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Abstract

Non-linearly perturbed semi-Markov processes in discrete time are considered. Asymptotic power series expansions for quasi-stationary distributions of such processes are presented and it is shown how the coefficients in these expansions can be computed from explicit recursive formulas. As a particular case, it is described how the results can be applied for discrete time Markov chains.

Keywords: Semi-Markov process, Perturbation, Quasi-stationary distribution, Asymptotic expansion, Renewal equation, Solidarity property, First hitting time, Markov chain.

MSC2010: Primary 60K15; Secondary 41A60, 60J10, 60K05.

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1 Introduction

The aim of this paper is to present asymptotic power series expansions for quasi-stationary distributions of non-linearly perturbed semi-Markov processes in discrete time, to show how the coefficients in these expansions can be calculated from explicit recursive formulas, and to illustrate the results in the special case of discrete time Markov chains.

Quasi-stationary distributions are useful for studies of stochastic systems with random lifetimes. Usually, for such systems, the evolution of some quantity of interest is described by some stochastic process and the lifetime of the system is the first time this process hits some absorbing subset of the state space. For such processes, the stationary distribution will be concentrated on this absorbing subset. However, if we expect that the system will persist for a long time, the stationary distribution may not be an appropriate measure for describing the long time behaviour of the process. Instead, it might be more relevant to consider so-called quasi-stationary distributions. This type of distributions are obtained as limits of transition probabilities which are conditioned on that the process has not yet been absorbed.

Models of the type described above arise in many areas of applications such as epidemics, genetics, population dynamics, queuing theory, reliability, and risk theory. For example, the number of individuals in some population may be modelled by some stochastic process and we can consider the extinction time of the population as the lifetime. In epidemic models, the process may describe the evolution of the number of infected individuals in some population and we can regard the end of the epidemic as the lifetime.

We consider, for every $\varepsilon \geq 0$, a discrete time semi-Markov process $\xi^{(\varepsilon)}(n)$, $n = 0, 1, \ldots$, on a finite state space $X = \{0, 1, \ldots, N\}$. It is assumed that the process $\xi^{(\varepsilon)}(n)$ depends on ε in such a way that its transition probabilities are functions of ε which converge pointwise to the transition probabilities for the limiting process $\xi^{(0)}(n)$. Thus, we can interpret $\xi^{(\varepsilon)}(n)$, for $\varepsilon > 0$, as a perturbation of $\xi^{(0)}(n)$. Furthermore, it is assumed that the states $\{1, \ldots, N\}$ is a communicating class for ε small enough.

Under conditions mentioned above, some additional assumptions of finite exponential moments for distributions of transition times, and a non-periodicity condition for the limiting semi-Markov process, a unique quasi-stationary distribution, independent of the initial state, can be defined for each sufficiently small ε by the following relation,

$$\pi_j^{(\varepsilon)} = \lim_{n \to \infty} \mathsf{P}_i\{\xi^{(\varepsilon)}(n) = j \,|\, \mu_0^{(\varepsilon)} > n\}, \ i, j \neq 0,$$

where $\mu_0^{(\varepsilon)}$ is the first hitting time of state 0.

In the present paper we are interested in the asymptotics of the quasi-stationary distribution as the perturbation parameter ε tends to zero. Specifically, an asymptotic power series expansion for the quasi-stationary distribution is constructed.

We allow for non-linear perturbations, i.e., the transition probabilities may be non-linear functions of ε. We do however restrict our consideration to smooth perturbations by assuming that certain mixed power-exponential moment functionals for transition probabilities, up to some order k, can be expanded in asymptotic power series with respect to ε. In this case, we show that the quasi-stationary distribution has the following asymptotic expansion,

$$\pi_j^{(\varepsilon)} = \pi_j^{(0)} + \pi_j[1]\varepsilon + \cdots + \pi_j[k]\varepsilon^k + o(\varepsilon^k), \ j \neq 0, \tag{1.1}$$

where the coefficients $\pi_j[1], \ldots, \pi_j[k]$, can be calculated from explicit recursive formulas. These formulas are functions of the coefficients in the expansions of the moment functionals mentioned above. The existence of the expansion (1.1) and the algorithm for computing the coefficients in this expansion is the main result of this paper.

It is worth mentioning that the asymptotics given by relation (1.1) simultaneously cover three different cases. In the simplest case, there exists $\varepsilon_0 > 0$ such that transitions to state 0 is not possible for any $\varepsilon \in [0, \varepsilon_0]$. In this case, relation (1.1) gives asymptotic expansions for stationary distributions. Then, we have an intermediate case where transitions to state 0 is possible for all $\varepsilon \in (0, \varepsilon_0]$ but not possible for $\varepsilon = 0$. In this case we have that $\mu_0^{(\varepsilon)} \to \infty$ in probability as $\varepsilon \to 0$. In the mathematically most difficult case, we have that transitions to state 0 is possible for all $\varepsilon \in [0, \varepsilon_0]$. In this case, the random variables $\mu_0^{(\varepsilon)}$ are stochastically bounded as $\varepsilon \to 0$.

The expansion (1.1) is proved for continuous time semi-Markov processes in Gyllenberg and Silvestrov (1999, 2008). However, the discrete time case is interesting in its own right and deserves a special treatment. In particular, a discrete time model is often a natural choice in applications where measures of some quantity of interest are only available at given time points, for example days or months. The proof of the continuous time case, as well as the proofs of the present paper, are based on the theory of non-linearly perturbed renewal equations. For results related to continuous time in this line of research, we refer to the comprehensive book by Gyllenberg and Silvestrov (2008), which also contains an extensive bibliography of work in related areas. The corresponding theory for discrete time renewal equations has been developed in Gyllenberg and Silvestrov (1994), Englund and Silvestrov (1997), Silvestrov and Petersson (2013), and Petersson (2014a, b, 2015).

Quasi-stationary distributions have been studied extensively since the 1960's. For some of the early works on Markov chains and semi-Markov processes, see, for example, Vere-Jones (1962), Kingman (1963), Darroch and Seneta (1965), Seneta and Vere-Jones (1966), Cheong (1968, 1970), and Flaspohler and Holmes (1972). A survey of quasi-stationary distributions for models with discrete state spaces and more references can be found in van Doorn and Pollett (2013).

Studies of asymptotics for first hitting times, stationary distributions, and other characteristics for Markov chains with linear and analytic perturbations have attracted a lot of attention, see, for example, Simon and Ando (1961), Schweitzer (1968), Gaĭtsgori and Pervozvanskiĭ (1975), Courtois and Louchard (1976), Latouche and Louchard (1978), Delebecque (1983), Latouche (1991), Stewart (1991), Hassin and Haviv (1992), Yin and Zhang (1998, 2003), Altman, Avrachenkov, and Núñez-Queija (2004), Avrachenkov and Haviv (2004), and Avrachenkov, Filar, and Howlett (2013). Recently, some of the results of these papers have been extended to non-linearly perturbed semi-Markov processes. Using a method of sequential phase space reduction, asymptotic expansions for expected first hitting times and stationary distributions are given in Silvestrov and Silvestrov (2015). This paper also contains an extensive bibliography.

Let us now briefly outline the structure of the present paper. In Section 2 we define perturbed discrete time semi-Markov processes and formulate our main result. Then, systems of linear equations for some important moment functionals are derived in Section 3 and in Section 4 we give some asymptotic solidarity properties. In Section 5 it is shown how we can use the renewal theorem to get a formula for the quasi-stationary distribution. In Section 6 we construct asymptotic expansions for some mixed power-exponential moment functionals. These expansions are fundamental for the proof of the main result which is given in Section 7. Finally, we illustrate the results applied to discrete time Markov chains in Section 8.

2 Main Result

In this section we define perturbed discrete time semi-Markov processes and formulate the main result of the present paper.

For every $\varepsilon \geq 0$, let $(\eta_n^{(\varepsilon)}, \kappa_n^{(\varepsilon)})$, $n = 0, 1, \ldots$, be a discrete time Markov renewal process, i.e., a homogeneous Markov chain with state space $X \times \mathbb{N}$, where $X = \{0, 1, \ldots, N\}$ and $\mathbb{N} = \{1, 2, \ldots\}$, an initial distribution $Q_i^{(\varepsilon)} = \mathsf{P}\{\eta_0^{(\varepsilon)} = i\}$, $i \in X$, and transition probabilities which do not depend on the current value of the second component, given by

$$Q_{ij}^{(\varepsilon)}(k) = \mathsf{P}\{\eta_{n+1}^{(\varepsilon)} = j,\ \kappa_{n+1}^{(\varepsilon)} = k \,|\, \eta_n^{(\varepsilon)} = i,\ \kappa_n^{(\varepsilon)} = l\},\ i, j \in X,\ k, l \in \mathbb{N}.$$

In this case, it is known that $\eta_n^{(\varepsilon)}$, $n = 0, 1, \ldots$, is also a Markov chain with state space X and transition probabilities,

$$p_{ij}^{(\varepsilon)} = \mathsf{P}\{\eta_{n+1}^{(\varepsilon)} = j \,|\, \eta_n^{(\varepsilon)} = i\} = \sum_{k=1}^{\infty} Q_{ij}^{(\varepsilon)}(k),\ i, j \in X.$$

Let us define $\tau^{(\varepsilon)}(0) = 0$ and $\tau^{(\varepsilon)}(n) = \kappa_1^{(\varepsilon)} + \cdots + \kappa_n^{(\varepsilon)}$, for $n \in \mathbb{N}$. Furthermore, for $n = 0, 1, \ldots$, we define $\nu^{(\varepsilon)}(n) = \max\{k : \tau^{(\varepsilon)}(k) \leq n\}$. The discrete time semi-Markov process associated with the Markov renewal process $(\eta_n^{(\varepsilon)}, \kappa_n^{(\varepsilon)})$ is defined by the following relation,

$$\xi^{(\varepsilon)}(n) = \eta_{\nu^{(\varepsilon)}(n)}^{(\varepsilon)},\ n = 0, 1, \ldots,$$

and we will refer to $Q_{ij}^{(\varepsilon)}(k)$ as the transition probabilities of this process.

For the semi-Markov process defined above we have that (i) $\kappa_n^{(\varepsilon)}$ are the times between successive moments of jumps, (ii) $\tau^{(\varepsilon)}(n)$ are the moments of the jumps, (iii) $\nu^{(\varepsilon)}(n)$ are the number of jumps in the interval $[0, n]$, and (iv) $\eta_n^{(\varepsilon)}$ is the embedded Markov chain.

It is sometimes convenient to write the transition probabilities of the semi-Markov process as $Q_{ij}^{(\varepsilon)}(k) = p_{ij}^{(\varepsilon)} f_{ij}^{(\varepsilon)}(k)$, where

$$f_{ij}^{(\varepsilon)}(k) = \mathsf{P}\{\kappa_{n+1}^{(\varepsilon)} = k \,|\, \eta_n^{(\varepsilon)} = i,\ \eta_{n+1}^{(\varepsilon)} = j\},\ k \in \mathbb{N},\ i, j \in X,$$

are the distributions of transition times.

We now define random variables for first hitting times. For each $j \in X$, let $\nu_j^{(\varepsilon)} = \min\{n \geq 1 : \eta_n^{(\varepsilon)} = j\}$ and $\mu_j^{(\varepsilon)} = \tau(\nu_j^{(\varepsilon)})$. Then, $\nu_j^{(\varepsilon)}$ and $\mu_j^{(\varepsilon)}$ are the first hitting times of state j for the embedded Markov chain and the semi-Markov process, respectively. Note that $\nu_j^{(\varepsilon)}$ and $\mu_j^{(\varepsilon)}$ are possibly improper random variables taking values in the set $\{1, 2, \ldots, \infty\}$.

Let us define

$$g_{ij}^{(\varepsilon)}(n) = \mathsf{P}_i\{\mu_j^{(\varepsilon)} = n,\ \nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}\},\ n = 0, 1, \ldots,\ i, j \in X,$$

and

$$g_{ij}^{(\varepsilon)} = \mathsf{P}_i\{\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}\},\ i, j \in X.$$

Here, and in what follows, we write $\mathsf{P}_i(A^{(\varepsilon)}) = \mathsf{P}\{A^{(\varepsilon)} \,|\, \eta_0^{(\varepsilon)} = i\}$ for any event $A^{(\varepsilon)}$. Corresponding notation for conditional expectation will also be used.

