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How Risky is the Optimal Portfolio Which Maximizes the Sharpe Ratio?

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Abstract

In this paper, we investigate the properties of the optimal portfolio in the sense of maximizing the Sharpe ratio (SR) and develop a procedure for the calculation of the risk of this portfolio. This is achieved by constructing an optimal portfolio which minimizes the Value-at-Risk (VaR) and at the same time coincides with the tangency (market) portfolio on the efficient frontier which is related to the SR portfolio. The resulting significance level of the minimum VaR portfolio is then used in the determination of the risk of both the market portfolio and the corresponding SR portfolio. However, the expression of this significance level depends on the unknown parameters which have to be estimated in practice. It leads to an estimator of the significance level whose distributional properties are investigated in detail. Based on these results, a confidence interval for the suggested risk measure of the SR portfolio is constructed and applied to real data. Both theoretical and empirical findings document that the SR portfolio is very risky since the corresponding significance level is smaller than 90% in most of the considered cases.

JEL Classification: C13, C18, C44, C54, G11

Keywords: tangency portfolio, Sharpe ratio, value-at-risk, parameter uncertainty, elliptically contoured distributions.

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1 Introduction

The problem of optimal portfolio selection has become very popular since the seminal paper of Markowitz in 1952. The main portfolio characteristics, which are used in the portfolio optimization suggested by Markowitz, are the expected return and the risk. The expected portfolio return is usually determined as the mean return. In contrast to the expected return, the specification of the portfolio risk appears to be a more complicated task. In Markowitz's portfolio theory, the variance is taken as a risk measure. Then optimal portfolios are constructed by minimizing the variance for a given level of the expected return or by maximizing the expected return for a given value of the variance. However, these optimization problems take into account only one characteristic of the portfolio, while the second is fixed.

Another possibility is to maximize the Sharpe ratio (SR) which is defined as a ratio of the expected portfolio return to the standard deviation (cf., Sharpe (1966, 1994)). It is noted that the optimal portfolio in the sense of maximizing the SR belongs to the efficient frontier in the case without a risk-free asset, i.e. it can be obtained as a solution of Makowitz's optimization problem. However, the choice of the variance in the definition of the SR is heavily criticized by both researchers and practitioners of the financial sector. The main argument is that the variance is not always an appropriate measure of risk since high returns might increase the variance. Better risk measures are based on the probability or the value of losses. In other words, it is desirable to have measures which depend only on the positive values of the loss function or negative values of the return and are known as downside risk measures (cf. Krokhmal et al. (2011)). Recent developments in risk theory suggest that quantile-based measures are well-suited functions to quantify risk. The most popular of each are the Value-at-Risk (VaR) and Conditional VaR (CVaR), also known as the expected shortfall (cf. Pflug (2000)). The portfolio selection problems based on minimizing the portfolio VaR (CVaR) have recently been considered in a number of literature studies (cf. Alexander and Baptista (2002, 2004), Rockafellar et al. (2006a,b), Kilianová and Pflug (2009)), whereas Bodnar et al. (2012, 2013b) took into account the problem of parameter uncertainty when the minimum VaR and the minimum CVaR portfolios are constructed.

Because the variance (standard deviation) is not an appropriate risk measure, several modifications of the SR were suggested in the literature. The most popular ones are the Treynor ratio and the Sortino ratio, defined as ratios of the expected return to the portfolio beta and the portfolio semi deviation, respectively. Recently, Bodnar and Zabolotskyy (2013) considered a generalization of the SR where the standard deviation is replaced by the VaR. Moreover, it was shown that under the assumption of normality imposed on the asset return distribution, the portfolio with the highest SR coincides with the portfolio that maximizes the modification of the SR based on the VaR. Similar results are also true in case of the modification of the SR where the portfolio standard deviation is replaced by the CVaR. Moreover, it appears that the resulting portfolio does not depend on the confidence level used in the calculation of the VaR (CVaR). This finding is in line with the results presented in Rockafellar et al. (2006a,b,c, 2007) who proved that the one-fund theorem, the capital asset pricing model (CAPM), and the existence of a market equilibrium hold for general deviations measures like standard lower semideviation, lower range semideviation, CVaR deviation, etc.

Alexander and Baptista (2002, 2004) showed that the minimum VaR (CVaR) optimal portfolio lies on the efficient frontier constructed in the case without a risk-free asset, if the asset returns are normally distributed. The inverse implication is also true, i.e. each Markowitz's (mean-variance) optimal portfolio can be presented as a solution of the minimization problem based on the VaR (CVaR). Because the optimal portfolio in the sense of maximizing the SR is also mean-variance efficient, there exists a confidence level for which this portfolio is also the minimum VaR (CVaR) portfolio (cf. Bodnar and Zabolotskyy (2013)). This means that the problem of optimal portfolio selection based on maximizing the SR can be interpreted as the problem of VaR (CVaR) minimization. Determining the confidence level of the VaR (CVaR) in the latter optimization problem is the main goal of the present paper. Because the confidence level used in the calculation of the VaR (CVaR) is directly related to the portfolio risk, it allows us to answer the question how risky the optimal portfolio with the highest SR is. Moreover, using the recommended values of the confidence level specified by the Basel Committee on Banking Supervision enables us to draw a conclusion wether the optimal portfolio in the sense of maximizing the SR could be considered as an investment opportunity at all or wether it leads to the portfolio which could possess high losses with a relatively high probability. This would not be acceptable.

Furthermore, we take the problem of parameter uncertainty into account when the confidence level of the optimal portfolio in the sense of maximizing the SR is determined. It appears that under the assumption of elliptically contoured distributed asset returns, this confidence level is only a function of the mean vector and the covariance matrix of the asset returns. However, these two quantities are unknown in practice and have to be replaced by the corresponding estimators. It leads to an estimator of the confidence level. In this paper, we derive the asymptotic distribution of the resulting estimator which is then used for determining the confidence interval. Moreover, using a one-to-one correspondence between interval estimation (confidence interval) and test theory (Lehmann and Romano (2005)), the obtained results directly lead to a test for the confidence level of the optimal portfolio with the highest SR. The theoretical results of this paper are implemented in an empirical study. Here, we show that the optimal portfolio in the sense of maximizing the SR is usually very risky in comparison to the recommended (for example Basel Committee on Banking Supervision, Risk Metrics) levels of risk which should be used in practice.

The rest of the paper is organized as follows. In the next section, we link the optimal portfolio in the sense of maximizing the SR with the minimum VaR portfolio which lies on the same place of the efficient frontier constructed in the case without a risk-free asset as the tangency portfolio which is related to the SR portfolio. The confidence level used in the calculation of the minimum VaR portfolio is determined in Theorem 1. In Section 3, we take the parameter uncertainty into account and construct an estimator for the suggested risk measure of the SR portfolio. Its distributional properties are investigated in Theorem 2 and an asymptotic confidence interval is constructed. In Section 4, the theoretical results are implemented in an empirical study. Final remarks are presented in Section 5.

2 Maximum SR Portfolio as a Minimum VaR Portfolio

Let $\mathbf{X}$ be the k-dimensional vector of risky asset returns with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $Var(\mathbf{X}) = \boldsymbol{\Sigma}$. We assume that a risk-free asset with return r_0 is available and the investor constructs his portfolio by investing into the k risky assets and the risk-free asset.

The objective of the investor is to maximize the Sharpe ratio given by

$$SR_{\mathbf{w}} = \frac{R_{\mathbf{w}} - r_0}{\sqrt{V_{\mathbf{w}}}}\,, \tag{1}$$

where $R_{\mathbf{w}} = \boldsymbol{\mu}'\mathbf{w} + w_0 r_0$ and $V_{\mathbf{w}} = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ are the expected return and the variance of the portfolio with the weights $(\mathbf{w}', w_0)'$. Throughout the paper it is assumed that the whole investor wealth is invested into the selected risky assets and the risk-free asset as well as that the short selling is allowed, i.e. $\mathbf{w}'\mathbf{1} + w_0 = 1$ where $\mathbf{1}$ stands for the k-dimensional vector of ones.

The maximization of (1) under the condition $\mathbf{w}'\mathbf{1} + w_0 = 1$ leads to the set of optimal portfolios known as the efficient frontier in the case of the presence of a risk-free asset. The weights of these optimal portfolios are given by (see, e.g., Luenberger (1998))

$$\mathbf{w}_{SR} = \begin{pmatrix} \eta\boldsymbol{\mu}'\mathbf{w}_{TP} \\ (1-\eta) \end{pmatrix}, \tag{2}$$

where

$$\mathbf{w}_{TP} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})} \tag{3}$$

are the weights of the tangency portfolio. This portfolio lies on the intersection of the efficient frontier constructed in the case without a risk-free asset, which is a parabola in the mean-variance space (c.f. Merton (1972)), and the efficient frontier in the case with a risk-free asset, which is a tangency line to the above parabola. It is also known as the market portfolio. The coefficient η specifies the optimal portfolio from the efficient frontier constructed in the case with a risk-free asset. If $\eta \in (0,1)$, then a part of the investor wealth is invested into the risk-free asset which reduces the portfolio risk. If $\eta > 1$, then the investor is more risky than the one who invests into the TP.

