\psi(x) = \log(m_{r^2}(x))$. Equaling the derivative of (38) to zero leads to

$$s - \frac{\gamma_{exp}\kappa s}{E(r^2)}\psi'\left(\frac{\gamma_{exp}^2(V_{GMV}+\kappa^2 s)}{2E(r^2)}\right) = 0\,.$$

Hence, κ is the solution of

$$\kappa\psi'\left(\frac{\gamma_{exp}^2(V_{GMV}+\kappa^2 s)}{2E(r^2)}\right) = \frac{E(r^2)}{\gamma_{exp}}\,.$$

The lemma is proved.

Proof of Theorem 2: The equality for γ_{quad} follows from Theorem 1 and the fact that the expression for the weights of the minimum VaR portfolio in case of elliptically contoured distributions can be obtained from those in the case of normally distributed asset returns by replacing $z_{1-\alpha}^2$ with $d_{1-\alpha}^2/E(r^2)$, while the weights of the portfolio in the sense of maximizing the expected quadratic utility portfolio do not depend on the type of elliptically contoured distributions. The second result follows from (22) and Lemma 1. The theorem is proved.

References

[1] Alexander, G. J. and M. A. Baptista, (2002), Economic implication of using a mean-VaR model for portfolio selection: A comparison with mean-variance analysis. *Journal of Economic Dynamics & Control* **26**, 1159-1193.

[2] Alexander, G. J. and M. A. Baptista, (2004), A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. *Management Science* **50**, 1261-1273.

[3] Berk, J.B., (1997), Necessary conditions for the CAPM. *Journal of Economic Theory* **73**, 245-257.

[4] Bodnar, T. and A.K. Gupta, (2009), Construction and inferences of the efficient frontier in elliptical models, *Journal of the Japan Statistical Society* **39**, 193-207.

[5] Bodnar, T., Parolya, N., and W. Schmid, (2013a), On the equivalence of quadratic optimization problems commonly used in portfolio theory. *European Journal of Operational Research* **229**, 637-644.

[6] Bodnar, T. and W. Schmid, (2008a), A test for the weights of the global minimum variance portfolio in an elliptical model. *Metrika* **67**, 127-143.

[7] Bodnar, T. and W. Schmid, (2008b), Estimation of optimal portfolio compositions for gaussian returns. *Statistics & Decisions* **26**, 179-201.

[8] Bodnar, T. and W. Schmid, (2009), Econometrical analysis of the sample efficient frontier. *The European Journal of Finance* **15**, 317-335.

[9] Bodnar T., W. Schmid and T. Zabolotskyy, (2012), Minimum VaR and minimum CVaR optimal portfolios: estimators, confidence regions, and tests. *Statistics & Risk Modeling* **29**, 281-314.

[10] Bollerslev, T., M. Gibson, and H. Zhou, (2011), Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities, *Journal of Econometrics* **160**, 235-245.

[11] Brandt, M., and Santa-Clara, (2006), Dynamic portfolio selection by augmenting the asset space. *The Journal of Finance* **61**, 2187-2217.

[12] Chamberlain, G.A. (1983) A characterization of the distributions that imply mean-variance utility functions. *Journal of Economic Theory* **29**, 185-201.

[13] Eltoft, T., T. Kim, and T.-W. Lee, (2006), On the multivariate Laplace distribution, *IEEE Signal Processing Letters* **13**, 300-303.

[14] Fabozzi, F. J., D. Huang, and G. Zhou, (2010), Robust portfolios: contributions from operations research and finance, *Annals of Operations Research* **176**, 191-220.

[15] Fang, K.T. and Y.T. Zhang, (1990), *Generalized Multivariate Analysis.* Berlin: Springer-Verlag and Beijing: Science Press.

[16] Gibbons, M.R., S.A. Ross and J. Shanken, (1989), A test of the efficiency of a given portfolio. *Econometrica* **57**, 1121-1152.

[17] Hodgson D. J., O. Linton and K. Vorkink, (2002), Testing the capital asset pricing model efficiency under elliptical symmetry: a semiparametric approach. *Journal of Applied Econometrics* **17**, 617-639.

[18] Ingersoll, J.E., (1987), *Theory of Financial Decision Making*, Rowman&Littlefield Publishers.

[19] Jackwerth, J.C., (2000), Recovering risk aversion from option prices and realized returns, *Review of Financial Studies* **13**, 433-451.

[20] Kotz, S., T. Kozubowski, K. Podgórski, (2001), *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*, Birkhäuser: Boston.

[21] Kroll Y., H. Levy, and H.M. Markowitz, (1984), Mean-variance versus direct utility maximization. *The Journal of Finance* **39**, 47-61.

[22] Levy, H. and H. M. Markowitz, (1979), Approximating expected utility by a function of mean and variance. *American Economic Review* **69**, 308-317.

[23] Merton, R. C., (1969), Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *Review of Economics and Statistics* **50**, 247-257.

[24] Merton, R. C., (1972), An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis* **7**, 1851-1872.

[25] Popescu, I., (2007), Robust mean-covariance solutions for stochastic optimization, *Operations Research* **55**, 98-112.

[26] Okhrin Y. and W. Schmid, (2006), Distributional properties of portfolio weights. *Journal of Econometrics* **134**, 235-256.

[27] Okhrin Y. and W. Schmid, (2008), Estimation of optimal portfolio weights. *International Journal of Theoretical and Applied Finance* **11**, 249-276.

[28] Owen, J. and R. Rabinovitch, (1983), On the class of elliptical distributions and their applications to the theory of portfolio choice. *The Journal of Finance* **38**, 745-752.

[29] Tobin, J., (1958), Liquidity preference as behavior towards risk, *Review of Economic Studies* **25**, 65-86.

[30] von Neumann, J. and O. Morgenstern, (1944), *Theory of Games and Economic Behavior*, Princeton, New Jersey.

[31] Zhou, G., (1993), Asset-pricing tests under alternative distributions. *The Journal of Finance* **48**, 1927-1942.