The functions $g_{ij}^{(\varepsilon)}(n)$ define discrete probability distributions which may be improper, i.e., $\sum_{n=0}^{\infty} g_{ij}^{(\varepsilon)}(n) = g_{ij}^{(\varepsilon)} \leq 1$.

Moment generating functions for distributions of first hitting times are defined by

$$\phi_{ij}^{(\varepsilon)}(\rho) = \sum_{n=0}^{\infty} e^{\rho n} g_{ij}^{(\varepsilon)}(n) = \mathsf{E}_i e^{\rho \mu_j^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}), \ \rho \in \mathbb{R}, \ i,j \in X. \qquad (2.1)$$

We also define the following mixed power-exponential moment functionals for transition probabilities,

$$p_{ij}^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} Q_{ij}^{(\varepsilon)}(n), \ \rho \in \mathbb{R}, \ r = 0, 1, \ldots, \ i,j \in X,$$

where we define $Q_{ij}^{(\varepsilon)}(0) = 0$. For convenience, we define $p_{ij}^{(\varepsilon)}(\rho) = p_{ij}^{(\varepsilon)}(\rho, 0)$.

Let us now introduce the following conditions, which we will refer to frequently throughout the paper:

A: (**a**) $p_{ij}^{(\varepsilon)} \to p_{ij}^{(0)}$, as $\varepsilon \to 0$, $i \neq 0$, $j \in X$.

(**b**) $f_{ij}^{(\varepsilon)}(n) \to f_{ij}^{(0)}(n)$, as $\varepsilon \to 0$, $n \in \mathbb{N}$, $i \neq 0$, $j \in X$.

B: $g_{ij}^{(0)} > 0$, $i, j \neq 0$.

C: There exists $\beta > 0$ such that:

(**a**) $\limsup_{0 \leq \varepsilon \to 0} p_{ij}^{(\varepsilon)}(\beta) < \infty$, for all $i \neq 0$, $j \in X$.

(**b**) $\phi_{ii}^{(0)}(\beta_i) \in (1, \infty)$, for some $i \neq 0$ and $\beta_i \leq \beta$.

D: $g_{ii}^{(0)}(n)$ is a non-periodic distribution for some $i \neq 0$.

It follows from conditions **A** and **B** that $\{1, \ldots, N\}$ is a communicating class of states for sufficiently small ε. Let us also remark that if $p_{i0}^{(0)} = 0$ for all $i \neq 0$, it can be shown that part (**b**) of condition **C** always holds under conditions **A**, **B**, and **C**(**a**).

Under the conditions stated above, there exists, for sufficiently small ε, so-called quasi-stationary distributions, which are independent of the initial state $i \neq 0$, and given by the relation

$$\pi_j^{(\varepsilon)} = \lim_{n \to \infty} \mathsf{P}_i \{ \xi^{(\varepsilon)}(n) = j \,|\, \mu_0^{(\varepsilon)} > n \}, \ j \neq 0. \qquad (2.2)$$

In order to construct an asymptotic expansion for the quasi-stationary distribution, we need a perturbation condition for the transition probabilities $Q_{ij}^{(\varepsilon)}(k)$ which is stronger than **A**. This condition is formulated in terms of the moment functionals $p_{ij}^{(\varepsilon)}(\rho^{(0)}, r)$, where $\rho^{(0)}$ is the solution of the characteristic equation $\phi_{ii}^{(0)}(\rho) = 1$, which is independent of $i \neq 0$ (see Section 4).

$\mathbf{P_k}$: $p_{ij}^{(\varepsilon)}(\rho^{(0)}, r) = p_{ij}^{(0)}(\rho^{(0)}, r) + p_{ij}[\rho^{(0)}, r, 1]\varepsilon + \cdots + p_{ij}[\rho^{(0)}, r, k-r]\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, $i \neq 0$, $j \in X$, where $|p_{ij}[\rho^{(0)}, r, n]| < \infty$, for $r = 0, \ldots, k$, $n = 1, \ldots, k-r$, $i \neq 0$, $j \in X$.

The following theorem is the main result of this paper. The proof is given in Section 7.

Theorem 2.1. *If conditions* **A**–**D** *and* $\mathbf{P_{k+1}}$ *hold, then we have the following asymptotic expansion,*

$$\pi_j^{(\varepsilon)} = \pi_j^{(0)} + \pi_j[1]\varepsilon + \cdots + \pi_j[k]\varepsilon^k + o(\varepsilon^k), \ j \neq 0,$$

where the coefficients $\pi_j[n]$, $n = 1, \ldots, k$, $j \neq 0$, *can be calculated from explicit recursive formulas given by Lemmas 6.1–6.4 and Lemmas 7.1–7.3.*

3 Systems of Linear Equations

In this section we derive systems of linear equations for some moment functionals that will play an important role in what follows.

We first consider the moment generating functions $\phi_{ij}^{(\varepsilon)}(\rho)$, defined by equation (2.1). By conditioning on $(\eta_1^{(\varepsilon)}, \kappa_1^{(\varepsilon)})$, we get for each $i, j \neq 0$,

$$\begin{aligned} \phi_{ij}^{(\varepsilon)}(\rho) &= \sum_{l \in X} \sum_{k=1}^{\infty} \mathsf{E}_i(e^{\rho \mu_j^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}) | \eta_1^{(\varepsilon)} = l, \ \kappa_1^{(\varepsilon)} = k) Q_{il}^{(\varepsilon)}(k) \\ &= \sum_{k=1}^{\infty} e^{\rho k} Q_{ij}^{(\varepsilon)}(k) + \sum_{l \neq 0, j} \sum_{k=1}^{\infty} \mathsf{E}_l e^{\rho(k + \mu_j^{(\varepsilon)})} \chi(\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}) Q_{il}^{(\varepsilon)}(k). \end{aligned} \tag{3.1}$$

Relation (3.1) gives us the following system of linear equations,

$$\phi_{ij}^{(\varepsilon)}(\rho) = p_{ij}^{(\varepsilon)}(\rho) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho) \phi_{lj}^{(\varepsilon)}(\rho), \ i, j \neq 0. \tag{3.2}$$

In what follows it will often be convenient to use matrix notation. Let us introduce the following column vectors,

$$\mathbf{\Phi}_j^{(\varepsilon)}(\rho) = \begin{bmatrix} \phi_{1j}^{(\varepsilon)}(\rho) & \cdots & \phi_{Nj}^{(\varepsilon)}(\rho) \end{bmatrix}^T, \ j \neq 0, \tag{3.3}$$

$$\mathbf{p}_j^{(\varepsilon)}(\rho) = \begin{bmatrix} p_{1j}^{(\varepsilon)}(\rho) & \cdots & p_{Nj}^{(\varepsilon)}(\rho) \end{bmatrix}^T, \ j \in X. \tag{3.4}$$

For each $j \neq 0$, we also define $N \times N$-matrices ${}_j\mathbf{P}^{(\varepsilon)}(\rho) = \|{}_jp_{ik}^{(\varepsilon)}(\rho)\|$ where the elements are given by

$${}_jp_{ik}^{(\varepsilon)}(\rho) = \begin{cases} p_{ik}^{(\varepsilon)}(\rho) & i = 1, \ldots, N, \ k \neq j, \\ 0 & i = 1, \ldots, N, \ k = j. \end{cases} \tag{3.5}$$

Using (3.3), (3.4), and (3.5), we can write the system (3.2) in the following matrix form,

$$\Phi_j^{(\varepsilon)}(\rho) = \mathbf{p}_j^{(\varepsilon)}(\rho) + {}_j\mathbf{P}^{(\varepsilon)}(\rho)\Phi_j^{(\varepsilon)}(\rho), \ j \neq 0. \tag{3.6}$$

Note that the relations given above hold for all $\rho \in \mathbb{R}$ even in the case where some of the quantities involved take the value infinity. In this case we use the convention $0 \cdot \infty = 0$ and the equalities may take the form $\infty = \infty$.

Let us now derive a similar type of system for the following moment functionals,

$$\omega_{ijs}^{(\varepsilon)}(\rho) = \sum_{n=0}^{\infty} e^{\rho n} \mathsf{P}_i\{\xi^{(\varepsilon)}(n) = s, \ \mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} > n\}, \ \rho \in \mathbb{R}, \ i, j, s \in X.$$

First note that

$$\begin{aligned}
\omega_{ijs}^{(\varepsilon)}(\rho) &= \mathsf{E}_i \sum_{n=0}^{\infty} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s, \ \mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} > n) \\
&= \mathsf{E}_i \sum_{n=0}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s).
\end{aligned}$$

We now decompose $\omega_{ijs}^{(\varepsilon)}(\rho)$ into two parts,

$$\omega_{ijs}^{(\varepsilon)}(\rho) = \mathsf{E}_i \sum_{n=0}^{\kappa_1^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) + \mathsf{E}_i \sum_{n=\kappa_1^{(\varepsilon)}}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s). \tag{3.7}$$

Let us first rewrite the first term on the right hand side of equation (3.7). By conditioning on $\kappa_1^{(\varepsilon)}$ we get, for $i, s \neq 0$,

$$\begin{aligned}
&\mathsf{E}_i \sum_{n=0}^{\kappa_1^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) \\
&\quad = \sum_{k=1}^{\infty} \mathsf{E}_i \left(\sum_{n=0}^{\kappa_1^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) \,\middle|\, \kappa_1^{(\varepsilon)} = k \right) \mathsf{P}_i\{\kappa_1^{(\varepsilon)} = k\} \\
&\quad = \sum_{k=1}^{\infty} \delta(i, s) \left(\sum_{n=0}^{k-1} e^{\rho n} \right) \mathsf{P}_i\{\kappa_1^{(\varepsilon)} = k\}.
\end{aligned}$$

It follows that

$$\mathsf{E}_i \sum_{n=0}^{\kappa_1^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) = \delta(i, s) \varphi_i^{(\varepsilon)}(\rho), \ i, s \neq 0. \tag{3.8}$$

where

$$\varphi_i^{(\varepsilon)}(\rho) = \begin{cases} \mathsf{E}_i \kappa_1^{(\varepsilon)} & \rho = 0, \\ (\mathsf{E}_i e^{\rho \kappa_1^{(\varepsilon)}} - 1)/(e^{\rho} - 1) & \rho \neq 0. \end{cases} \tag{3.9}$$

Let us now consider the second term on the right hand side of equation (3.7). By conditioning on $(\eta_1^{(\varepsilon)}, \kappa_1^{(\varepsilon)})$ we get, for $i, j, s \neq 0$,

$$\begin{aligned} \mathsf{E}_i & \sum_{n=\kappa_1^{(\varepsilon)}}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) \\ & = \sum_{l \neq 0, j} \sum_{k=1}^{\infty} \mathsf{E}_i \left(\sum_{n=\kappa_1^{(\varepsilon)}}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) \,\middle|\, \eta_1^{(\varepsilon)} = l, \ \kappa_1^{(\varepsilon)} = k \right) Q_{il}^{(\varepsilon)}(k) \\ & = \sum_{l \neq 0, j} \sum_{k=1}^{\infty} \mathsf{E}_l \left(\sum_{n=0}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho (k+n)} \chi(\xi^{(\varepsilon)}(n) = s) \right) Q_{il}^{(\varepsilon)}(k). \end{aligned}$$

It follows that

$$\mathsf{E}_i \sum_{n=\kappa_1^{(\varepsilon)}}^{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} - 1} e^{\rho n} \chi(\xi^{(\varepsilon)}(n) = s) = \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho) \omega_{ljs}^{(\varepsilon)}(\rho), \ i, j, s \neq 0. \tag{3.10}$$

From (3.7), (3.8), and (3.10) we now get the following system of linear equations,

$$\omega_{ijs}^{(\varepsilon)}(\rho) = \delta(i, s) \varphi_i^{(\varepsilon)}(\rho) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho) \omega_{ljs}^{(\varepsilon)}(\rho), \ i, j, s \neq 0. \tag{3.11}$$

In order to write this system in matrix form, let us define the following column vectors,

$$\widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho) = \left[\delta(1, s) \varphi_1^{(\varepsilon)}(\rho) \ \ \cdots \ \ \delta(N, s) \varphi_N^{(\varepsilon)}(\rho) \right]^T, \ s \neq 0, \tag{3.12}$$

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) = \left[\omega_{1js}^{(\varepsilon)}(\rho) \ \ \cdots \ \ \omega_{Njs}^{(\varepsilon)}(\rho) \right]^T, \ j, s \neq 0. \tag{3.13}$$

Using (3.5), (3.12), and (3.13), the system (3.11) can be written in the following matrix form,

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) = \widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho) + {}_j\mathbf{P}^{(\varepsilon)}(\rho) \boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho), \ j, s \neq 0. \tag{3.14}$$

We close this section with a lemma which will be important in what follows.