In order to capture the risk of portfolio (2), we first specify the risk of the TP used in its construction. The weights of the TP can be rewritten in the following way

$$\begin{aligned}
\mathbf{w}_{TP} &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \left(\frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right) \\
&= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}\left(\frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} - \boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right) \\
&= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{V_{GMV}}{R_{GMV} - r_0}\mathbf{Q}\boldsymbol{\mu}\,,
\end{aligned} \tag{4}$$

where $\mathbf{Q} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$; $R_{GMV} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$ and $V_{GMV} = 1/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$ are the expected return and the variance of the global minimum variance (GMV) portfolio which is the optimal portfolio with the smallest possible variance in case without a risk-free asset. The expression (4) determines also the weights of the optimal portfolio which maximizes the mean-variance utility function $\boldsymbol{\mu}'\mathbf{w} - \frac{\gamma}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ (cf. Bodnar and Schmid (2011)) with the coefficient of the investor's risk aversion equal to $\gamma = (R_{GMV} - r_0)/V_{GMV}$. It is noted that the maximization of the mean-variance utility leads to the same set of optimal portfolios which is obtained by solving Markowitz's optimization problem in the case without a risk-free asset under some conditions (see, e.g., Baron (1977), Bodnar et al. (2013a)). As a result, the TP lies on the efficient frontier in case without a risk-free asset, i.e. it is mean-variance efficient.

Recently, Alexander and Baptista (2002) proved that any mean-variance optimal portfolio in the case without a risk-free asset can also be obtained as a solution to the problem of minimizing the portfolio VaR at some confidence level α. The VaR at the confidence level $\alpha \in (0.5, 1)$, namely VaR_α, is formally defined as the rate of return such that

$$P\{X_{\mathbf{w}} < -VaR_\alpha\} = 1 - \alpha\,,$$

where $X_{\mathbf{w}} = \mathbf{X}'\mathbf{w}$. Changing α, the investor gets different optimal portfolios from the efficient frontier in the case without a risk-free asset, where $\alpha = 1$ corresponds to the GMV portfolio.

If $\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the VaR can be presented as a function of the first two moments of $\mathbf{X}$. It holds that

$$VaR_\alpha(X_{\mathbf{w}}) = -\mathbf{w}'\boldsymbol{\mu} - z_{1-\alpha}\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}\,, \tag{5}$$

where $z_\beta = \Phi^{-1}(\beta)$ is the β-quantile of the standard normal distribution. In a general situation, the quantile $z_{1-\alpha}$ should be replaced by $d_{1-\alpha}(\mathbf{w})$ satisfying

$$P\left\{\frac{X_{\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} < d_{1-\alpha}(\mathbf{w})\right\} = 1 - \alpha\,.$$

This leads to

$$VaR_\alpha(X_{\mathbf{w}}) = -\mathbf{w}'\boldsymbol{\mu} - d_{1-\alpha}(\mathbf{w})\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}\,. \tag{6}$$

In the following, we assume that the quantity $d_{1-\alpha}(\mathbf{w})$ is not a function of $\mathbf{w}$, i.e. $d_{1-\alpha}(\mathbf{w}) = d_{1-\alpha}$. This property applies to many distributions used to model the asset returns, like, e.g., for elliptically contoured distributions (cf. Fang and Zhang (1990, Theorem 2.6.3), Gupta et al. (2013)). In this case, $d_{1-\alpha}(\mathbf{w})$ does not depend on $\mathbf{w}$ and it can be taken from the table for a chosen type of elliptically contoured distribution or approximated via a Monte Carlo study where only the type of elliptically contoured distribution has to be fixed. It is remarkable to note that the expressions (5) and (6) are also used in the calculation of CVaR under elliptically contoured distributions where the constant $z_{1-\alpha}$ ($d_{1-\alpha}$) should be replaced by another one which is the function of the significance level α only (cf., Fang and Zhang (1990)). This is an important property which allows us to deal with the VaR as a risk measure only and note that similar results can also be obtained in the case of CVaR.

For an arbitrary distribution of the asset returns, the expression (6) can also be considered as an upper bound of the VaR, obtained by applying Chebyshev's inequality (see Alexander and Baptista (2002, Section 3.2)). See also, Bonami and Lejeune (2009) and Grechuk et al. (2010) for further applications of the classical and generalized Chebyshev's inequalities in finance.

In this case, $d_{1-\alpha}(\mathbf{w}) = 1/\sqrt{1-\alpha}$ and the constructed portfolio corresponds to the worst-case scenario. For instance, the upper bound for the VaR at the confidence level $\alpha = 0.95$ is obtained by (6) with $d_{1-\alpha}(\mathbf{w}) = 4.4721$ (cf. Alexander and Baptista (2002, p. 1179)).

The minimum VaR portfolio at the confidence level α is obtained by minimizing the portfolio VaR subject to the constraint that the whole wealth is invested into the selected risky assets (cf. Alexander and Baptista (2002)). It leads to

$$VaR_\alpha(X_{\mathbf{w}}) \to \min, \quad \text{subject to } \mathbf{1}'\mathbf{w} = 1\,. \tag{7}$$

The exact solution of (7) was derived by Alexander and Baptista (2002) assuming that the asset returns are multivariate normally distributed. Here, we make use of the equivalent expression suggested by Bodnar et al. (2013b) obtained under the assumption of elliptically contoured distributions. It is given by

$$\mathbf{w}_{VaR} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + \frac{\sqrt{V_{GMV}}}{\sqrt{d_{1-\alpha}^2 - s}}\mathbf{Q}\boldsymbol{\mu}, \tag{8}$$

where

$$s = \boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \frac{R_{GMV}^2}{V_{GMV}}$$

is the slope parameter of the efficient frontier in the case without a risk-free asset. Because both the TP and the minimum VaR portfolio lie on the efficient frontier, then there exists such an α_{TP} that these portfolios coincide. We summarize this result in Theorem 1.

Theorem 1. *Let* $\mathbf{X}$ *be elliptically contoured distributed with the quantile function* d_β *of* $\dfrac{X_{\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu}}{\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}}$ *for some* $\mathbf{w}$ *and* $X_{\mathbf{w}} = \mathbf{w}'\mathbf{X}$*. We assume that* $\boldsymbol{\mu} \neq \mu\mathbf{1}$*. Then*[1]

$$d_{1-\alpha_{TP}}^2 = (\boldsymbol{\mu} - r_0\mathbf{1})'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})\,. \tag{9}$$

The proof of Theorem 1 follows from the definitions of R_{GMV}, V_{GMV}, and s as well as from the equality

$$\frac{V_{GMV}}{R_{GMV} - r_f} = \frac{\sqrt{V_{GMV}}}{\sqrt{d_{1-\alpha_{TP}}^2 - s}}$$

and from the fact that

$$s = \boldsymbol{\mu}'\mathbf{Q}\boldsymbol{\mu} = (\boldsymbol{\mu} - r_0\mathbf{1})'\mathbf{Q}(\boldsymbol{\mu} - r_0\mathbf{1})$$

because $\mathbf{Q}\mathbf{1} = \mathbf{0}$. It is remarkable that if $\boldsymbol{\mu} = \mu\mathbf{1}$, then the TP coincides with the GMV portfolio. In this case, the confidence level of the TP is the same as one of the GMV portfolio, i.e. $\alpha_{TP} = 1$ if $\boldsymbol{\mu} = \mu\mathbf{1}$ independently of r_0.

[1] From the properties of elliptically contoured distribution we get that the distribution of $\dfrac{X_{\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu}}{\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}}$ is independent of the vector $\mathbf{w}$.

Let F denote the univariate marginal distribution function of the elements in $\mathbf{X}$. Then from Theorem 1 we get

$$\alpha_{TP} = F\left(\gamma\sqrt{(\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}\right) . \tag{10}$$

The factor $\gamma = \sqrt{-\psi'(0)/2}$, where the function ψ is defined in the expression of the characteristic function of $\mathbf{X}$ by $E\left(\exp(i\mathbf{t}'\mathbf{X})\right) = \exp(i\boldsymbol{\mu}'\mathbf{t})\psi(\mathbf{t}'\mathbf{t})$, appears in the expression since the marginal quantiles of an elliptical distribution are defined from the distribution of $\dfrac{X_{\mathbf{w}} - \mathbf{w}'\boldsymbol{\mu}}{\sqrt{\mathbf{w}'\mathbf{D}\mathbf{w}}}$, where $\mathbf{D} = \boldsymbol{\Sigma}/\gamma$ is a dispersion matrix.

It is noted that the quantity α_{TP} can be interpreted as a risk measure of the TP in the sense that the TP minimizes the VaR at confidence level α_{TP}. If another confidence level is used in the calculation of VaR, then another portfolio has to be chosen because the TP is no longer optimal in this case. In particular, if α_{TP} is significantly smaller than 1, say $\alpha_{TP} < 0.9$, then the application of the TP becomes questionable since it could lead to large losses with a high probability.

Since the weights of the SR portfolio in (2) are defined as a linear combination of the TP weights, we also obtain that for any η the expected return of the SR portfolio is a linear function of the TP return. As a result, for any η, the risk of the SR portfolio can be defined by α_{TP} following the discussion above. This observation follows directly from the fact that for any other $\alpha \neq \alpha_{TP}$ the TP does not minimizes the VaR and, consequently, the corresponding SR portfolio would not be optimal in terms of the VaR.

Another important finding which follows from Theorem 1 is the determination of $r_0(\alpha)$ for which the TP minimizes the VaR at confidence level α. This result is obtained by noting that if r_0 decreases, then the TP moves towards the GMV portfolio along the efficient frontier in the case without a risk-free asset. We observe a similar behaviour for the minimum VaR portfolio as α increases. Hence, we get

$$r_0(\alpha) \leq R_{GMV} - \sqrt{V_{GMV}}\sqrt{d_{1-\alpha}^2 - s}\,. \tag{11}$$

Finally, we point out that a similar result to the one given in Theorem 1 can also be obtained for other optimal portfolios which are obtained by maximizing a modification of the SR (cf., Jorion (1997), Campbell et al. (2001), Bodnar and Zabolotskyy (2013)). For instance, the solution of the portfolio optimization problem based on the analogue of the SR where the standard deviation is replaced by the VaR coincides with the solution obtained by maximizing (1) in the case of normally distributed asset returns independently of the confidence level used in the computation of VaR. Similarly, the result presented in Theorem 1 can be extended to other portfolio optimization problems where the variance (standard deviation) is replaced by more sophisticated measures of risk (see, e.g., Rockafellar et al. (2006a,b)). The latter problem is not treated in the present paper and it is left for future research.

3 Estimation of the Confidence Level for the SR Portoflio

Although we obtain a closed-form expression for the confidence level of the TP portfolio in (10), this formula cannot be applied directly in practice since both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown parameters of the asset return distribution. These two quantities have to be estimated using historical data. Here, we make use of the sample estimators expressed as

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i \qquad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{X}_i - \hat{\boldsymbol{\mu}})(\mathbf{X}_i - \hat{\boldsymbol{\mu}})' \tag{12}$$

where $\mathbf{X}_1, ..., \mathbf{X}_n$ are independent realizations of the vector of asset returns.

Replacing the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (10) by the corresponding estimators given in (12), we get an estimator of α_{TP} denoted by $\hat{\alpha}_{TP}$ which is expressed as

$$\hat{\alpha}_{TP} = F\left(\gamma\sqrt{(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})}\right). \tag{13}$$

In this section, we also consider an estimator for the upper bound of $r_0(\alpha)$ given in (11). It is expressed as

$$\widehat{UB}(\alpha) = \hat{R}_{GMV} - \sqrt{\hat{V}_{GMV}}\sqrt{d^2_{1-\alpha_{TP}} - \hat{s}}\,. \tag{14}$$

Assuming that the asset returns are independently and multivariate elliptically contoured distributed, the asymptotic distributions of $\hat{\alpha}_{TP}$ and $\widehat{UB}(\alpha)$ are derived. These results are summarized in Theorem 2.

Theorem 2. *Let $\mathbf{X}_1, ..., \mathbf{X}_n$ be independently and elliptically contoured distributed with univariate marginal distribution function F (univariate marginal density f) and let $\boldsymbol{\mu} \neq \mu\mathbf{1}$. Let the characteristic function of $\mathbf{X}$ be given by $E(\exp(i\mathbf{t}'\mathbf{X})) = \exp(i\boldsymbol{\mu}'\mathbf{t})\psi(\mathbf{t}'\mathbf{t})$. Then,*

a) the asymptotic distribution of $\hat{\alpha}_{TP}$ is given by

$$\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP}) \xrightarrow{d} \mathcal{N}(0, \sigma^2_\alpha) \tag{15}$$

with

$$\sigma^2_\alpha = \gamma^2\left(1 + \frac{\psi''(0)}{2\left(\psi'(0)\right)^2}(\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})\right) f^2\left(\gamma\sqrt{(\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})}\right). \tag{16}$$

b) the asymptotic distribution of $\widehat{UB}(\alpha)$ is given by

$$\sqrt{n}(\widehat{UB}(\alpha) - UB(\alpha)) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{UB}) \tag{17}$$

with

$$\sigma^2_{UB} = V_{GMV}d^2_{1-\alpha_{TP}}\frac{2 + d^2_{1-\alpha_{TP}}\frac{\psi''(0)}{(\psi'(0))^2}}{2(d^2_{1-\alpha_{TP}} - s)}\,. \tag{18}$$

Proof. From Theorem 1.2.18 of Muirhead (1982) and Theorem 3.15 of Gupta et al. (2013), we get

$$\sqrt{n}\begin{pmatrix} \hat{\boldsymbol{\mu}}-\boldsymbol{\mu} \\ \text{vech}(\hat{\boldsymbol{\Sigma}})-\text{vech}(\boldsymbol{\Sigma}) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\mathbf{0},\boldsymbol{\Omega}\right)$$

as $n \longrightarrow \infty$ where the symbol vech stands for the **vech** operator, i.e., $\text{vech}(\mathbf{A}) = (a_{11},...,a_{k1},..., a_{ii},...,a_{ki},...a_{kk})'$ for an arbitrary symmetric matrix $\mathbf{A}=(a_{ij})$, and

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{\psi''(0)}{(\psi'(0))^2}\mathbf{D}_k^{+}(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+}{}' \end{pmatrix}.$$

In the following we make use of the operator **vec** defined by $\text{vec}(\mathbf{A}) = (a_{11},...,a_{k1},...,a_{1i},...,a_{ki}, a_{1k},...,a_{kk})'$. The symbol $\mathbf{I}_{k^2}$ denotes the identity matrix of order k^2; $\mathbf{K}_k$ is a commutation matrix; $\mathbf{D}_k$ is a $k^2\times k(k+1)/2$ duplication matrix such that $\mathbf{D}_k\text{vech}(\mathbf{A}) = \text{vec}(\mathbf{A})$ and $\mathbf{D}_k^{+} = (\mathbf{D}_k'\mathbf{D}_k)^{-1}\mathbf{D}_k'$ with the property $\mathbf{D}_k^{+}\text{vec}(\mathbf{A}) = \text{vech}(\mathbf{A})$ (c.f., Harville (1997)). Finally, the symbol $\otimes$ denotes the Kronecker product.

Next, we derive the asymptotic distribution of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ which is further used in the proof of the theorem. From the proof of Theorem 1 in Bodnar et al. (2009), we get

$$\begin{aligned}
\frac{\partial R_{GMV}}{\partial\boldsymbol{\mu}} &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,, \\
\frac{\partial V_{GMV}}{\partial\boldsymbol{\mu}} &= \mathbf{0}\,, \\
\frac{\partial s}{\partial\boldsymbol{\mu}} &= 2\mathbf{Q}\boldsymbol{\mu}\,, \\
\frac{\partial R_{GMV}}{\partial \text{vech}(\boldsymbol{\Sigma})} &= \frac{\partial(\text{vec}\boldsymbol{\Sigma}^{-1})'}{\partial\text{vech}\boldsymbol{\Sigma}}\left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right)\,, \\
\frac{\partial V_{GMV}}{\partial \text{vech}(\boldsymbol{\Sigma})} &= -V_{GMV}^{-2}\frac{\partial(\text{vec}\boldsymbol{\Sigma}^{-1})'}{\partial\text{vech}\boldsymbol{\Sigma}}(\mathbf{1}\otimes\mathbf{1})\,, \\
\frac{\partial s}{\partial \text{vech}(\boldsymbol{\Sigma})} &= -\frac{\partial(vec\boldsymbol{\Sigma}^{-1})'}{\partial\text{vech}\boldsymbol{\Sigma}}\Big(R_{GMV}^2(\mathbf{1}\otimes\mathbf{1}) - 2R_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) + (\boldsymbol{\mu}\otimes\boldsymbol{\mu})\Big)\,,
\end{aligned}$$

where (cf., Harville (1997, p. 368))

$$\frac{\partial(\text{vec}(\boldsymbol{\Sigma}^{-1}))'}{\partial\text{vech}\boldsymbol{\Sigma}} = -\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+}{}'\mathbf{D}_k'\,. \tag{19}$$

The application of the delta-method (cf., Theorem 3.7 in DasGupta (2008)) leads to

$$\sqrt{n}\left(\begin{pmatrix}\hat{R}_{GMV}\\ \hat{V}_{GMV}\\ \hat{s}\end{pmatrix} - \begin{pmatrix}R_{GMV}\\ V_{GMV}\\ s\end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_2^2 & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_3^2\end{pmatrix}\right),$$

where

$$
\begin{aligned}
\sigma_1^2 &= ((\partial R_{GMV}/\partial \boldsymbol{\mu})' \ (\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial R_{GMV}/\partial \boldsymbol{\mu})' \ (\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')', \\
\sigma_{12} &= ((\partial R_{GMV}/\partial \boldsymbol{\mu})' \ (\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial V_{GMV}/\partial \boldsymbol{\mu})' \ (\partial V_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')', \\
\sigma_{13} &= ((\partial R_{GMV}/\partial \boldsymbol{\mu})' \ (\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial s/\partial \boldsymbol{\mu})' \ (\partial s/\partial(\text{vech}\boldsymbol{\Sigma}))')', \\
\sigma_2^2 &= ((\partial V_{GMV}/\partial \boldsymbol{\mu})' \ (\partial V_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial V_{GMV}/\partial \boldsymbol{\mu})' \ (\partial V_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')', \\
\sigma_{23} &= ((\partial V_{GMV}/\partial \boldsymbol{\mu})' \ (\partial V_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial s/\partial \boldsymbol{\mu})' \ (\partial s/\partial(\text{vech}\boldsymbol{\Sigma}))')', \\
\sigma_3^2 &= ((\partial s/\partial \boldsymbol{\mu})' \ (\partial s/\partial(\text{vech}\boldsymbol{\Sigma}))')\boldsymbol{\Omega}((\partial s/\partial \boldsymbol{\mu})' \ (\partial s/\partial(\text{vech}\boldsymbol{\Sigma}))')'.
\end{aligned}
$$

We get that

$$
\begin{aligned}
\sigma_1^2 &= (\partial R_{GMV}/\partial \boldsymbol{\mu})'\boldsymbol{\Sigma}(\partial R_{GMV}/\partial \boldsymbol{\mu}) \\
&+ \frac{\psi''(0)}{(\psi'(0))^2}(\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma}))'\mathbf{D}_k^+(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}(\partial R_{GMV}/\partial(\text{vech}\boldsymbol{\Sigma})) \\
&= \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{\psi''(0)}{(\psi'(0))^2}\left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right)'\left(\frac{\partial(\text{vec}\boldsymbol{\Sigma}^{-1})'}{\partial\text{vech}\boldsymbol{\Sigma}}\right)' \\
&\times \mathbf{D}_k^+(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\frac{\partial(\text{vec}\boldsymbol{\Sigma}^{-1})'}{\partial\text{vech}\boldsymbol{\Sigma}}\left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right) \\
&= V_{GMV} + \frac{\psi''(0)}{(\psi'(0))^2}\left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right)' \\
&\times \mathbf{D}_k\mathbf{D}_k^+(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}_k\mathbf{D}_k^+(\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k' \\
&\times \left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right).
\end{aligned}
$$