Lemma 3.1. *Assume that we for some $\varepsilon \geq 0$ and $\rho \in \mathbb{R}$ have that $g_{ik}^{(\varepsilon)} > 0$, $i, k \neq 0$ and $p_{ik}^{(\varepsilon)}(\rho) < \infty$, $i \neq 0$, $k \in X$. Then, for any $j \neq 0$, the following statements are equivalent:*

(**a**) $\Phi_j^{(\varepsilon)}(\rho) < \infty$.

(**b**) $\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) < \infty$, $s \neq 0$.

(**c**) *The inverse matrix* $(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}$ *exists.*

Proof. For each $j \neq 0$, let us define a matrix valued function ${}_j\mathbf{A}^{(\varepsilon)}(\rho) = \|{}_ja_{ik}^{(\varepsilon)}(\rho)\|$ by the relation

$$ {}_j\mathbf{A}^{(\varepsilon)}(\rho) = \mathbf{I} + {}_j\mathbf{P}^{(\varepsilon)}(\rho) + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^2 + \cdots, \ \rho \in \mathbb{R}. \tag{3.15} $$

Since each term on the right hand side of (3.15) is non-negative, it follows that the elements ${}_ja_{ik}^{(\varepsilon)}(\rho)$ are well defined and take values in the set $[0, \infty]$. Furthermore, the elements can be written in the following form which gives a probabilistic interpretation,

$$ {}_ja_{ik}^{(\varepsilon)}(\rho) = \mathsf{E}_i \sum_{n=0}^{\infty} e^{\rho\tau^{(\varepsilon)}(n)} \chi(\nu_0^{(\varepsilon)} \wedge \nu_j^{(\varepsilon)} > n, \ \eta_n^{(\varepsilon)} = k), \ i, k \neq 0. \tag{3.16} $$

Let us now show that

$$ \Phi_j^{(\varepsilon)}(\rho) = {}_j\mathbf{A}^{(\varepsilon)}(\rho)\mathbf{p}_j^{(\varepsilon)}(\rho), \ \rho \in \mathbb{R}, \ j \neq 0. \tag{3.17} $$

In order to do this, first note that, for $j \neq 0$,

$$ \chi(\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}) = \sum_{n=0}^{\infty} \sum_{k \neq 0} \chi(\nu_0^{(\varepsilon)} \wedge \nu_j^{(\varepsilon)} > n, \ \eta_n^{(\varepsilon)} = k, \ \eta_{n+1}^{(\varepsilon)} = j). \tag{3.18} $$

Using (3.18) and the regenerative property of the semi-Markov process, the following is obtained, for $i, j \neq 0$,

$$ \begin{aligned} \phi_{ij}^{(\varepsilon)}(\rho) &= \sum_{n=0}^{\infty} \sum_{k \neq 0} \mathsf{E}_i e^{\rho\mu_j^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} \wedge \nu_j^{(\varepsilon)} > n, \ \eta_n^{(\varepsilon)} = k, \ \eta_{n+1}^{(\varepsilon)} = j) \\ &= \sum_{n=0}^{\infty} \sum_{k \neq 0} \mathsf{E}_i e^{\rho\tau^{(\varepsilon)}(n)} \chi(\nu_0^{(\varepsilon)} \wedge \nu_j^{(\varepsilon)} > n, \ \eta_n^{(\varepsilon)} = k) p_{kj}^{(\varepsilon)}(\rho). \end{aligned} \tag{3.19} $$

From (3.16) and (3.19) we get

$$ \phi_{ij}^{(\varepsilon)}(\rho) = \sum_{k \neq 0} {}_ja_{ik}^{(\varepsilon)}(\rho) p_{kj}^{(\varepsilon)}(\rho), \ i, j \neq 0, \tag{3.20} $$

and this proves (3.17).

Let us now define

$$\omega_{ij}^{(\varepsilon)}(\rho) = \sum_{s\neq 0} \omega_{ijs}^{(\varepsilon)}(\rho) = \sum_{n=0}^{\infty} e^{\rho n} \mathsf{P}_i\{\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} > n\},\ \rho \in \mathbb{R},\quad i,j \neq 0. \tag{3.21}$$

Then, we have

$$\omega_{ij}^{(\varepsilon)}(\rho) = \begin{cases} \mathsf{E}_i(\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)}) & \rho = 0, \\ (\mathsf{E}_i e^{\rho(\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)})} - 1)/(e^{\rho} - 1) & \rho \neq 0. \end{cases} \tag{3.22}$$

Also notice that

$$\mathsf{E}_i e^{\rho(\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)})} = \mathsf{E}_i e^{\rho \mu_j^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} > \nu_j^{(\varepsilon)}) + \mathsf{E}_i e^{\rho \mu_0^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} < \nu_j^{(\varepsilon)}),\ i,j \neq 0. \tag{3.23}$$

Using similar calculations as above, it can be shown that

$$\mathsf{E}_i e^{\rho \mu_0^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} < \nu_j^{(\varepsilon)}) = \sum_{k\neq 0} {}_j a_{ik}^{(\varepsilon)}(\rho) p_{k0}^{(\varepsilon)}(\rho),\ i,j \neq 0. \tag{3.24}$$

It follows from (3.20), (3.23), and (3.24) that

$$\mathsf{E}_i e^{\rho(\mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)})} = \sum_{k\neq 0} {}_j a_{ik}^{(\varepsilon)}(\rho) \left(p_{kj}^{(\varepsilon)}(\rho) + p_{k0}^{(\varepsilon)}(\rho) \right),\ i,j \neq 0. \tag{3.25}$$

Let us now show that (**a**) implies (**b**).

By iterating relation (3.6) we obtain,

$$\begin{aligned} \boldsymbol{\Phi}_j^{(\varepsilon)}(\rho) &= (\mathbf{I} + {}_j\mathbf{P}^{(\varepsilon)}(\rho) + \cdots + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^n)\mathbf{p}_j^{(\varepsilon)}(\rho) \\ &\quad + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^{n+1} \boldsymbol{\Phi}_j^{(\varepsilon)}(\rho),\ n = 1,2,\ldots \end{aligned} \tag{3.26}$$

Since $\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho) < \infty$, it follows from (3.26) that

$$({}_j\mathbf{P}^{(\varepsilon)}(\rho))^{n+1} \boldsymbol{\Phi}_j^{(\varepsilon)}(\rho) \to \mathbf{0},\ \text{as } n \to \infty. \tag{3.27}$$

The assumptions of the lemma guarantee that $\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho) > 0$. From this and relation (3.27) we can conclude that $({}_j\mathbf{P}^{(\varepsilon)}(\rho))^{n+1} \to \mathbf{0}$, as $n \to \infty$. It is known that this holds if and only if the matrix series (3.15) converges in norms, that is, ${}_j\mathbf{A}^{(\varepsilon)}(\rho)$ is finite. From this and relations (3.21), (3.22), and (3.25) it follows that (**b**) holds.

Next we show that (**b**) implies (**c**).

By summing over all $s \neq 0$ in relation (3.14) it follows that

$$\boldsymbol{\omega}_j^{(\varepsilon)}(\rho) = \boldsymbol{\varphi}^{(\varepsilon)}(\rho) + {}_j\mathbf{P}^{(\varepsilon)}(\rho)\boldsymbol{\omega}_j^{(\varepsilon)}(\rho),\ \rho \in \mathbb{R}, \tag{3.28}$$

where

$$\boldsymbol{\omega}_j^{(\varepsilon)}(\rho) = \left[\omega_{1j}^{(\varepsilon)}(\rho) \quad \cdots \quad \omega_{Nj}^{(\varepsilon)}(\rho)\right]^T, \ j \neq 0,$$

and

$$\boldsymbol{\varphi}^{(\varepsilon)}(\rho) = \left[\varphi_1^{(\varepsilon)}(\rho) \quad \cdots \quad \varphi_N^{(\varepsilon)}(\rho)\right]^T.$$

By iterating relation (3.28) we get

$$\begin{aligned} \boldsymbol{\omega}_j^{(\varepsilon)}(\rho) &= (\mathbf{I} + {}_j\mathbf{P}^{(\varepsilon)}(\rho) + \cdots + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^n)\boldsymbol{\varphi}^{(\varepsilon)}(\rho) \\ &\quad + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^{n+1}\boldsymbol{\omega}_j^{(\varepsilon)}(\rho), \ n = 1, 2, \ldots \end{aligned} \tag{3.29}$$

It follows from (**b**) and the definition of $\omega_{ij}^{(\varepsilon)}(\rho)$ that $0 < \boldsymbol{\omega}_j^{(\varepsilon)}(\rho) < \infty$. So, letting $n \to \infty$ in (3.29) and using similar arguments as above, it follows that the matrix series (3.15) converges in norms. It is then known that the inverse matrix $(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}$ exists, that is, (**c**) holds.

Let us finally argue that (**c**) implies (**a**).

If $(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}$ exists, then the following relation holds,

$$(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1} = \mathbf{I} + {}_j\mathbf{P}^{(\varepsilon)}(\rho)(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}. \tag{3.30}$$

Iteration of (3.30) gives

$$\begin{aligned} (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1} &= \mathbf{I} + {}_j\mathbf{P}^{(\varepsilon)}(\rho) + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^2 + \cdots + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^n \\ &\quad + ({}_j\mathbf{P}^{(\varepsilon)}(\rho))^{n+1}(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}, \ n = 1, 2, \ldots \end{aligned} \tag{3.31}$$

Letting $n \to \infty$ in (3.31) it follows that ${}_j\mathbf{A}^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1} < \infty$. From (3.17) we now see that (**a**) holds. □

4 Asymptotic Solidarity Properties

In this section we prove some asymptotic solidarity properties for moment generating functions of first hitting times.

Let us define

$${}_k\phi_{ij}^{(\varepsilon)}(\rho) = \mathsf{E}_i e^{\rho\mu_j^{(\varepsilon)}} \chi(\nu_0^{(\varepsilon)} \wedge \nu_k^{(\varepsilon)} > \nu_j^{(\varepsilon)}), \ \rho \in \mathbb{R}, \ i, j, k \in X.$$

If the states $\{1, \ldots, N\}$ is a communicating class and $\phi_{ii}^{(\varepsilon)}(\rho) \leq 1$ for some $i \neq 0$, then it can be shown (see, for example, Petersson (2015)) that the following relation holds for all $j \neq 0$,

$$(1 - \phi_{ii}^{(\varepsilon)}(\rho))(1 - {}_i\phi_{jj}^{(\varepsilon)}(\rho)) = (1 - \phi_{jj}^{(\varepsilon)}(\rho))(1 - {}_j\phi_{ii}^{(\varepsilon)}(\rho)). \tag{4.1}$$

Relation (4.1) is useful in order to prove various solidarity properties for semi-Markov processes. In particular, if $\phi_{ii}^{(\varepsilon)}(\rho) = 1$, relation (4.1) reduces to

$$(1 - \phi_{jj}^{(\varepsilon)}(\rho))(1 - {}_j\phi_{ii}^{(\varepsilon)}(\rho)) = 0. \tag{4.2}$$

From the regenerative property of the semi-Markov process it follows that

$$\phi_{ii}^{(\varepsilon)}(\rho) = {}_j\phi_{ii}^{(\varepsilon)}(\rho) + {}_i\phi_{ij}^{(\varepsilon)}(\rho)\phi_{ji}^{(\varepsilon)}(\rho), \ j \neq 0, i. \tag{4.3}$$

Since $\{1, \ldots, N\}$ is a communicating class, we have ${}_i\phi_{ij}^{(\varepsilon)}(\rho) > 0$ and $\phi_{ji}^{(\varepsilon)}(\rho) > 0$. So, if $\phi_{ii}^{(\varepsilon)}(\rho) = 1$ it follows from (4.3) that ${}_j\phi_{ii}^{(\varepsilon)}(\rho) < 1$. From this and (4.2) we can conclude that $\phi_{jj}^{(\varepsilon)}(\rho) = 1$ for all $j \neq 0$. Thus, we have the following lemma:

Lemma 4.1. *Assume that we for some $\varepsilon \geq 0$ have that $g_{kj}^{(\varepsilon)} > 0$ for all $k, j \neq 0$. Then, if we for some $i \neq 0$ and $\rho \in \mathbb{R}$, have that $\phi_{ii}^{(\varepsilon)}(\rho) = 1$, it follows that $\phi_{jj}^{(\varepsilon)}(\rho) = 1$ for all $j \neq 0$.*

Let us now define the following characteristic equation,

$$\phi_{ii}^{(\varepsilon)}(\rho) = 1. \tag{4.4}$$

where $i \neq 0$. The root of equation (4.4) plays an important role for the quasi-stationary distribution.