Taking into account that for arbitrary $k \times k$ matrix $\mathbf{A}$ the following equalities hold

$$
\begin{aligned}
\mathbf{D}_k\mathbf{D}_k^+ &= \frac{1}{2}(\mathbf{I}_{k^2}+\mathbf{K}_k) = \mathbf{N}_k, \\
\mathbf{N}_k(\mathbf{A}\otimes\mathbf{A}) &= (\mathbf{A}\otimes\mathbf{A})\mathbf{N}_k, \\
\mathbf{D}_k\mathbf{D}_k^+(\mathbf{A}\otimes\mathbf{A})\mathbf{D}_k &= (\mathbf{A}\otimes\mathbf{A})\mathbf{D}_k, \\
\mathbf{N}_k &= \mathbf{N}_k^2 = \mathbf{N}_k'
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathbf{D}_k\mathbf{D}_k^+(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}_k\mathbf{D}_k^+\mathbf{N}_k(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\mathbf{D}_k^{+\prime}\mathbf{D}_k'(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}_k^{+\prime}\mathbf{D}_k' \\
&\quad = (\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{N}_k(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}) = \mathbf{N}_k(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\sigma_1^2 &= V_{GMV} + \frac{\psi''(0)}{(\psi'(0))^2}\left(V_{GMV}(\mathbf{1}'\otimes\boldsymbol{\mu}') - R_{GMV}V_{GMV}(\mathbf{1}'\otimes\mathbf{1}')\right) \\
&\times (\mathbf{I}_{k^2}+\mathbf{K}_k)(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1})\left(V_{GMV}(\mathbf{1}\otimes\boldsymbol{\mu}) - R_{GMV}V_{GMV}(\mathbf{1}\otimes\mathbf{1})\right) \\
&= V_{GMV} + \frac{\psi''(0)}{(\psi'(0))^2}sV_{GMV},
\end{aligned}
$$

where we use the property that for an arbitrary $k \times n$ matrix $\mathbf{A}$ and an arbitrary $k \times 1$ vector $\mathbf{a}$ it holds that $\mathbf{K}_k(\mathbf{A}\otimes\mathbf{a}) = (\mathbf{a}\otimes\mathbf{A})$.

Analogically, the quantities σ_{12}, σ_{13}, σ_{23}, σ_2^2, and σ_3^2 are calculated. Hence,

$$\sqrt{n}\left(\left(\begin{array}{c}\hat{R}_{GMV}\\ \hat{V}_{GMV}\\ \hat{s}\end{array}\right)-\left(\begin{array}{c}R_{GMV}\\ V_{GMV}\\ s\end{array}\right)\right)\xrightarrow{d}\mathcal{N}\left(\mathbf{0},\boldsymbol{\Xi}\right)\,. \tag{20}$$

$$\boldsymbol{\Xi}=\left(\begin{array}{ccc}V_{GMV}(1+s\frac{\psi''(0)}{(\psi'(0))^2}) & 0 & 0\\ 0 & 2V_{GMV}^2\frac{\psi''(0)}{(\psi'(0))^2} & 0\\ 0 & 0 & 4s+2s^2\frac{\psi''(0)}{(\psi'(0))^2}\end{array}\right)$$

a) Rewriting (13), we get

$$\hat{\alpha}_{TP}=F\left(\gamma\sqrt{(\hat{\boldsymbol{\mu}}-r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}}-r_0\mathbf{1})}\right)=F\left(\gamma\sqrt{\hat{s}+\frac{(\hat{R}_{GMV}-r_0)^2}{\hat{V}_{GMV}}}\right)\,.$$

The application of (20) and the delta method lead to

$$\sqrt{n}(\hat{\alpha}_{TP}-\alpha_{TP})\xrightarrow{d}\mathcal{N}(0,\sigma_\alpha^2)$$

where

$$\begin{aligned}
\sigma_\alpha^2 &= \left(\frac{f\left(\gamma\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}\right)}{2\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}}\right)^2\gamma^2\\
&\times\left(2\frac{(R_{GMV}-r_0)}{V_{GMV}},\,-\frac{(R_{GMV}-r_0)^2}{V_{GMV}^2},\,1\right)\boldsymbol{\Xi}\left(\begin{array}{c}2(R_{GMV}-r_0)/V_{GMV}\\ -(R_{GMV}-r_0)^2/V_{GMV}^2\\ 1\end{array}\right)\\
&= \left(\frac{f\left(\gamma\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}\right)}{2\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}}\right)^2\gamma^2\\
&\times\left(4\frac{(R_{GMV}-r_0)^2}{V_{GMV}}\left(1+s\frac{\psi''(0)}{(\psi'(0))^2}\right)+2\frac{(R_{GMV}-r_0)^4}{V_{GMV}^2}\frac{\psi''(0)}{(\psi'(0))^2}+4s+2s^2\frac{\psi''(0)}{(\psi'(0))^2}\right)\\
&= \frac{2\frac{\psi''(0)}{(\psi'(0))^2}\left(s+(R_{GMV}-r_0)^2/V_{GMV}\right)^2+4\left(s+(R_{GMV}-r_0)^2/V_{GMV}\right)}{4\left(s+(R_{GMV}-r_0)^2/V_{GMV}\right)}\\
&\times f^2\left(\gamma\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}\right)\gamma^2\\
&= \left(1+\frac{1}{2}\left(s+(R_{GMV}-r_0)^2/V_{GMV}\right)\frac{\psi''(0)}{(\psi'(0))^2}\right)f^2\left(\gamma\sqrt{s+(R_{GMV}-r_0)^2/V_{GMV}}\right)\gamma^2\\
&= \left(1+\frac{\psi''(0)}{2\left(\psi'(0)\right)^2}(\boldsymbol{\mu}-r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-r_0\mathbf{1})\right)f^2\left(\gamma\sqrt{(\boldsymbol{\mu}-r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-r_0\mathbf{1})}\right)\gamma^2\,.
\end{aligned}$$

b) Similarly, we get

$$\sqrt{n}(\widehat{UB}(\alpha)-UB(\alpha))\xrightarrow{d}\mathcal{N}(0,\sigma_{UB}^2)$$

where

$$\sigma^2_{UB} = \left(1, -\frac{\sqrt{d^2_{1-\alpha_{TP}} - s}}{2\sqrt{V_{GMV}}}, \frac{\sqrt{V_{GMV}}}{2\sqrt{d^2_{1-\alpha_{TP}} - s}}\right) \boldsymbol{\Xi} \begin{pmatrix} 1 \\ -\frac{\sqrt{d^2_{1-\alpha_{TP}}-s}}{2\sqrt{V_{GMV}}} \\ \frac{\sqrt{V_{GMV}}}{2\sqrt{d^2_{1-\alpha_{TP}}-s}} \end{pmatrix}$$

$$= V_{GMV}\left(1 + s\frac{\psi''(0)}{(\psi'(0))^2}\right) + \frac{d^2_{1-\alpha_{TP}} - s}{2} V_{GMV}\frac{\psi''(0)}{(\psi'(0))^2}$$

$$+ \left(4s + 2s^2\frac{\psi''(0)}{(\psi'(0))^2}\right)\frac{V_{GMV}}{4(d^2_{1-\alpha_{TP}} - s)}$$

$$= V_{GMV} d^2_{1-\alpha_{TP}} \frac{2 + d^2_{1-\alpha_{TP}}\frac{\psi''(0)}{(\psi'(0))^2}}{2(d^2_{1-\alpha_{TP}} - s)}$$

□

Consistent estimators for σ^2_α and σ^2_{UB} are provided in Theorem 3.

Theorem 3. *Let $\mathbf{X}_1, ..., \mathbf{X}_n$ be independently and elliptically contoured distributed with univariate marginal distribution function F (univariate marginal density f) and let $\boldsymbol{\mu} \neq \mu\mathbf{1}$. Let $\psi(.)$ denote the characteristic function of $\mathbf{X}$, i.e. $\psi(\mathbf{t}) = E(\exp(i\mathbf{t}'\mathbf{X}))$. Then, for $n \to \infty$*

$$\hat{\sigma}^2_\alpha = \left(1 + \frac{\psi''(0)}{2(\psi'(0))^2}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})\right) f^2\left(\sqrt{(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})}\right) \xrightarrow{a.s.} \sigma^2_\alpha \quad (21)$$

and

$$\hat{\sigma}^2_{UB} = \hat{V}_{GMV} d^2_{1-\alpha_{TP}} \frac{2 + d^2_{1-\alpha_{TP}}\frac{\psi''(0)}{(\psi'(0))^2}}{2(d^2_{1-\alpha_{TP}} - \hat{s})} \xrightarrow{a.s.} \sigma^2_{UB}\,. \quad (22)$$

Proof. The statement of the theorem follows directly from the proof of Theorem 2 and the Continuous Mapping Theorem (see, e.g., Theorem 1.14 in DasGupta (2008)). □

The application of Theorems 2 and 3 leads to the following $(1-\beta)$-confidence interval for α_{SR} expressed as

$$\left[F\left(\gamma\sqrt{(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})}\right) - \frac{\hat{\sigma}_\alpha}{\sqrt{n}} z_{1-\beta/2}, F\left(\gamma\sqrt{(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})}\right) + \frac{\hat{\sigma}_\alpha}{\sqrt{n}} z_{1-\beta/2}\right], \quad (23)$$

where $\hat{\sigma}_\alpha$ is given in (22). In a similar way, we construct the confidence interval for the upper bound of r_0. However, in this case we are interested in the upper one-sided interval which is expressed by

$$\left(-\infty, \hat{R}_{GMV} - \sqrt{\hat{V}_{GMV}}\sqrt{d^2_{1-\alpha_{TP}} - \hat{s}} + \frac{\hat{\sigma}_{UB}}{\sqrt{n}} z_{1-\beta}\right]. \quad (24)$$

It is noted that the results of Theorems 2 and 3 also hold if the assumption of independence is replaced by the condition of weak dependence or by the assumption of a strictly stationary process (cf. Bodnar et al. (2013b)).