The following lemma shows, in particular, that the characteristic equation has a unique non-negative solution for sufficiently small ε, which does not depend on i.

Lemma 4.2. *If conditions* **A**–**C** *hold, then there exists $\delta \in (0, \beta]$ such that the following holds:*

- **(i)** $\phi_{kj}^{(\varepsilon)}(\rho) \to \phi_{kj}^{(0)}(\rho) < \infty$, *as* $\varepsilon \to 0$, $\rho \leq \delta$, $k, j \neq 0$.
- **(ii)** $\omega_{kjs}^{(\varepsilon)}(\rho) \to \omega_{kjs}^{(0)}(\rho) < \infty$, *as* $\varepsilon \to 0$, $\rho \leq \delta$, $k, j, s \neq 0$.
- **(iii)** $\phi_{jj}^{(0)}(\delta) \in (1, \infty)$, $j \neq 0$.
- **(iv)** *For sufficiently small ε, there exists a unique non-negative root $\rho^{(\varepsilon)}$ of the characteristic equation (4.4) which does not depend on i.*
- **(v)** $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta$ *as* $\varepsilon \to 0$.

Proof. Let $i \neq 0$ and $\beta_i \leq \beta$ be the values given in condition **C**. It follows from conditions **B** and **C** that $\phi^{(0)}_{ii}(\rho)$ is a continuous and strictly increasing function for $\rho \leq \beta_i$. Since $\phi^{(0)}_{ii}(0) = g^{(0)}_{ii} \leq 1$ and $\phi^{(0)}_{ii}(\beta_i) > 1$, there exists a unique $\rho' \in [0, \beta_i)$ such that $\phi^{(0)}_{ii}(\rho') = 1$. Moreover, by Lemma 4.1,

$$\phi^{(0)}_{jj}(\rho') = 1, \ j \neq 0. \tag{4.5}$$

For all $j \neq 0$, we have

$$\phi^{(0)}_{jj}(\rho') = {}_k\phi^{(0)}_{jj}(\rho') + {}_j\phi^{(0)}_{jk}(\rho')\phi^{(0)}_{kj}(\rho'), \ k \neq 0, j. \tag{4.6}$$

It follows from (4.5), (4.6), and condition **B**, that

$$\phi^{(0)}_{kj}(\rho') < \infty, \ k, j \neq 0. \tag{4.7}$$

From (4.7) and Lemma 3.1 we get that $\det(\mathbf{I} - {}_j\mathbf{P}^{(0)}(\rho')) \neq 0$, for $j \neq 0$. Under condition **C**, the elements of $\mathbf{I} - {}_j\mathbf{P}^{(0)}(\rho)$ are continuous functions for $\rho \leq \beta$. This implies that we for each $j \neq 0$ can find $\beta_j \in (\rho', \beta_i]$ such that $\det(\mathbf{I} - {}_j\mathbf{P}^{(0)}(\beta_j)) \neq 0$. By condition **C** we also have that $p^{(0)}_{kj}(\beta_j) < \infty$ for $k \neq 0$, $j \in X$. It now follows from Lemma 3.1 that $\phi^{(0)}_{kj}(\beta_j) < \infty$, $k, j \neq 0$. If we define $\delta = \min\{\beta_1, \ldots, \beta_N\}$, it follows that

$$\phi^{(0)}_{kj}(\rho) < \infty, \ \rho \leq \delta, \ k, j \neq 0. \tag{4.8}$$

Now, let $\rho \leq \delta$ be fixed. Relation (4.8) and Lemma 3.1 imply that

$$\det(\mathbf{I} - {}_j\mathbf{P}^{(0)}(\rho)) \neq 0, \ j \neq 0. \tag{4.9}$$

Note that we have

$$p^{(\varepsilon)}_{kj}(\rho) = p^{(\varepsilon)}_{kj} \sum_{n=0}^{\infty} e^{\rho n} f^{(\varepsilon)}_{kj}(n), \ k, j \in X. \tag{4.10}$$

Since $f^{(\varepsilon)}_{kj}(n)$ are proper probability distributions, it follows from (4.10) and conditions **A** and **C** that

$$p^{(\varepsilon)}_{kj}(\rho) \to p^{(0)}_{kj}(\rho) < \infty, \ \text{as } \varepsilon \to 0, \ k \neq 0, \ j \in X. \tag{4.11}$$

It follows from (4.9) and (4.11) that there exists $\varepsilon_1 > 0$ such that we for all $\varepsilon \leq \varepsilon_1$ have that $\det(\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho)) \neq 0$ and $p^{(\varepsilon)}_{kj}(\rho) < \infty$, for all $k, j \neq 0$. Using Lemma 3.1 once again, it now follows that $\phi^{(\varepsilon)}_{kj}(\rho) < \infty$, $k, j \neq 0$, for all $\varepsilon \leq \varepsilon_1$. Moreover, in this case, the system of linear equations (3.6) has a unique solution for $\varepsilon \leq \varepsilon_1$ given by

$$\mathbf{\Phi}_j^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}\mathbf{p}_j^{(\varepsilon)}(\rho), \ j \neq 0. \tag{4.12}$$

From (4.11) and (4.12) it follows that

$$\phi_{kj}^{(\varepsilon)}(\rho) \to \phi_{kj}^{(0)}(\rho) < \infty, \text{ as } \varepsilon \to 0, \ k, j \neq 0.$$

This completes the proof of part (**i**).

For the proof of part (**ii**) we first note that, since $\phi_{kj}^{(\varepsilon)}(\rho) < \infty$ for $\varepsilon \leq \varepsilon_1$, $k, j \neq 0$, it follows from Lemma 3.1 that $\omega_{kjs}^{(\varepsilon)}(\rho) < \infty$ for $\varepsilon \leq \varepsilon_1$, $k, j, s \neq 0$. From this, and arguments given above, we see that the system of linear equations given by relation (3.14) has a unique solution for $\varepsilon \leq \varepsilon_1$ given by

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}\widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho), \ j, s \neq 0. \tag{4.13}$$

Now, since $\mathsf{E}_i e^{\rho\kappa_1^{(\varepsilon)}} = \sum_{j \in X} p_{ij}^{(\varepsilon)}(\rho)$, it follows from (3.9) and (4.11) that $\varphi_i^{(\varepsilon)}(\rho) \to \varphi_i^{(0)}(\rho) < \infty$ as $\varepsilon \to 0$, $i \neq 0$. Using this and relations (4.11) and (4.13) we can conclude that part (**ii**) holds.

By part (**i**) we have, in particular, $\phi_{jj}^{(\varepsilon)}(\delta) \to \phi_{jj}^{(0)}(\delta) < \infty$ as $\varepsilon \to 0$, for all $j \neq 0$. Furthermore, since $\rho' < \delta$ and $\phi_{jj}^{(0)}(\rho)$ is strictly increasing for $\rho \leq \delta$, it follows from (4.5) that $\phi_{jj}^{(0)}(\delta) > 1$, $j \neq 0$. This proves part (**iii**).

Let us now prove part (**iv**).

It follows from (**i**) and (**iii**) that we can find $\varepsilon_2 > 0$ such that $\phi_{jj}^{(\varepsilon)}(\delta) \in (1, \infty)$, $j \neq 0$, for all $\varepsilon \leq \varepsilon_2$. By conditions **A** and **B** there exists $\varepsilon_3 > 0$ such that, for $i \neq 0$ and $\varepsilon \leq \varepsilon_3$, the functions $g_{ii}^{(\varepsilon)}(n)$ are not concentrated at zero. Thus, for $i \neq 0$ and $\varepsilon \leq \min\{\varepsilon_2, \varepsilon_3\}$, we have that $\phi_{ii}^{(\varepsilon)}(\rho)$ are continuous and strictly increasing functions for $\rho \in [0, \delta]$. Since $\phi_{ii}^{(\varepsilon)}(0) = g_{ii}^{(\varepsilon)} \leq 1$ and $\phi_{ii}^{(\varepsilon)}(\delta) > 1$, there exists a unique $\rho_i^{(\varepsilon)} \in [0, \delta)$ such that $\phi_{ii}^{(\varepsilon)}(\rho_i^{(\varepsilon)}) = 1$. By Lemma 4.1, the root of the characteristic equation does not depend on i so we can write $\rho^{(\varepsilon)}$ instead of $\rho_i^{(\varepsilon)}$. This proves part (**iv**).

Finally we show that $\rho^{(\varepsilon)} \to \rho^{(0)}$ as $\varepsilon \to 0$.

Let $\gamma > 0$ such that $\rho^{(0)} + \gamma \leq \delta$ be arbitrary. Then $\phi_{ii}^{(0)}(\rho^{(0)} - \gamma) < 1$ and $\phi_{ii}^{(0)}(\rho^{(0)} + \gamma) > 1$. From this and part (**i**) we get that there exists $\varepsilon_4 > 0$ such that $\phi_{ii}^{(\varepsilon)}(\rho^{(0)} - \gamma) < 1$ and $\phi_{ii}^{(\varepsilon)}(\rho^{(0)} + \gamma) > 1$, for all $\varepsilon \leq \varepsilon_4$. It follows that $|\rho^{(\varepsilon)} - \rho^{(0)}| < \gamma$ for $\varepsilon \leq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$. This completes the proof of Lemma 4.2. □

5 Quasi-Stationary Distributions

In this section we use renewal theory in order to get a formula for the quasi-stationary distribution.

The probabilities $P_{ij}^{(\varepsilon)}(n) = \mathsf{P}_i\{\xi^{(\varepsilon)}(n) = j,\ \mu_0^{(\varepsilon)} > n\}$, $i, j \neq 0$, satisfy the following discrete time renewal equation,

$$P_{ij}^{(\varepsilon)}(n) = h_{ij}^{(\varepsilon)}(n) + \sum_{k=0}^{n} P_{ij}^{(\varepsilon)}(n-k) g_{ii}^{(\varepsilon)}(k),\ n = 0, 1, \ldots, \tag{5.1}$$

where

$$h_{ij}^{(\varepsilon)}(n) = \mathsf{P}_i\{\xi^{(\varepsilon)}(n) = j,\ \mu_0^{(\varepsilon)} \wedge \mu_i^{(\varepsilon)} > n\}.$$

Since $\sum_{n=0}^{\infty} g_{ii}^{(\varepsilon)}(n) = g_{ii}^{(\varepsilon)} \leq 1$, relation (5.1) defines a possibly improper renewal equation.

Let us now, for each $n = 0, 1, \ldots$, multiply both sides of (5.1) by $e^{\rho^{(\varepsilon)} n}$, where $\rho^{(\varepsilon)}$ is the root of the characteristic equation $\phi_{ii}^{(\varepsilon)}(\rho) = 1$. Then, we get

$$\widetilde{P}_{ij}^{(\varepsilon)}(n) = \widetilde{h}_{ij}^{(\varepsilon)}(n) + \sum_{k=0}^{n} \widetilde{P}_{ij}^{(\varepsilon)}(n-k) \widetilde{g}_{ii}^{(\varepsilon)}(k),\ n = 0, 1, \ldots, \tag{5.2}$$

where

$$\widetilde{P}_{ij}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)} n} P_{ij}^{(\varepsilon)}(n),\ \widetilde{h}_{ij}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)} n} h_{ij}^{(\varepsilon)}(n),\ \widetilde{g}_{ii}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)} n} g_{ii}^{(\varepsilon)}(n).$$

By the definition of the root of the characteristic equation, relation (5.2) defines a proper renewal equation. We can now use the classical renewal theorem in order to get a formula for the quasi-stationary distribution.

Lemma 5.1. *Assume that conditions* **A**–**D** *hold. Then:*

(i) *For sufficiently small ε, the quasi stationary distribution $\pi_j^{(\varepsilon)}$, given by relation* (2.2)*, have the following representation,*

$$\pi_j^{(\varepsilon)} = \frac{\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)})}{\omega_{ii1}^{(\varepsilon)}(\rho^{(\varepsilon)}) + \cdots + \omega_{iiN}^{(\varepsilon)}(\rho^{(\varepsilon)})},\ i, j \neq 0. \tag{5.3}$$

(ii) *For $j = 1, \ldots, N$, we have*

$$\pi_j^{(\varepsilon)} \to \pi_j^{(0)},\ \text{as } \varepsilon \to 0.$$

Proof. Under condition **D**, the functions $g_{ii}^{(0)}(n)$ are non-periodic for all $i \neq 0$. By Lemma 4.2 we have that $\phi_{ii}^{(\varepsilon)}(\rho) \to \phi_{ii}^{(0)}(\rho)$ as $\varepsilon \to 0$, for $\rho \leq \delta$, $i \neq 0$. From this it follows that $g_{ii}^{(\varepsilon)}(n) \to g_{ii}^{(0)}(n)$ as $\varepsilon \to 0$, for $n \geq 0$, $i \neq 0$. Thus, we can conclude that there exists $\varepsilon_1 > 0$ such that the functions $\widetilde{g}_{ii}^{(\varepsilon)}(n)$, $i \neq 0$, are non-periodic for all $\varepsilon \leq \varepsilon_1$.

Now choose γ such that $\rho^{(0)} < \gamma < \delta$. Using Lemma 4.2, we get the following for all $i \neq 0$,

$$\begin{aligned}\limsup_{0 \leq \varepsilon \to 0} \sum_{n=0}^{\infty} n \widetilde{g}_{ii}^{(\varepsilon)}(n) &\leq \limsup_{0 \leq \varepsilon \to 0} \sum_{n=0}^{\infty} n e^{\gamma n} g_{ii}^{(\varepsilon)}(n) \\ &\leq \left(\sup_{n \geq 0} n e^{-(\delta - \gamma) n} \right) \phi_{ii}^{(0)}(\delta) < \infty.\end{aligned}$$

Thus, there exists $\varepsilon_2 > 0$ such that the distributions $\widetilde{g}_{ii}^{(\varepsilon)}(n)$, $i \neq 0$, have finite mean for all $\varepsilon \leq \varepsilon_2$.

Furthermore, it follows from Lemma 4.2 that, for all $i, j \neq 0$,

$$\limsup_{0 \leq \varepsilon \to 0} \sum_{n=0}^{\infty} \widetilde{h}_{ij}^{(\varepsilon)}(n) \leq \limsup_{0 \leq \varepsilon \to 0} \sum_{n=0}^{\infty} e^{\delta n} h_{ij}^{(\varepsilon)}(n) = \omega_{iij}^{(0)}(\delta) < \infty,$$

so there exists $\varepsilon_3 > 0$ such that $\sum_{n=0}^{\infty} \widetilde{h}_{ij}^{(\varepsilon)}(n) < \infty$, $i, j \neq 0$, for all $\varepsilon \leq \varepsilon_3$.

Now, let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. For all $\varepsilon \leq \varepsilon_0$, the assumptions of the discrete time renewal theorem are satisfied for the renewal equation defined by (5.2). This yields

$$\widetilde{P}_{ij}^{(\varepsilon)}(n) \to \frac{\sum_{k=0}^{\infty} \widetilde{h}_{ij}(k)}{\sum_{k=0}^{\infty} k \widetilde{g}_{ii}^{(\varepsilon)}(k)}, \text{ as } n \to \infty, \ i, j \neq 0, \ \varepsilon \leq \varepsilon_0. \tag{5.4}$$

Note that we have

$$\mathsf{P}_i\{\xi^{(\varepsilon)}(n) = j \,|\, \mu_0^{(\varepsilon)} > n\} = \frac{\widetilde{P}_{ij}^{(\varepsilon)}(n)}{\sum_{k=1}^{N} \widetilde{P}_{ik}^{(\varepsilon)}(n)}, \ n = 0, 1, \ldots, \ i, j \neq 0. \tag{5.5}$$

It follows from (5.4) and (5.5) that, for $\varepsilon \leq \varepsilon_0$,

$$\mathsf{P}_i\{\xi^{(\varepsilon)}(n) = j \,|\, \mu_0^{(\varepsilon)} > n\} \to \frac{\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)})}{\sum_{k=1}^{N} \omega_{iik}^{(\varepsilon)}(\rho^{(\varepsilon)})}, \text{ as } n \to \infty, \ i, j \neq 0.$$

This proves part (**i**).

For the proof of part (**ii**), first note that,

$$
\begin{aligned}
0 &\le \limsup_{0 \le \varepsilon \to 0} \sum_{n=N}^{\infty} e^{\rho^{(\varepsilon)} n} h_{ij}^{(\varepsilon)}(n) \\
&\le \limsup_{0 \le \varepsilon \to 0} \sum_{n=N}^{\infty} e^{\gamma n} h_{ij}^{(\varepsilon)}(n) \\
&\le e^{-(\delta-\gamma)N} \omega_{iij}^{(0)}(\delta) < \infty, \ N = 1, 2, \ldots, \ i, j \neq 0.
\end{aligned}
\tag{5.6}
$$

Relation (5.6) implies that

$$
\lim_{N \to \infty} \limsup_{0 \le \varepsilon \to 0} \sum_{n=N}^{\infty} e^{\rho^{(\varepsilon)} n} h_{ij}^{(\varepsilon)}(n) = 0, \ i, j \neq 0. \tag{5.7}
$$

It follows from Lemma 4.2 that

$$
\rho^{(\varepsilon)} \to \rho^{(0)}, \text{ as } \varepsilon \to 0. \tag{5.8}
$$

Since $h_{ij}^{(\varepsilon)}(n)$, for each $n = 0, 1, \ldots$, can be written as a finite sum where each term in the sum is a continuous function of quantities given in condition **A**, we have

$$
h_{ij}^{(\varepsilon)}(n) \to h_{ij}^{(0)}(n), \text{ as } \varepsilon \to 0, \ i, j \neq 0. \tag{5.9}
$$

It now follows from (5.7), (5.8), and (5.9) that

$$
\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)}) \to \omega_{iij}^{(0)}(\rho^{(0)}), \text{ as } \varepsilon \to 0, \ i, j \neq 0. \tag{5.10}
$$

Relations (5.3) and (5.10) show that part (**ii**) of Lemma 5.1 holds. □

6 Expansions of Moment Functionals

In this section asymptotic expansions for mixed power-exponential moment functionals are constructed.

Let us define the following mixed power-exponential moment functionals for distributions of first hitting times,

$$
\phi_{ij}^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} g_{ij}^{(\varepsilon)}(n), \ \rho \in \mathbb{R}, \ r = 0, 1, \ldots, \ i, j \in X.
$$

By definition, $\phi_{ij}^{(\varepsilon)}(\rho, 0) = \phi_{ij}^{(\varepsilon)}(\rho)$.

Furthermore, let us recall from Section 2 that we define

$$
p_{ij}^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} Q_{ij}^{(\varepsilon)}(n), \ \rho \in \mathbb{R}, \ r = 0, 1, \ldots, \ i, j \in X.
$$

By definition, $p_{ij}^{(\varepsilon)}(\rho, 0) = p_{ij}^{(\varepsilon)}(\rho)$.

It follows from conditions **A**–**C** and Lemma 4.2 that, for $\rho < \delta$ and sufficiently small ε, the functions $\phi_{ij}^{(\varepsilon)}(\rho)$ and $p_{ij}^{(\varepsilon)}(\rho)$ are arbitrarily many times differentiable with respect to ρ, and the derivatives of order r are given by $\phi_{ij}^{(\varepsilon)}(\rho, r)$ and $p_{ij}^{(\varepsilon)}(\rho, r)$, respectively.

Recall from Section 3 that the following system of linear equations holds,

$$\phi_{ij}^{(\varepsilon)}(\rho) = p_{ij}^{(\varepsilon)}(\rho) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho) \phi_{lj}^{(\varepsilon)}(\rho), \ i, j \neq 0. \tag{6.1}$$

Differentiating relation (6.1) gives

$$\phi_{ij}^{(\varepsilon)}(\rho, r) = \lambda_{ij}^{(\varepsilon)}(\rho, r) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho) \phi_{lj}^{(\varepsilon)}(\rho, r), \ r = 1, 2, \ldots, \ i, j \neq 0, \tag{6.2}$$

where

$$\lambda_{ij}^{(\varepsilon)}(\rho, r) = p_{ij}^{(\varepsilon)}(\rho, r) + \sum_{m=1}^{r} \binom{r}{m} \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho, m) \phi_{lj}^{(\varepsilon)}(\rho, r - m). \tag{6.3}$$

In order to write relations (6.1), (6.2), and (6.3) in matrix form, let us define the following column vectors,

$$\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho, r) = \left[\phi_{1j}^{(\varepsilon)}(\rho, r) \quad \cdots \quad \phi_{Nj}^{(\varepsilon)}(\rho, r)\right]^T, \ j \neq 0, \tag{6.4}$$

$$\mathbf{p}_j^{(\varepsilon)}(\rho, r) = \left[p_{1j}^{(\varepsilon)}(\rho, r) \quad \cdots \quad p_{Nj}^{(\varepsilon)}(\rho, r)\right]^T, \ j \neq 0, \tag{6.5}$$

$$\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, r) = \left[\lambda_{1j}^{(\varepsilon)}(\rho, r) \quad \cdots \quad \lambda_{Nj}^{(\varepsilon)}(\rho, r)\right]^T, \ j \neq 0. \tag{6.6}$$

Let us also, for $j \neq 0$, define $N \times N$-matrices ${}_j\mathbf{P}^{(\varepsilon)}(\rho, r) = \| {}_j p_{ik}^{(\varepsilon)}(\rho, r) \|$ where the elements are given by

$${}_j p_{ik}^{(\varepsilon)}(\rho, r) = \begin{cases} p_{ik}^{(\varepsilon)}(\rho, r) & i = 1, \ldots, N, \ k \neq j, \\ 0 & i = 1, \ldots, N, \ k = j. \end{cases} \tag{6.7}$$

Using (6.1)–(6.7) we can for any $j \neq 0$ write the following recursive systems of linear equations,

$$\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho) = \mathbf{p}_j^{(\varepsilon)}(\rho) + {}_j\mathbf{P}^{(\varepsilon)}(\rho) \boldsymbol{\Phi}_j^{(\varepsilon)}(\rho), \tag{6.8}$$

and, for $r = 1, 2, \ldots$,

$$\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho, r) = \boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, r) + {}_j\mathbf{P}^{(\varepsilon)}(\rho) \boldsymbol{\Phi}_j^{(\varepsilon)}(\rho, r), \tag{6.9}$$

where

$$\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, r) = \mathbf{p}_j^{(\varepsilon)}(\rho, r) + \sum_{m=1}^{r} \binom{r}{m} {}_j\mathbf{P}^{(\varepsilon)}(\rho, m)\boldsymbol{\Phi}_j^{(\varepsilon)}(\rho, r-m). \tag{6.10}$$

Let us now introduce the following perturbation condition, which is assumed to hold for some $\rho < \delta$, where δ is the parameter in Lemma 4.2:

$\mathbf{P}^*_\mathbf{k}$: $p_{ij}^{(\varepsilon)}(\rho, r) = p_{ij}^{(0)}(\rho, r) + p_{ij}[\rho, r, 1]\varepsilon + \cdots + p_{ij}[\rho, r, k-r]\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, $i \neq 0$, $j \in X$, where $|p_{ij}[\rho, r, n]| < \infty$, for $r = 0, \ldots, k$, $n = 1, \ldots, k-r$, $i \neq 0$, $j \in X$.

For convenience we denote $p_{ij}^{(0)}(\rho, r) = p_{ij}[\rho, r, 0]$, for $r = 0, \ldots, k$.