3.1 Numerical Illustration

In this subsection, we analyse the finite sample distributional properties of $\hat{\alpha}_{TP}$. Theorem 2 shows that the asymptotic density function of $\hat{\alpha}_{TP}$ depends only on the type of elliptical distribution and on the quadratic form $\delta = (\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})$. Since the first quantity is assumed to be known in practical applications, the practitioners have only to calibrate the density function with respect to possible values of δ in order to study the asymptotic distributional properties of $\hat{\alpha}_{SR}$ via simulations. Another important question is to investigate how fast the finite sample distribution of $\hat{\alpha}_{TP}$ converges to the corresponding asymptotic one.

In Figures 1 and 2 we present the results by considering $k \in \{5, 15, 25\}$ and $n \in \{120, 250, 500, 1000\}$. Several values of δ are considered for each choice of k and n, namely $\delta \in \{0.1, 0.3\}$ in Figure 1 and $\delta \in \{0.5, 0.7\}$ in Figure 2. This range of the values for δ corresponds to the results of the empirical study presented in Section 4 (see Figure 4).

In all plots, we present the asymptotic density of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$ derived in Theorem 2 as well as the kernel density estimators with the Epanechnikov kernel calculated from the generated samples of size $n \in \{120, 250, 500, 1000\}$ from $k \in \{5, 15, 25\}$ dimensional multivariate t-distribution with 5 degrees of freedom. The location vector and the scale matrix of the t-distribution are fixed in such a way that the corresponding values of δ are obtained. The results of the simulation study are based on 10^4 independent repetitions.

The resulting densities appear to be roughly symmetric which is also true in case of a large-dimensional portfolio of 25 assets when the sample size is relatively small with respect to the portfolio dimension, namely $n = 120$. This finding illustrates that the finite sample density of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$ can be well approximated by a normal distribution and the resulting approximation performs very well also for moderate sample sizes. We also observe that the values of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$ are shifted to the right in case of $p = 15$ and $p = 25$. Consequently, the estimator $\hat{\alpha}_{TP}$ overestimates the true value of α_{TP}. The situation is improved when n increases.

In Table 1, we provide a further analysis of the stochastic behaviour of $\hat{\alpha}_{TP}$. Here, the probabilities $P(\hat{\alpha}_{TP} > \alpha)$ with $\alpha \in \{0.9, 0.95, 0.99\}$ are presented for different values of n ($n = 120$ – panel 1, $n = 250$ – panel 2, $n = 500$ – panel 3, $n = 1000$ – panel 4) and for the values of k and δ considered in Figures 1 and 2. We observe that the calculated probabilities are very close to zero in most of the considered cases. Significant deviations are present only for $k = 25$ and $\delta = 0.7$. This result illustrates that the optimal portfolio in the sense of maximizing the SR could be less risky only if a large-dimensional portfolio is constructed as well as the value of δ is sufficiently large. However, this observation seems to be related to the positive bias which is present in $\hat{\alpha}_{TP}$, but not to the true value of α_{TP}. The property disappears when n increases. Furthermore, also in this case the maximum SR portfolio corresponds to the minimum VaR portfolio at the significance value smaller than 0.9 with a high probability. This is an unpleasant property of the maximum SR portfolio and questions its practical applicability.

The positive bias observed in the densities of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$, especially for $k = 25$, is mainly related to the bias which is present in the estimation of $\delta = (\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})$. If

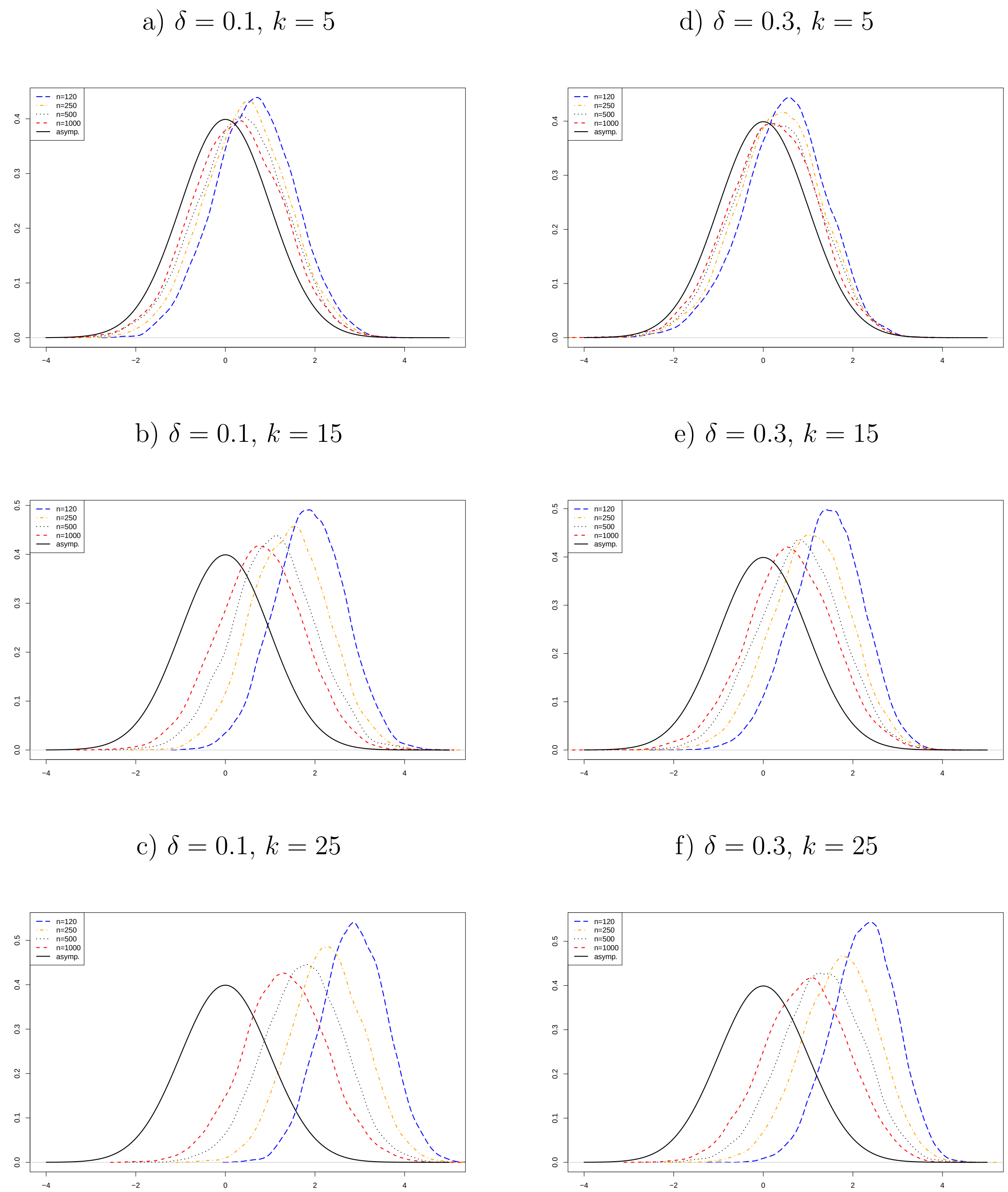

Figure 1: Exact and asymptotic densities of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$ for $k \in \{5, 15, 25\}$, $n \in \{120, 250, 500, 1000\}$, and $\delta \in \{0.1, 0.3\}$.

we additionally assumed that the asset returns are normally distributed, then it holds that

$$\frac{n(n-k)}{(n-1)k}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1}) \sim F_{k,n-k,n\delta}\,,$$

i.e. it has a non-central F-distribution. From the properties of the non-central F-distribution