Note that if condition $\mathbf{P}^*_\mathbf{k}$ holds, then, for $r = 0, \ldots, k$, we have the following asymptotic matrix expansions:

$${}_j\mathbf{P}^{(\varepsilon)}(\rho, r) = {}_j\mathbf{P}[\rho, r, 0] + {}_j\mathbf{P}[\rho, r, 1]\varepsilon + \cdots + {}_j\mathbf{P}[\rho, r, k-r]\varepsilon^{k-r} + \mathbf{o}(\varepsilon^{k-r}), \tag{6.11}$$

and

$$\mathbf{p}_j^{(\varepsilon)}(\rho, r) = \mathbf{p}_j[\rho, r, 0] + \mathbf{p}_j[\rho, r, 1]\varepsilon + \cdots + \mathbf{p}_j[\rho, r, k-r]\varepsilon^{k-r} + \mathbf{o}(\varepsilon^{k-r}). \tag{6.12}$$

Here, and in what follows, $\mathbf{o}(\varepsilon^p)$ denotes a matrix-valued function of ε where all elements are of order $o(\varepsilon^p)$. The coefficients in (6.11) are $N \times N$-matrices ${}_j\mathbf{P}[\rho, r, n] = \| {}_jp_{ik}[\rho, r, n] \|$ with elements given by

$${}_jp_{ik}[\rho, r, n] = \begin{cases} p_{ik}[\rho, r, n] & i = 1, \ldots, N, \ k \neq j, \\ 0 & i = 1, \ldots, N, \ k = j, \end{cases}$$

and the coefficients in (6.12) are column vectors defined by

$$\mathbf{p}_j[\rho, r, n] = \begin{bmatrix} p_{1j}[\rho, r, n] & \cdots & p_{Nj}[\rho, r, n] \end{bmatrix}^T.$$

Let us now define the following matrix, which will play an important role in what follows,

$${}_j\mathbf{U}^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}.$$

Under conditions **A**–**C**, it follows from Lemmas 3.1 and 4.2 that ${}_j\mathbf{U}^{(\varepsilon)}(\rho)$ is well defined for $\rho \leq \delta$ and sufficiently small ε.

The following lemma gives an asymptotic expansion for ${}_j\mathbf{U}^{(\varepsilon)}(\rho)$.

Lemma 6.1. *Assume that conditions* **A**–**C** *and* $\mathbf{P}^*_\mathbf{k}$ *hold. Then we have the following asymptotic expansion,*

$${}_j\mathbf{U}^{(\varepsilon)}(\rho) = {}_j\mathbf{U}[\rho, 0] + {}_j\mathbf{U}[\rho, 1]\varepsilon + \cdots + {}_j\mathbf{U}[\rho, k]\varepsilon^k + \mathbf{o}(\varepsilon^k), \tag{6.13}$$

where

$${}_j\mathbf{U}[\rho, n] = \begin{cases} (\mathbf{I} - {}_j\mathbf{P}^{(0)}(\rho))^{-1} & n = 0, \\ {}_j\mathbf{U}[\rho, 0] \sum_{q=1}^{n} {}_j\mathbf{P}[\rho, 0, q] {}_j\mathbf{U}[\rho, n-q] & n = 1, \ldots, k. \end{cases} \tag{6.14}$$

Proof. As already mentioned above, conditions **A**–**C** ensure us that the inverse ${}_j\mathbf{U}^{(\varepsilon)}(\rho)$ exists for sufficiently small ε. In this case, it is known that the expansion (6.13) exists under condition $\mathbf{P}^*_\mathbf{k}$. To see that the coefficients are given by (6.14), first note that

$$\begin{aligned}\mathbf{I} &= (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))\,{}_j\mathbf{U}^{(\varepsilon)}(\rho) \\ &= (\mathbf{I} - {}_j\mathbf{P}^{(0)}(\rho) - {}_j\mathbf{P}[\rho,0,1]\varepsilon - \cdots - {}_j\mathbf{P}[\rho,0,k]\varepsilon^k + \mathbf{o}(\varepsilon^k)) \\ &\times ({}_j\mathbf{U}[\rho,0] + {}_j\mathbf{U}[\rho,1]\varepsilon + \cdots + {}_j\mathbf{U}[\rho,k]\varepsilon^k + \mathbf{o}(\varepsilon^k)).\end{aligned} \tag{6.15}$$

By first expanding both sides of equation (6.15) and then, for $n = 0, 1, \ldots, k$, equating coefficients of ε^n in the left and right hand sides, we get formula (6.14). □

We are now ready to construct asymptotic expansions for $\mathbf{\Phi}_j^{(\varepsilon)}(\rho, r)$.

Lemma 6.2. *Assume that conditions* **A**–**C** *and* $\mathbf{P}^*_\mathbf{k}$ *hold. Then:*

(i) *We have the following asymptotic expansion,*

$$\mathbf{\Phi}_j^{(\varepsilon)}(\rho) = \mathbf{\Phi}_j[\rho,0,0] + \mathbf{\Phi}_j[\rho,0,1]\varepsilon + \cdots + \mathbf{\Phi}_j[\rho,0,k]\varepsilon^k + \mathbf{o}(\varepsilon^k),$$

where

$$\mathbf{\Phi}_j[\rho,0,n] = \begin{cases} \mathbf{\Phi}_j^{(0)}(\rho) & n = 0, \\ \sum_{q=0}^n {}_j\mathbf{U}[\rho,q]\mathbf{p}_j[\rho,0,n-q] & n = 1,\ldots,k. \end{cases}$$

(ii) *For $r = 1, \ldots, k$, we have the following asymptotic expansions,*

$$\mathbf{\Phi}_j^{(\varepsilon)}(\rho,r) = \mathbf{\Phi}_j[\rho,r,0] + \mathbf{\Phi}_j[\rho,r,1]\varepsilon + \cdots + \mathbf{\Phi}_j[\rho,r,k-r]\varepsilon^{k-r} + \mathbf{o}(\varepsilon^{k-r}),$$

where

$$\mathbf{\Phi}_j[\rho,r,n] = \begin{cases} \mathbf{\Phi}_j^{(0)}(\rho,r) & n = 0, \\ \sum_{q=0}^n {}_j\mathbf{U}[\rho,q]\boldsymbol{\lambda}_j[\rho,r,n-q] & n = 1,\ldots,k-r, \end{cases}$$

and, for $t = 0, \ldots, k-r$,

$$\boldsymbol{\lambda}_j[\rho,r,t] = \mathbf{p}_j[\rho,r,t] + \sum_{m=1}^r \binom{r}{m} \sum_{q=0}^t {}_j\mathbf{P}[\rho,m,q]\mathbf{\Phi}_j[\rho,r-m,t-q].$$

Proof. Under conditions **A**–**C**, we have, for sufficiently small ε, that the recursive systems of linear equations given by relations (6.8), (6.9), and

(6.10), all have finite components. Moreover, the inverse matrix ${}_j\mathbf{U}^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}$ exists, so these systems have unique solutions.

It follows from (6.8), Lemma 6.1, and condition $\mathbf{P}^*_\mathbf{k}$ that

$$\begin{aligned}\mathbf{\Phi}_j^{(\varepsilon)}(\rho) &= {}_j\mathbf{U}^{(\varepsilon)}(\rho)\mathbf{p}_j^{(\varepsilon)}(\rho) \\ &= ({}_j\mathbf{U}[\rho, 0] + {}_j\mathbf{U}[\rho, 1]\varepsilon + \cdots + {}_j\mathbf{U}[\rho, k]\varepsilon^k + \mathbf{o}(\varepsilon^k)) \\ &\quad \times (\mathbf{p}_j[\rho, 0, 0] + \mathbf{p}_j[\rho, 0, 1]\varepsilon + \cdots + \mathbf{p}_j[\rho, 0, k]\varepsilon^k + \mathbf{o}(\varepsilon^k)).\end{aligned} \tag{6.16}$$

By expanding the right hand side of equation (6.16), we see that part (**i**) of Lemma 6.2 holds.

With $r = 1$, relation (6.10) takes the form

$$\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, 1) = \mathbf{p}_j^{(\varepsilon)}(\rho, 1) + {}_j\mathbf{P}^{(\varepsilon)}(\rho, 1)\mathbf{\Phi}_j^{(\varepsilon)}(\rho). \tag{6.17}$$

From (6.17), condition $\mathbf{P}^*_\mathbf{k}$, and part (**i**), we get

$$\begin{aligned}\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, 1) &= \mathbf{p}_j[\rho, 1, 0] + \cdots + \mathbf{p}_j[\rho, 1, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1}) \\ &\quad + ({}_j\mathbf{P}[\rho, 1, 0] + \cdots + {}_j\mathbf{P}[\rho, 1, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1})) \\ &\quad \times (\mathbf{\Phi}_j[\rho, 0, 0] + \cdots + \mathbf{\Phi}_j[\rho, 0, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1})).\end{aligned} \tag{6.18}$$

Expanding the right hand side of (6.18) gives

$$\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, 1) = \boldsymbol{\lambda}_j[\rho, 1, 0] + \boldsymbol{\lambda}_j[\rho, 1, 1]\varepsilon + \cdots + \boldsymbol{\lambda}_j[\rho, 1, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1}), \tag{6.19}$$

where

$$\boldsymbol{\lambda}_j[\rho, 1, t] = \mathbf{p}_j[\rho, 1, t] + \sum_{q=0}^{t} {}_j\mathbf{P}[\rho, 1, q]\mathbf{\Phi}_j[\rho, 0, t-q], \ t = 0, \ldots, k-1.$$

It now follows from (6.9), (6.19), and Lemma 6.1 that

$$\begin{aligned}\mathbf{\Phi}_j^{(\varepsilon)}(\rho, 1) &= {}_j\mathbf{U}^{(\varepsilon)}(\rho)\boldsymbol{\lambda}_j^{(\varepsilon)}(\rho, 1) \\ &= ({}_j\mathbf{U}[\rho, 0] + \cdots + {}_j\mathbf{U}[\rho, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1})) \\ &\quad \times (\boldsymbol{\lambda}_j[\rho, 1, 0] + \cdots + \boldsymbol{\lambda}_j[\rho, 1, k-1]\varepsilon^{k-1} + \mathbf{o}(\varepsilon^{k-1})).\end{aligned} \tag{6.20}$$

By expanding the right hand side of equation (6.20) we get the expansion in part (**ii**) for $r = 1$. If $k = 1$, this concludes the proof. If $k \geq 2$, we can repeat the steps above, successively, for $r = 2, \ldots, k$. This gives the expansions and formulas given in part (**ii**). □

Let us now define the following mixed power exponential moment functionals, for $i, j, s \in X$,

$$\omega_{ijs}^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} \mathsf{P}_i\{\xi^{(\varepsilon)}(n) = s,\ \mu_0^{(\varepsilon)} \wedge \mu_j^{(\varepsilon)} > n\},\ \rho \in \mathbb{R},\ r = 0, 1, \ldots$$

Notice that $\omega_{ijs}^{(\varepsilon)}(\rho, 0) = \omega_{ijs}^{(\varepsilon)}(\rho)$.

It follows from conditions **A**–**C** and Lemma 4.2 that for $\rho < \delta$ and sufficiently small ε, the functions $\omega_{ijs}^{(\varepsilon)}(\rho)$ and $p_{ij}^{(\varepsilon)}(\rho)$ are arbitrarily many times differentiable with respect to ρ, and the derivatives of order r are given by $\omega_{ijs}^{(\varepsilon)}(\rho, r)$ and $p_{ij}^{(\varepsilon)}(\rho, r)$, respectively. Under these conditions we also have that the functions $\varphi_i^{(\varepsilon)}(\rho)$, defined by equation (3.9), are differentiable. Let us denote the corresponding derivatives by $\varphi_i^{(\varepsilon)}(\rho, r)$.

Recall from Section 3 that the functions $\omega_{ijs}^{(\varepsilon)}(\rho)$ satisfy the following system of linear equations:

$$\omega_{ijs}^{(\varepsilon)}(\rho) = \delta(i, s)\varphi_i^{(\varepsilon)}(\rho) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho)\omega_{ljs}^{(\varepsilon)}(\rho),\ i, j, s \neq 0. \tag{6.21}$$

Differentiating relation (6.21) gives

$$\omega_{ijs}^{(\varepsilon)}(\rho, r) = \theta_{ijs}^{(\varepsilon)}(\rho, r) + \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho)\omega_{ljs}^{(\varepsilon)}(\rho, r),\ r = 1, 2, \ldots,\ i, j, s \neq 0, \tag{6.22}$$

where

$$\theta_{ijs}^{(\varepsilon)}(\rho, r) = \delta(i, s)\varphi_i^{(\varepsilon)}(\rho, r) + \sum_{m=1}^{r} \binom{r}{m} \sum_{l \neq 0, j} p_{il}^{(\varepsilon)}(\rho, m)\omega_{ljs}^{(\varepsilon)}(\rho, r - m). \tag{6.23}$$

In order to rewrite these systems in matrix form, we define the following column vectors,

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r) = \begin{bmatrix}\omega_{1js}^{(\varepsilon)}(\rho, r) & \cdots & \omega_{Njs}^{(\varepsilon)}(\rho, r)\end{bmatrix}^T,\ j, s \neq 0, \tag{6.24}$$

$$\boldsymbol{\theta}_{js}^{(\varepsilon)}(\rho, r) = \begin{bmatrix}\theta_{1js}^{(\varepsilon)}(\rho, r) & \cdots & \theta_{Njs}^{(\varepsilon)}(\rho, r)\end{bmatrix}^T,\ j, s \neq 0, \tag{6.25}$$

$$\widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho, r) = \begin{bmatrix}\delta(1, s)\varphi_1^{(\varepsilon)}(\rho, r) & \cdots & \delta(N, s)\varphi_N^{(\varepsilon)}(\rho, r)\end{bmatrix}^T,\ s \neq 0. \tag{6.26}$$

Using (6.7) and (6.21)–(6.26), we can for each $j, s \neq 0$ write the following recursive systems of linear equations,

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) = \widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho) + {}_j\mathbf{P}^{(\varepsilon)}(\rho)\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho), \tag{6.27}$$

and, for $r = 1, 2, \ldots,$

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r) = \boldsymbol{\theta}_{js}^{(\varepsilon)}(\rho, r) + {}_j\mathbf{P}^{(\varepsilon)}(\rho)\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r), \tag{6.28}$$

where

$$\boldsymbol{\theta}_{js}^{(\varepsilon)}(\rho, r) = \widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho, r) + \sum_{m=1}^{r} \binom{r}{m} {}_j\mathbf{P}^{(\varepsilon)}(\rho, m)\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r-m). \tag{6.29}$$

In order to construct asymptotic expansions for the vectors $\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r)$, we can use the same technique as in Lemma 6.2. However, a preliminary step needed in this case is to construct asymptotic expansions for the functions $\varphi_i^{(\varepsilon)}(\rho, r)$. In order to do this, we first derive an expression for these functions.

Let us define

$$\psi_i^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} \mathsf{P}_i\{\kappa_1^{(\varepsilon)} = n\}, \ \rho \in \mathbb{R}, \ r = 0, 1, \ldots, \ i \in X. \tag{6.30}$$

Note that

$$\psi_i^{(\varepsilon)}(\rho, r) = \sum_{j \in X} p_{ij}^{(\varepsilon)}(\rho, r), \ \rho \in \mathbb{R}, \ r = 0, 1, \ldots, \ i \in X. \tag{6.31}$$

Thus, the functions $\psi_i^{(\varepsilon)}(\rho, 0)$ are arbitrarily many times differentiable with respect to ρ and the corresponding derivatives are given by $\psi_i^{(\varepsilon)}(\rho, r)$.

The function $\varphi_i^{(\varepsilon)}(\rho)$, defined by equation (3.9), can be written as

$$\varphi_i^{(\varepsilon)}(\rho) = \begin{cases} \psi_i^{(\varepsilon)}(0, 1) & \rho = 0, \\ (\psi_i^{(\varepsilon)}(\rho, 0) - 1)/(e^\rho - 1) & \rho \neq 0. \end{cases} \tag{6.32}$$

From (6.30) and (6.32) it follows that

$$\psi_i^{(\varepsilon)}(\rho, 0) = (e^\rho - 1)\varphi_i^{(\varepsilon)}(\rho) + 1, \ \rho \in \mathbb{R}. \tag{6.33}$$

Differentiating both sides of (6.33) gives

$$\psi_i^{(\varepsilon)}(\rho, r) = (e^\rho - 1)\varphi_i^{(\varepsilon)}(\rho, r) + e^\rho \sum_{m=0}^{r-1} \binom{r}{m} \varphi_i^{(\varepsilon)}(\rho, m), \ r = 1, 2, \ldots \tag{6.34}$$

If $\rho = 0$, equation (6.34) implies

$$\psi_i^{(\varepsilon)}(0, r) = r\varphi_i^{(\varepsilon)}(0, r-1) + \sum_{m=0}^{r-2} \binom{r}{m} \varphi_i^{(\varepsilon)}(0, m), \ r = 2, 3, \ldots$$

From this it follows that, for $r = 1, 2, \ldots,$

$$\varphi_i^{(\varepsilon)}(0, r) = \frac{1}{r+1}\left(\psi_i^{(\varepsilon)}(0, r+1) - \sum_{m=0}^{r-1}\binom{r+1}{m}\varphi_i^{(\varepsilon)}(0, m)\right). \tag{6.35}$$

If $\rho \neq 0$, equation (6.34) gives, for $r = 1, 2, \ldots,$

$$\varphi_i^{(\varepsilon)}(\rho, r) = \frac{1}{e^\rho - 1}\left(\psi_i^{(\varepsilon)}(\rho, r) - e^\rho \sum_{m=0}^{r-1}\binom{r}{m}\varphi_i^{(\varepsilon)}(\rho, m)\right). \tag{6.36}$$

Using relations (6.31), (6.35), and (6.36), we can recursively calculate the derivatives of $\varphi_i^{(\varepsilon)}(\rho)$. Furthermore, it follows directly from these formulas that we can construct asymptotic expansions for these derivatives. The formulas are given in the following lemma.

Lemma 6.3. *Assume that conditions* **A**–**C** *hold.*

- **(i)** *If, in addition, condition* $\mathbf{P}^*_\mathbf{k}$ *holds, then for each* $i \neq 0$ *and* $r = 0, \ldots, k$ *we have the following asymptotic expansion,*

$$\psi_i^{(\varepsilon)}(\rho, r) = \psi_i[\rho, r, 0] + \psi_i[\rho, r, 1]\varepsilon + \cdots + \psi_i[\rho, r, k-r]\varepsilon^{k-r} + o(\varepsilon^{k-r}),$$

where

$$\psi_i[\rho, r, n] = \sum_{j \in X} p_{ij}[\rho, r, n], \ \ n = 0, \ldots, k-r.$$

- **(ii)** *If, in addition,* $\rho = 0$ *and condition* $\mathbf{P}^*_{\mathbf{k+1}}$ *holds, then for each* $i \neq 0$ *and* $r = 0, \ldots, k$ *we have the following asymptotic expansion,*

$$\varphi_i^{(\varepsilon)}(0, r) = \varphi_i[0, r, 0] + \varphi_i[0, r, 1]\varepsilon + \cdots + \varphi_i[0, r, k-r]\varepsilon^{k-r} + o(\varepsilon^{k-r}),$$

where, for $n = 0, \ldots, k-r,$

$$\varphi_i[0, r, n] = \frac{1}{r+1}\left(\psi_i[0, r+1, n] - \sum_{m=0}^{r-1}\binom{r+1}{m}\varphi_i[0, m, n]\right).$$

- **(iii)** *If, in addition,* $\rho \neq 0$ *and condition* $\mathbf{P}^*_\mathbf{k}$ *holds, then for each* $i \neq 0$ *and* $r = 0, \ldots, k$ *we have the following asymptotic expansion,*

$$\varphi_i^{(\varepsilon)}(\rho, r) = \varphi_i[\rho, r, 0] + \varphi_i[\rho, r, 1]\varepsilon + \cdots + \varphi_i[\rho, r, k-r]\varepsilon^{k-r} + o(\varepsilon^{k-r}),$$

where, for $n = 0, \ldots, k-r,$

$$\varphi_i[\rho, r, n] = \frac{1}{e^\rho - 1}\left(\psi_i[\rho, r, n] - e^\rho \sum_{m=0}^{r-1}\binom{r}{m}\varphi_i[\rho, m, n]\right).$$

Using (6.26) and Lemma 6.3 we can now construct the following asymptotic expansions, for $r = 0, \ldots, k$, and $s \neq 0$,

$$\widehat{\boldsymbol{\varphi}}_s^{(\varepsilon)}(\rho, r) = \widehat{\boldsymbol{\varphi}}_s[\rho, r, 0] + \widehat{\boldsymbol{\varphi}}_s[\rho, r, 1]\varepsilon + \cdots + \widehat{\boldsymbol{\varphi}}_s[\rho, r, k-r]\varepsilon^{k-r} + \mathbf{o}(\varepsilon^{k-r}). \quad (6.37)$$

The next lemma gives asymptotic expansions for $\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r)$.

Lemma 6.4. *Assume that conditions* **A**–**C** *hold. If* $\rho = 0$, *we also assume that condition* $\mathbf{P}^*_{\mathbf{k+1}}$ *holds. If* $\rho \neq 0$, *we also assume that condition* $\mathbf{P}^*_{\mathbf{k}}$ *holds. Then:*

(**i**) *We have the following asymptotic expansion,*

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho) = \boldsymbol{\omega}_{js}[\rho, 0, 0] + \boldsymbol{\omega}_{js}[\rho, 0, 1]\varepsilon + \cdots + \boldsymbol{\omega}_{js}[\rho, 0, k]\varepsilon^k + \mathbf{o}(\varepsilon^k),$$

where

$$\boldsymbol{\omega}_{js}[\rho, 0, n] = \begin{cases} \boldsymbol{\omega}_{js}^{(0)}(\rho) & n = 0, \\ \sum_{q=0}^{n} {}_j\mathbf{U}[\rho, q]\widehat{\boldsymbol{\varphi}}_s[\rho, 0, n-q] & n = 1, \ldots, k. \end{cases}$$

(**ii**) *For* $r = 1, \ldots, k$, *we have the following asymptotic expansions,*

$$\boldsymbol{\omega}_{js}^{(\varepsilon)}(\rho, r) = \boldsymbol{\omega}_{js}[\rho, r, 0] + \boldsymbol{\omega}_{js}[\rho, r, 1]\varepsilon + \cdots + \boldsymbol{\omega}_{js}[\rho, r, k-r]\varepsilon^{k-r} + \mathbf{o}(\varepsilon^{k-r}),$$

where

$$\boldsymbol{\omega}_{js}[\rho, r, n] = \begin{cases} \boldsymbol{\omega}_{js}^{(0)}(\rho, r) & n = 0, \\ \sum_{q=0}^{n} {}_j\mathbf{U}[\rho, q]\boldsymbol{\theta}_{js}[\rho, r, n-q] & n = 1, \ldots, k-r, \end{cases}$$

and, for $t = 0, \ldots, k-r$,

$$\boldsymbol{\theta}_{js}[\rho, r, t] = \widehat{\boldsymbol{\varphi}}_s[\rho, r, t] + \sum_{m=1}^{r} \binom{r}{m} \sum_{q=0}^{t} {}_j\mathbf{P}[\rho, m, q]\boldsymbol{\omega}_{js}[\rho, r-m, t-q].$$

Proof. Under conditions **A**–**C**, we have, for sufficiently small ε, that the recursive systems of linear equations given by relations (6.27), (6.28), and (6.29), all have finite components. Moreover, the inverse matrix ${}_j\mathbf{U}^{(\varepsilon)}(\rho) = (\mathbf{I} - {}_j\mathbf{P}^{(\varepsilon)}(\rho))^{-1}$ exists, so these systems have unique solutions. Since we, by Lemma 6.3, have the expansions given in equation (6.37), the proof is from this point analogous to the proof of Lemma 6.2. □

7 Proof of the Main Result

In this section we prove Theorem 2.1.

Throughout this section, it is assumed that conditions **A**–**D** and $\mathbf{P_{k+1}}$ hold.

Let us first note that it follows from Lemmas 6.1–6.4 that we for $r = 0, \dots, k$ and $i, j \neq 0$ have the following asymptotic expansions,

$$\omega_{iij}^{(\varepsilon)}(\rho^{(0)}, r) = a_{ij}[r, 0] + a_{ij}[r, 1]\varepsilon + \cdots + a_{ij}[r, k - r]\varepsilon^{k-r} + o(\varepsilon^{k-r}) \quad (7.1)$$

and

$$\phi_{ii}^{(\varepsilon)}(\rho^{(0)}, r) = b_i[r, 0] + b_i[r, 1]\varepsilon + \cdots + b_i[r, k - r]\varepsilon^{k-r} + o(\varepsilon^{k-r}), \quad (7.2)$$

where the coefficients in these expansions can be calculated from the formulas given in these lemmas. Furthermore, from Lemma 6.4 we see that in the case where $\rho^{(0)} > 0$, condition $\mathbf{P_{k+1}}$ can be replaced by condition $\mathbf{P_k}$.