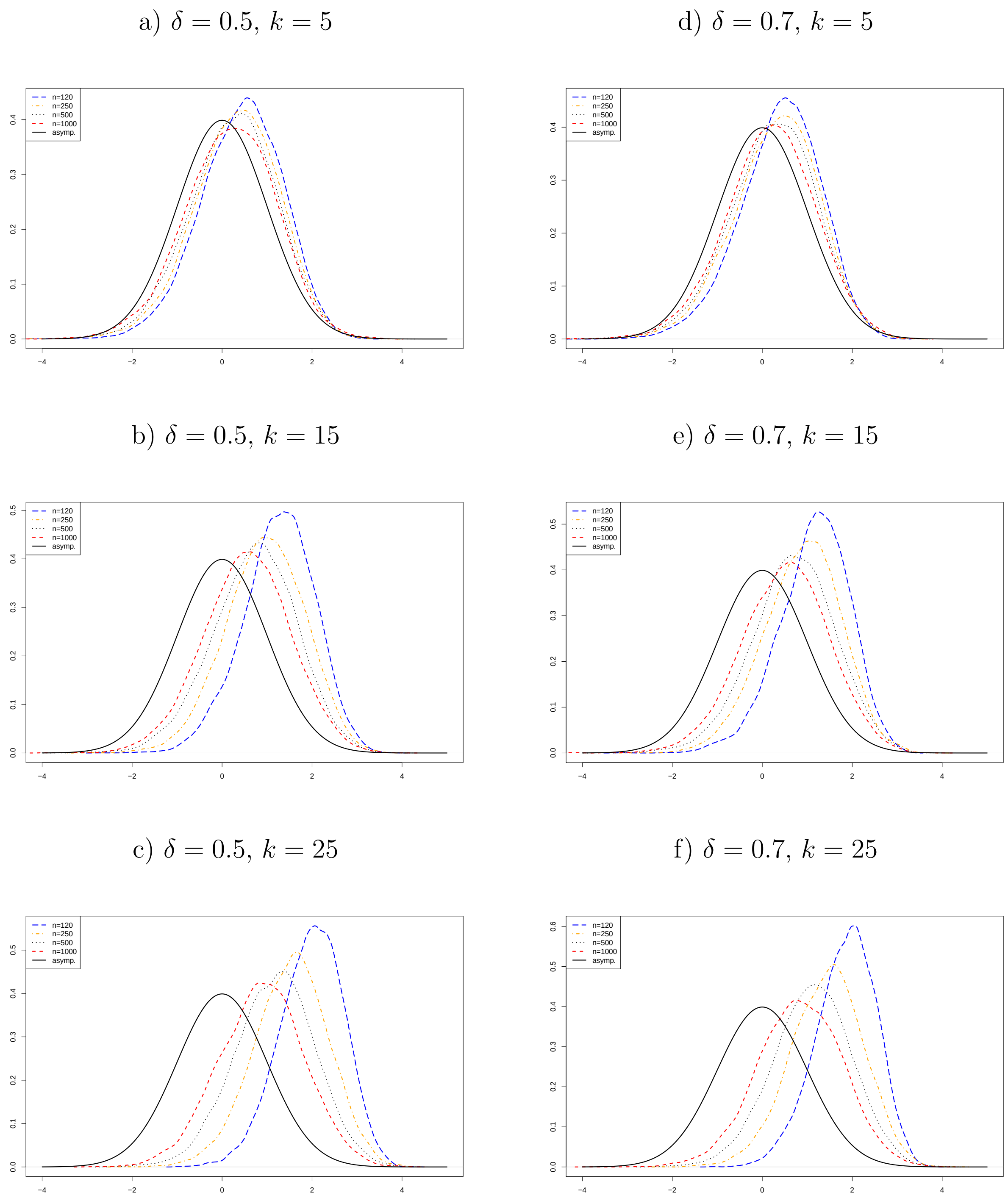

Figure 2: Exact and asymptotic densities of $\sqrt{n}(\hat{\alpha}_{TP} - \alpha_{TP})/\sigma_\alpha$ for $k \in \{5, 15, 25\}$, $n \in \{120, 250, 500, 1000\}$, and $\delta \in \{0.5, 0.7\}$.

(cf., Johnson et al. (1995, p. 481)) we get

$$E((\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})) = \frac{k(n-1)}{(n-k-2)n} + \frac{n-1}{n-k-2}\delta\,,$$

and, hence, the unbiased estimator for δ is given by

$$\frac{n-k-2}{n-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1}) - \frac{k}{n}\,. \tag{25}$$

$n=120$												
	$k=5$				$k=15$				$k=25$			
	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$
$\alpha=0.9$	0	0	0.0017	0.0326	0	0.0004	0.022	0.1759	0	0.0091	0.1474	0.4956
$\alpha=0.95$	0	0	0	0	0	0	0	0	0	0	0.0002	0.0036
$\alpha=0.99$	0	0	0	0	0	0	0	0	0	0	0	0
$n=250$												
	$k=5$				$k=15$				$k=25$			
	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$
$\alpha=0.9$	0	0	0	0.0009	0	0	0.0004	0.0095	0	0	0.0011	0.0433
$\alpha=0.95$	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha=0.99$	0	0	0	0	0	0	0	0	0	0	0	0
$n=500$												
	$k=5$				$k=15$				$k=25$			
	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$
$\alpha=0.9$	0	0	0	0	0	0	0	0	0	0	0	0.0001
$\alpha=0.95$	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha=0.99$	0	0	0	0	0	0	0	0	0	0	0	0
$n=1000$												
	$k=5$				$k=15$				$k=25$			
	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$	$\delta=0.1$	$\delta=0.3$	$\delta=0.5$	$\delta=0.7$
$\alpha=0.9$	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha=0.95$	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha=0.99$	0	0	0	0	0	0	0	0	0	0	0	0

Table 1: Probabilities $P(\hat{\alpha}_{SR} > \alpha)$ with $\alpha \in \{0.9, 0.95, 0.99\}$ for $k \in \{5, 10, 25, 50\}$, $\delta \in \{0.1, 0.3, 0.5, 0.7\}$, and $n \in \{120 \text{ (panel 1)}, 250 \text{ (panel 2)}, 500 \text{ (panel 3)}, 1000 \text{ (panel 4)}\}$.

However, the estimator (25) can also be negative with a positive probability, especially when n is not large enough. As a result, the following finite-sample adjusted version of $\hat{\alpha}_{TP}$ is given by

$$\hat{\alpha}^*_{TP} = F\left(\gamma\sqrt{\max\{0, \frac{n-k-2}{n-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1})'\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - r_0\mathbf{1}) - \frac{k}{n}\}}\right) . \tag{26}$$

In Figure 3, we present the performance of $\sqrt{n}(\hat{\alpha}^*_{TP} - \alpha_{TP})/\sigma_\alpha$. The calculations are done for the same of the parameters as the ones used in Figures 1 and 2 for $k = 25$. We observe a considerable improvement in the convergent rate of the finite-sample adjusted estimator for α_{TP} as given in (26) with respect to the sample one presented in (13).

4 Empirical Study

In this section we apply the theoretical results of the paper to real data which consist of weekly asset returns of 30 stocks included into the Dow Jones index from 01.01.2009 to 31.12.2013. As a risk-free asset we use the 3 month US treasury bill. Based on the considered data, several optimal portfolios of different sizes are constructed and their properties are investigated. In order to obtain a better understanding, we make use of the rolling window estimation with the window size of $n = 110$.

In Figure 4, the estimated values of the quadratic form $\delta = (\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})$ for $k \in \{5, 10, 15, 20, 25, 30\}$ are presented. We observe that these values are always smaller than 0.45 and they are increasing as k becomes larger. Moreover, the estimated values of the quadratic

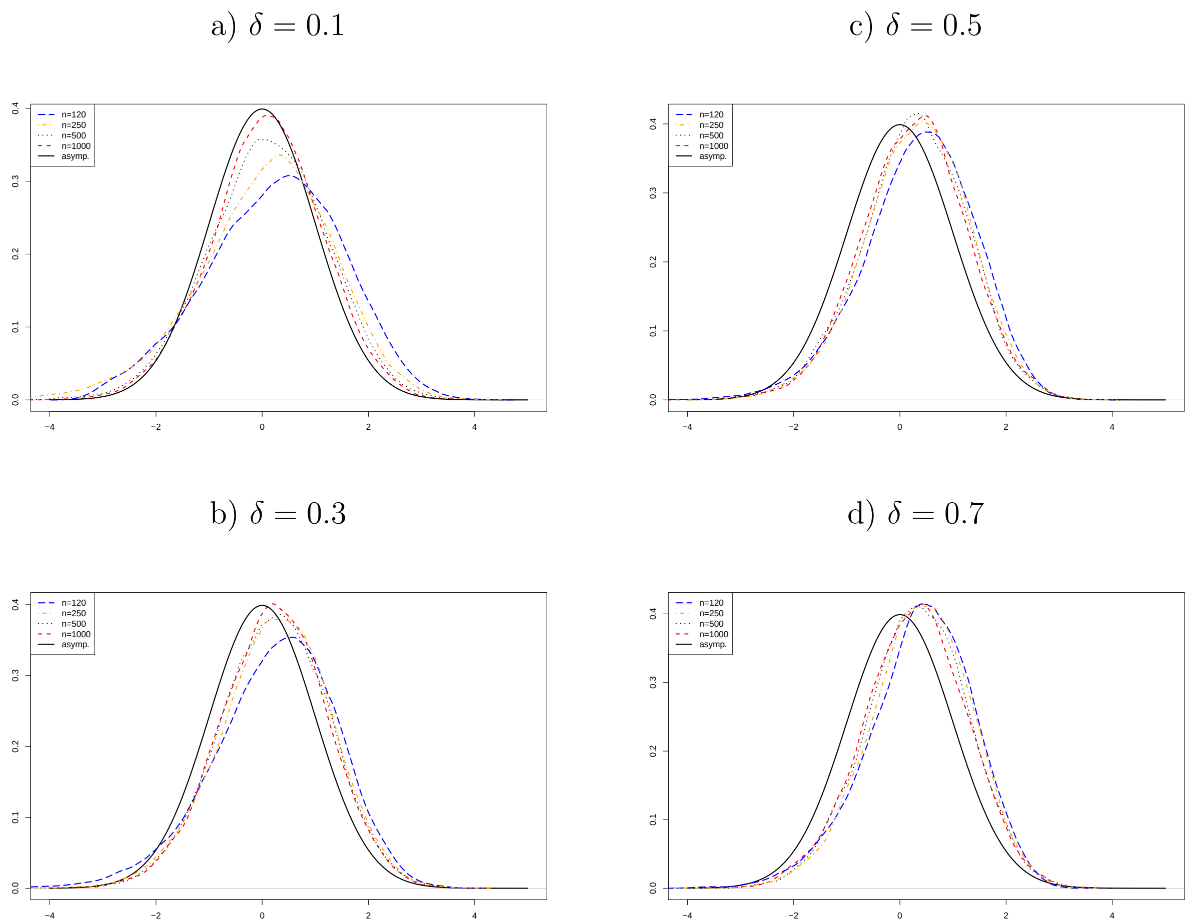

Figure 3: Exact and asymptotic densities of $\sqrt{n}(\hat{\alpha}^*_{TP} - \alpha_{TP})/\sigma_\alpha$ for $k = 25$, $n \in \{120, 250, 500, 1000\}$, and $\delta \in \{0.1, 0.3, 0.5, 0.7\}$.

form lie in the range of δ as considered in the numerical illustration of Section 3. Using the results presented in Table 1, a first conclusion can be drawn that the corresponding $\hat{\alpha}_{TP}$ with a high probability are smaller than 0.9, i.e. the corresponding SR portfolios are very risky and they are not in line with the recommendations of the Basel Committee on Banking Supervision.

More pronounced results are presented in Figure 5 where the confidence intervals at the significance level $\beta = 0.95$ are constructed by using the results of Section 3.1. The upper bounds of the calculated confidence intervals are always smaller than 0.85. This shows that not only the estimated confidence levels of the VaR calculated for the SR portfolio, but also that their population counterparts are significantly smaller than 0.9 with probability of 0.95.

Finally, in Figure 6 we present the estimated upper bounds for $r_0(0.95)$ with the upper limits of the corresponding upper one-sided confidence intervals. The results of the figure document that the SR portfolio is very risky and its risk cannot be reduced by considering a risk-free asset with return $r_0 \geq 0$. The results of the empirical study confirm our previous conclusions that the application of the SR portfolio in practice could lead to large losses with a relatively high probability.

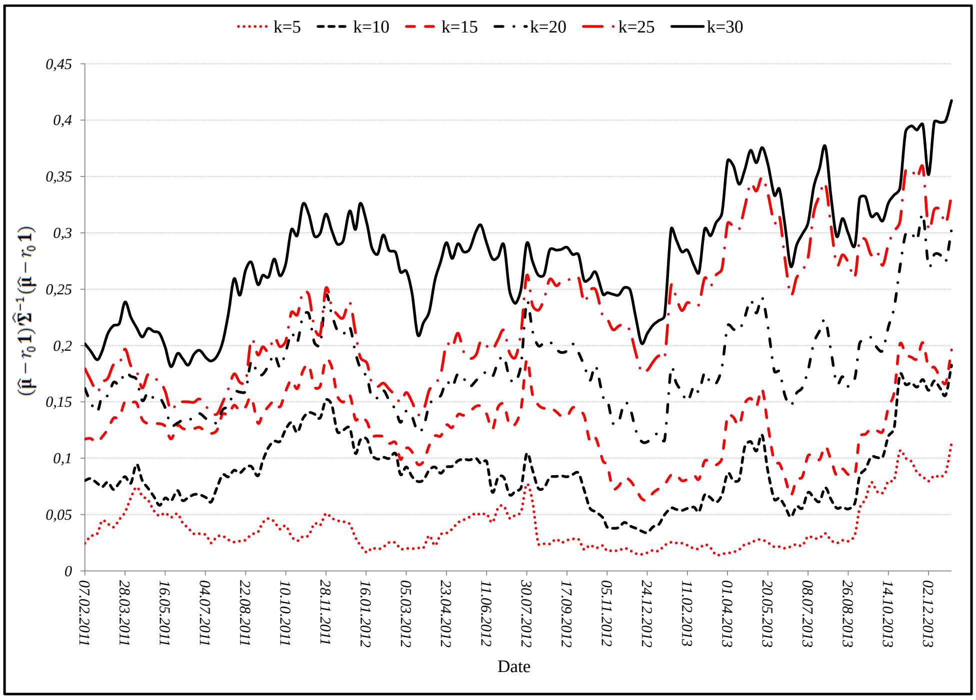

Figure 4: Estimator of $\delta = (\boldsymbol{\mu} - r_0\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_0\mathbf{1})$ for optimal portfolios of size $k \in \{5, 10, 15, 20, 25, 30\}$ calculated by using the weekly asset returns of 30 stocks included into the Dow Jones index from 01.01.2009 to 31.12.2013. The 3 month US treasury bill is used as a risk-free asset. The rolling window estimation with the window size of $n = 110$ is performed.

5 Summary

Both the SR and the optimal portfolio, which is obtained by maximizing the SR, are popular in the financial literature (see, e.g., Sharpe (1966, 1994), Schmid and Zabolotskyy (2008)). It provides the investor with a simple and intuitively understandable strategy how to determine an optimal portfolio. Moreover, the SR portfolio is in line with Markowitz's optimization problem since it lies on the efficient frontier in the mean-variance space in the case without a risk-free asset and provides a reasonable alternative to the global minimum variance portfolio.

In contrast, the estimated SR portfolio does not possess desirable distributional properties. Okhrin and Schmid (2006) proved that the distribution of its estimated weights are heavy-tailed. Moreover, the sample weights of the portfolio with the highest SR do not possess a first moment at all. An extension to this result was given by Schmid and Zabolotskyy (2008) who showed that an unbiased estimator for the weights of the optimal portfolio in the sense of maximizing SR does not exist. Finally, Bodnar and Schmid (2008) proved that the estimator of the expected return of the SR portfolio does not have a first moment, while the estimator for its variance does not possess moments of order greater than or equal to 1/2.

In the present paper we extend these results by showing that the difficulties already appear when the population SR portfolio is determined. This conclusion is achieved by comparing the TP related to the SR portfolio for the given return of the risk-free asset with the corresponding

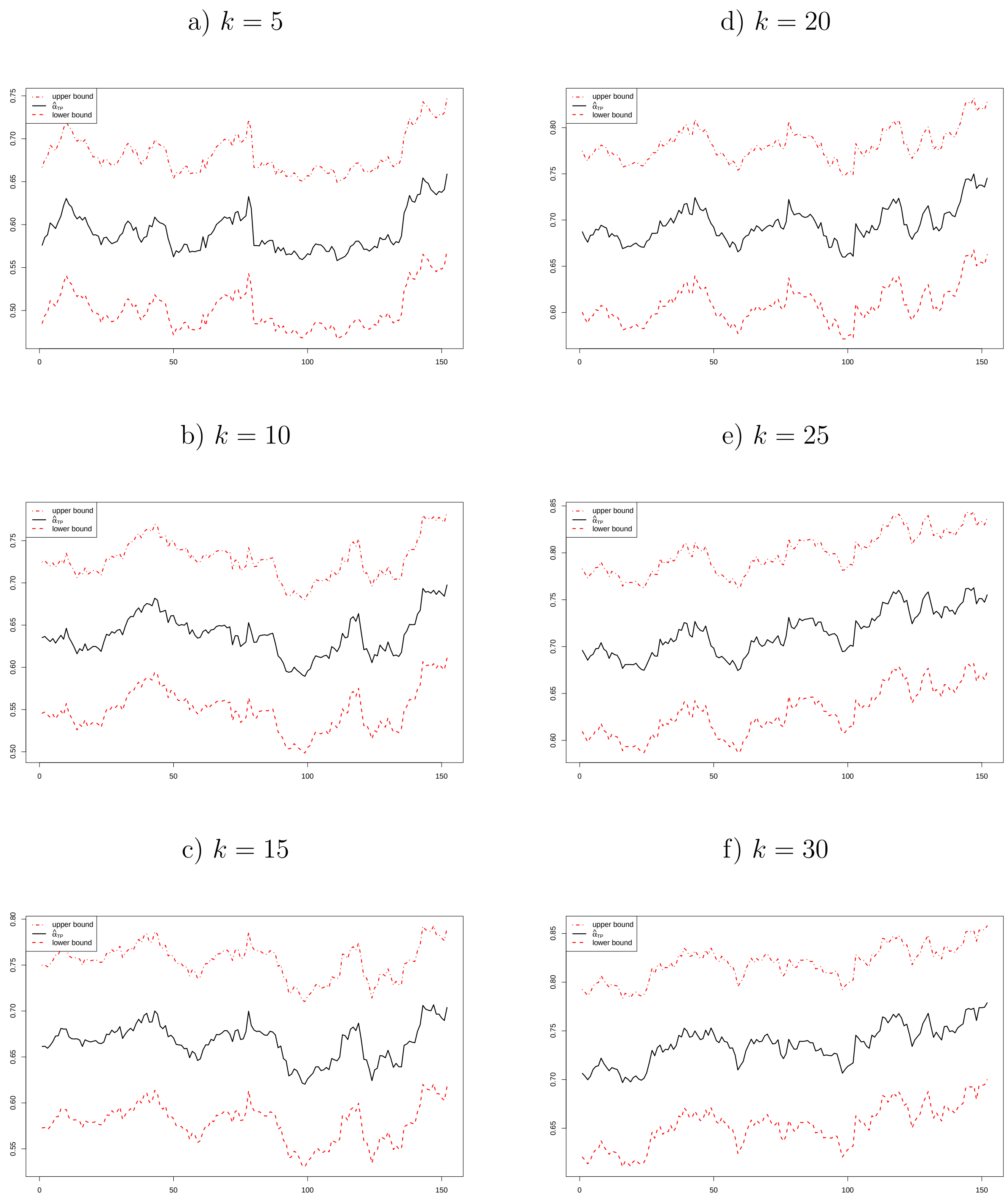

Figure 5: Confidence intervals for α_{TP} in case of the SR portfolios of size $k \in \{5, 10, 15, 20, 25, 30\}$ calculated by using the weekly asset returns of 30 stocks included into the Dow Jones index from 01.01.2009 to 31.12.2013. The rolling window estimation with the window size of $n = 110$ is used.

minimum VaR portfolio which also lies on the efficient frontier and coincides with the TP. The confidence level of the "equivalent" minimum VaR portfolio is then used to determine the risk of the TP as well as the related SR portfolio. Both theoretical and empirical results show that this significance level is smaller than 0.9 in almost all of the considered cases. As a result, the

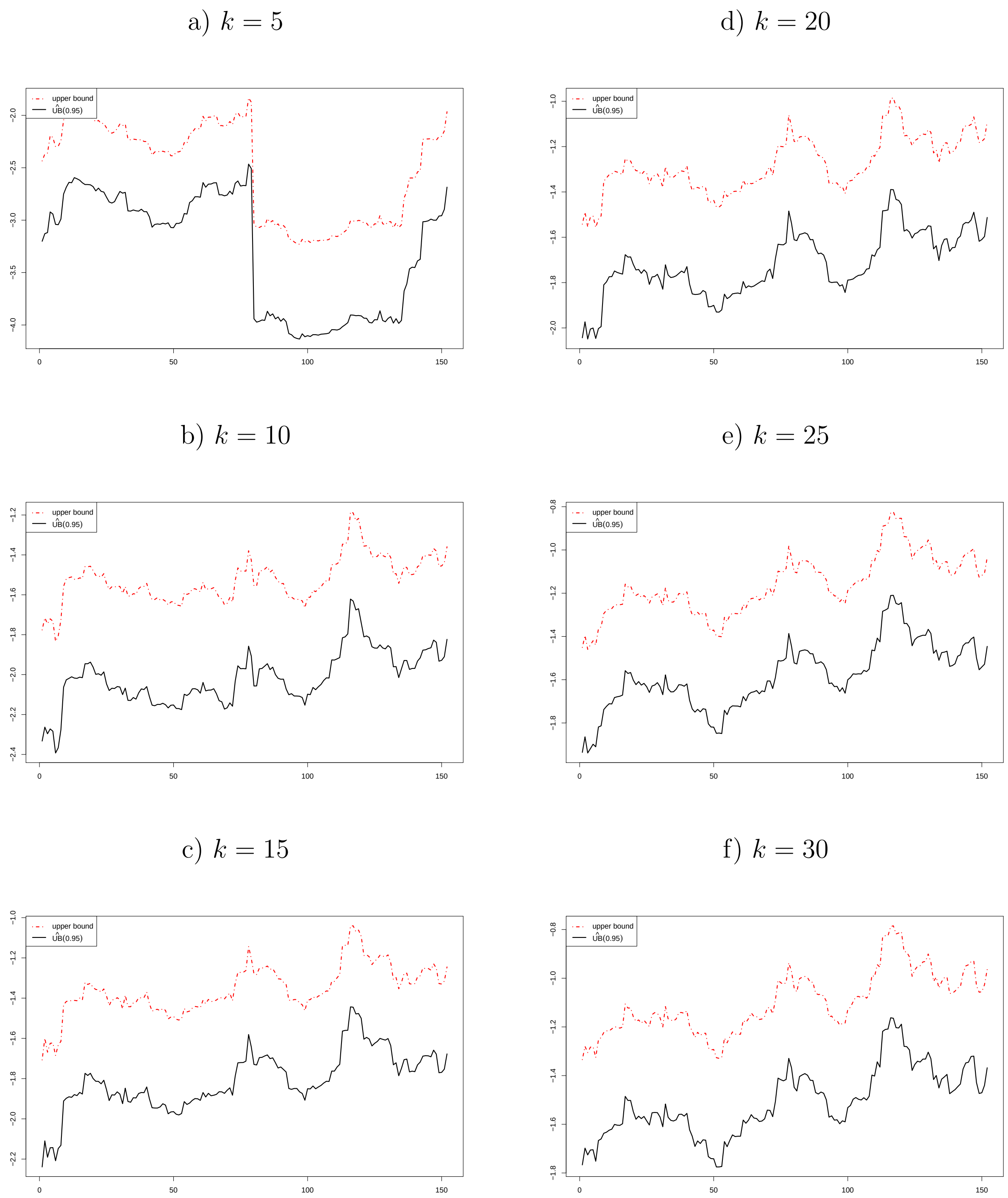

Figure 6: Confidence intervals for $r_0(0.95)$ in case of the SR portfolios of size $k \in \{5, 10, 15, 20, 25, 30\}$ calculated by using the weekly asset returns of 30 stocks included into the Dow Jones index from 01.01.2009 to 31.12.2013. The rolling window estimation with the window size of $n = 110$ is used.

SR portfolio appears to be a very risky investment strategy and it should be used very carefully in practice since its application could lead to large losses with a relatively high probability.

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References

[1] Alexander, G. J. and Baptista, M. A. (2002). Economic implication of using a mean-VaR model for portfolio selection: A comparison with mean-variance analysis. *Journal of Economic Dynamics & Control* **26**, 1159-1193.

[2] Alexander, G. J. and Baptista, M. A. (2004). A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. *Management Science* **50**, 1261-1273.

[3] Baron, D. P. (1977). On the utility theoretic foundations of the mean-variance analysis. *The Journal of Finance* **32**, 1683-1697.

[4] Bodnar, T. and Schmid, W. (2008). Mean-variance portfolio analysis under parameter uncertainty. *Statistics & Decisions* **26**, 179-201.

[5] Bodnar, T. and Schmid, W. (2011). On the exact distribution of the estimated expected utility portfolio weights: Theory and applications. *Statistics & Risk Modeling* **28**, 319-342.

[6] Bodnar, T., Parolya, N. and Schmid, W. (2013a). On the equivalence of quadratic optimization problems commonly used in portfolio theory. *European Journal of Operational Research* **229**, 637-644.

[7] Bodnar, T., Schmid, W. and Zabolotskyy, T. (2009). Statistical inference of the efficient frontier for dependent asset returns. *Statistical Papers*, **50**, 593-604.

[8] Bodnar, T., Schmid, W. and Zabolotskyy, T. (2012). Minimum VaR and minimum CVaR optimal portfolios: estimators, confidence regions, and tests. *Statistics & Risk Modeling* **29**, 281-314.

[9] Bodnar, T., Schmid, W. and Zabolotskyy, T. (2013b). Minimum VaR and minimum CVaR optimal portfolios: estimators, confidence regions, and tests. *Metrika* **76**, 1105-1134.

[10] Bodnar, T. and Zabolotskyy, T. (2013). Maximization of the Sharpe ratio of an asset portfolio in the context of risk minimization. *Economic Annals-XXI* **11-12**, 110-113.

[11] Bonami, P. and Lejeune, M. (2009). An exact solution approach for integer constrained portfolio optimization problems under stochastic constraints. *Operations Research* **57**, 650-670.

[12] Campbell, R., Huisman, R. and Koedijk, K. (2001). Optimal portfolio selection in a value-at-risk framework. *Journal of Banking & Finance* **25**, 1789-1804.

[13] DasGupta, A. (2008). *Asymptotic theory of statistics and probability.* Springer, New York.

[14] Fang, K.T. and Zhang, Y.T. (1990). *Generalized multivariate analysis.* Springer-Verlag, Berlin and Science Press, Beijing.

[15] Grechuk, B., Molyboha, A. and Zabarankin, M. (2010). Chebyshev inequalities with law invariant deviation measures, *Probability in the Engineering and Informational Sciences* **24**, 145-170.

[16] Gupta A.K., Varga, T. and Bodnar, T. (2013). *Elliptically contoured models in statistics and portfolio theory.* Springer, New York.

[17] Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995), *Continuous univariate distributions*, vol.2. Wiley, New York.

[18] Jorion, Ph. (1997). *Value at risk: the new benchmark for controlling market risk.* McGraw-Hill, New York.

[19] Kilianová, S. and Pflug, G. Ch. (2009). Optimal pension fund management under multi-period risk minimization, *Annals of Operations Research* **166**, 261-270.

[20] Krokhmal, P., Zabarankin, M. and S. Uryasev. (2011). Modeling and optimization of risk, *Surveys in Operations Research and Management Science* **16**, 49-66.

[21] Lehmann, E. L. and Romano, J. P. (2005). *Testing statistical hypotheses.* Springer, New York.

[22] Luenberger, D. (1998). *Investment science.* Oxford University Press, Oxford.

[23] Markowitz, H. (1952). Portfolio selection. *The Journal of Finance* **7**, 77-91.

[24] Merton, R. C. (1972). An analytical derivation of the efficient frontier. *Journal of Financial and Quantitative Analysis* **7**, 1851-1872.

[25] Muirhead, R.J. (1982). *Aspects of multivariate statistical theory.* Wiley, New York.

[26] Okhrin, Y. and Schmid, W. (2006). Distributional properties of optimal portfolio weights. *Journal of Econometrics* **134**, 235-256.

[27] Pflug, G. Ch. (2000). Some remarks on the value-at-risk and conditional value-at-risk. In: *Probabilistic Constrained Optimization: Methodology and Applications* (ed. S. Uryasev). Kluwer.

[28] Rockafellar, R. T. , Uryasev, S. and Zabarankin, M. (2006a). Optimality conditions in portfolio analysis with general deviation measures. *Mathematical Programming, Ser. B* **108**, 515-540.

[29] Rockafellar, R. T. , Uryasev, S. and Zabarankin, M. (2006b). Master funds in portfolio analysis with general deviation measures. *Journal of Banking & Finance* **30**, 743-778.

[30] Rockafellar, R. T. , Uryasev, S. and Zabarankin, M. (2006c). Generalized deviations in risk analysis. *Finance and Stochastics* **10**, 51-74.

[31] Rockafellar, R. T. , Uryasev, S. and Zabarankin, M. (2007). Equilibrium with investors using a diversity of deviation measures. *Journal of Banking & Finance* **31**, 3251-3268.

[32] Schmid, W. and Zabolotskyy, T. (2008). On the existence of unbiased estimators for the portfolio weights. *AStA - Advances in Statistical Analysis* **92**, 29-34.

[33] Sharpe, W. F. (1966). Mutual fund performance. *Journal of Business* **39**, 119-138.

[34] Sharpe, W. F. (1994). The Sharpe ratio. *The Journal of Portfolio Management* **21**, 49-58.