Let us also recall from Section 5 that the quasi-stationary distribution, for sufficiently small ε, has the following representation,

$$\pi_j^{(\varepsilon)} = \frac{\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)})}{\omega_{ii1}^{(\varepsilon)}(\rho^{(\varepsilon)}) + \cdots + \omega_{iiN}^{(\varepsilon)}(\rho^{(\varepsilon)})}, \ j = 1, \dots, N. \quad (7.3)$$

The construction of the asymptotic expansion for the quasi-stationary distribution will be realized in three steps. First we use the coefficients in the expansions given by (7.2) to build an asymptotic expansion for $\rho^{(\varepsilon)}$, the root of the characteristic equation. Then, the coefficients in this expansion and the coefficients in the expansions given by (7.1) are used to construct asymptotic expansions for $\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)})$. Finally, relation (7.3) is used to complete the proof.

We formulate these steps in the following three lemmas. Let us here remark that the proof of Lemma 7.1 is given in Silvestrov and Petersson (2013) in the context of general discrete time renewal equations and the proofs of Lemmas 7.2 and 7.3 are given in Petersson (2014b) in the context of quasi-stationary distributions for discrete time regenerative processes. In order to make the paper more self-contained, we also give the proofs here, in slightly reduced forms.

Lemma 7.1. *The root of the characteristic equation has the following asymptotic expansion,*

$$\rho^{(\varepsilon)} = \rho^{(0)} + c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k),$$

where $c_1 = -b_i[0,1]/b_i[1,0]$ *and, for* $n = 2, \dots, k$,

$$c_n = -\frac{1}{b_i[1,0]}\Bigg(b_i[0,n] + \sum_{q=1}^{n-1} b_i[1, n-q] c_q \\ + \sum_{m=2}^{n}\sum_{q=m}^{n} b_i[m, n-q] \cdot \sum_{n_1,\dots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!} \Bigg),$$

where $D_{m,q}$ *is the set of all non-negative integer solutions of the system*

$$n_1 + \cdots + n_{q-1} = m, \quad n_1 + 2n_2 + \cdots + (q-1)n_{q-1} = q.$$

Proof. Let $\Delta^{(\varepsilon)} = \rho^{(\varepsilon)} - \rho^{(0)}$. It follows from the Taylor expansion of the exponential function that, for $n = 0, 1, \dots,$

$$e^{\rho^{(\varepsilon)} n} = e^{\rho^{(0)} n} \left(\sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r n^r}{r!} + \frac{(\Delta^{(\varepsilon)})^{k+1} n^{k+1}}{(k+1)!} e^{|\Delta^{(\varepsilon)}| n} \zeta_{k+1}^{(\varepsilon)}(n) \right), \tag{7.4}$$

where $0 \le \zeta_{k+1}^{(\varepsilon)}(n) \le 1$.

If we multiply both sides of (7.4) by $g_{ii}^{(\varepsilon)}(n)$, sum over all n, and use that $\rho^{(\varepsilon)}$ is the root of the characteristic equation, we get

$$1 = \sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!} \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, r) + (\Delta^{(\varepsilon)})^{k+1} M_{k+1}^{(\varepsilon)}, \tag{7.5}$$

where

$$M_{k+1}^{(\varepsilon)} = \frac{1}{(k+1)!} \sum_{n=0}^{\infty} n^{k+1} e^{(\rho^{(0)} + |\Delta^{(\varepsilon)}|) n} \zeta_{k+1}^{(\varepsilon)}(n) g_{ii}^{(\varepsilon)}(n). \tag{7.6}$$

It follows from Lemma 4.2 that $|\Delta^{(\varepsilon)}| \to 0$ as $\varepsilon \to 0$, so there exist $\beta > 0$ and $\varepsilon_1(\beta) > 0$ such that

$$\rho^{(0)} + |\Delta^{(\varepsilon)}| \le \beta < \delta, \ \varepsilon \le \varepsilon_1(\beta). \tag{7.7}$$

From Lemma 4.2 it also follows that there exists $\varepsilon_2(\beta) > 0$ such that

$$\phi_{ii}^{(\varepsilon)}(\beta, r) < \infty, \ r = 0, 1, \dots, \ \varepsilon \le \varepsilon_2(\beta). \tag{7.8}$$

Let $\varepsilon_0 = \varepsilon_0(\beta) = \min\{\varepsilon_1(\beta), \varepsilon_2(\beta)\}$. Then, relations (7.6), (7.7), and (7.8) imply that

$$M_{k+1}^{(\varepsilon)} \le \frac{1}{(k+1)!} \phi_{ii}^{(\varepsilon)}(\beta, k+1) < \infty, \ \varepsilon \le \varepsilon_0. \tag{7.9}$$

It follows from (7.9) that we can rewrite (7.5) as

$$1 = \sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!} \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, r) + (\Delta^{(\varepsilon)})^{k+1} M_{k+1} \zeta_{k+1}^{(\varepsilon)}, \tag{7.10}$$

where $M_{k+1} = \sup_{\varepsilon \leq \varepsilon_0} M_{k+1}^{(\varepsilon)} < \infty$ and $0 \leq \zeta_{k+1}^{(\varepsilon)} \leq 1$.

From relation (7.10) we can successively construct the asymptotic expansion for the root of the characteristic equation.

Let us first assume that $k = 1$. In this case (7.10) implies that

$$1 = \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, 0) + \Delta^{(\varepsilon)} \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, 1) + (\Delta^{(\varepsilon)})^2 O(1). \tag{7.11}$$

Using (7.2), (7.11), and that $\Delta^{(\varepsilon)} \to 0$ as $\varepsilon \to 0$, it follows that

$$-b_i[0,1]\varepsilon = \Delta^{(\varepsilon)}(b_i[1,0] + o(1)) + o(\varepsilon). \tag{7.12}$$

Dividing both sides of equation (7.12) by ε and letting ε tend to zero we can conclude that $\Delta^{(\varepsilon)}/\varepsilon \to -b_i[0,1]/b_i[1,0]$ as $\varepsilon \to 0$. From this it follows that we have the representation

$$\Delta^{(\varepsilon)} = c_1 \varepsilon + \Delta_1^{(\varepsilon)}, \tag{7.13}$$

where $c_1 = -b_i[0,1]/b_i[1,0]$ and $\Delta_1^{(\varepsilon)}/\varepsilon \to 0$ as $\varepsilon \to 0$.

This proves Lemma 7.1 for the case $k = 1$.

Let us now assume that $k = 2$. In this case relation (7.10) implies that

$$1 = \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, 0) + \Delta^{(\varepsilon)} \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, 1) + \frac{(\Delta^{(\varepsilon)})^2}{2} \phi_{ii}^{(\varepsilon)}(\rho^{(0)}, 2) + (\Delta^{(\varepsilon)})^3 O(1). \tag{7.14}$$

Using (7.2) and (7.13) in relation (7.14) and rearranging gives

$$-\left(b_i[0,2] + b_i[1,1]c_1 + \frac{b_i[2,0]c_1^2}{2} \right) \varepsilon^2 = \Delta_1^{(\varepsilon)}(b_i[1,0] + o(1)) + o(\varepsilon^2). \tag{7.15}$$

Dividing both sides of equation (7.15) by ε^2 and letting ε tend to zero we can conclude that $\Delta_1^{(\varepsilon)}/\varepsilon^2 \to c_2$ as $\varepsilon \to 0$, where

$$c_2 = -\frac{1}{b_i[1,0]} \left(b_i[0,2] + b_i[1,1]c_1 + \frac{b_i[2,0]c_1^2}{2} \right).$$

From this and (7.13) it follows that we have the representation

$$\Delta^{(\varepsilon)} = c_1 \varepsilon + c_2 \varepsilon^2 + \Delta_2^{(\varepsilon)},$$

where $\Delta_2^{(\varepsilon)}/\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

This proves Lemma 7.1 for the case $k = 2$.

Continuing in this way we can prove the lemma for any positive integer k. However, once it is known that the expansion exists, the coefficients can be obtained in a simpler way. From (7.2) and (7.10) we get the following formal equation,

$$\begin{aligned} &- (b_i[0,1]\varepsilon + b_i[0,2]\varepsilon^2 + \cdots) \\ &\quad = (c_1\varepsilon + c_2\varepsilon^2 + \cdots)(b_i[1,0] + b_i[1,1]\varepsilon + \cdots) \\ &\quad + (1/2!)(c_1\varepsilon + c_2\varepsilon^2 + \cdots)^2(b_i[2,0] + b_i[2,1]\varepsilon + \cdots) + \cdots \end{aligned} \tag{7.16}$$

By expanding the right hand side of (7.16) and then equating coefficients of equal powers of ε in the left and right hand sides, we obtain the formulas given in Lemma 7.1. □

Lemma 7.2. *For any $i, j \neq 0$, we have the following asymptotic expansion,*

$$\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)}) = \omega_{iij}^{(0)}(\rho^{(0)}) + d_{ij}[1]\varepsilon + \cdots + d_{ij}[k]\varepsilon^k + o(\varepsilon^k),$$

where $d_{ij}[1] = a_{ij}[0,1] + a_{ij}[1,0]c_1$, and, for $n = 2, \ldots, k$,

$$\begin{aligned} d_{ij}[n] &= a_{ij}[0,n] + \sum_{q=1}^{n} a_{ij}[1, n-q]c_q \\ &+ \sum_{m=2}^{n}\sum_{q=m}^{n} a_{ij}[m, n-q] \cdot \sum_{n_1,\ldots,n_{q-1}\in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!}, \end{aligned}$$

where $D_{m,q}$ is the set of all non-negative integer solutions of the system

$$n_1 + \cdots + n_{q-1} = m, \quad n_1 + 2n_2 + \cdots + (q-1)n_{q-1} = q.$$

Proof. Let us again use relation (7.4) given in the proof of Lemma 7.1. Multiplying both sides of (7.4) by $h_{ij}^{(\varepsilon)}(n)$ and summing over all n we get

$$\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)}) = \sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!}\omega_{iij}^{(\varepsilon)}(\rho^{(0)}, r) + (\Delta^{(\varepsilon)})^{k+1}\widetilde{M}_{k+1}^{(\varepsilon)}, \tag{7.17}$$

where

$$\widetilde{M}_{k+1}^{(\varepsilon)} = \frac{1}{(k+1)!}\sum_{n=0}^{\infty} n^{k+1}e^{(\rho^{(0)}+|\Delta^{(\varepsilon)}|)n}\zeta_{k+1}^{(\varepsilon)}(n)h_{ij}^{(\varepsilon)}(n).$$

Using similar arguments as in the proof of Lemma 7.1 we can rewrite (7.17) as

$$\omega_{iij}^{(\varepsilon)}(\rho^{(\varepsilon)}) = \sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!}\omega_{iij}^{(\varepsilon)}(\rho^{(0)}, r) + (\Delta^{(\varepsilon)})^{k+1}\widetilde{M}_{k+1}\zeta_{k+1}^{(\varepsilon)}, \tag{7.18}$$

where $\widetilde{M}_{k+1} = \sup_{\varepsilon \leq \varepsilon_0} \widetilde{M}^{(\varepsilon)}_{k+1} < \infty$, for some $\varepsilon_0 > 0$, and $0 \leq \zeta^{(\varepsilon)}_{k+1} \leq 1$.

From Lemma 7.1 we have the following asymptotic expansion,

$$\Delta^{(\varepsilon)} = c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k). \tag{7.19}$$

Substituting the expansions (7.1) and (7.19) into relation (7.18) yields

$$\begin{aligned}\omega^{(\varepsilon)}_{iij}(\rho^{(\varepsilon)}) = \omega^{(0)}_{iij}(\rho^{(0)}) &+ a_{ij}[0,1]\varepsilon + \cdots + a_{ij}[0,k]\varepsilon^k + o(\varepsilon^k) \\ &+ (c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k)) \\ &\times (a_{ij}[1,0] + a_{ij}[1,1]\varepsilon + \cdots + a_{ij}[1,k-1]\varepsilon^{k-1} + o(\varepsilon^{k-1})) \\ &+ \cdots + \\ &+ (1/k!)(c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k))^k(a_{ij}[k,0] + o(1)).\end{aligned} \tag{7.20}$$

By expanding the right hand side of (7.20) and grouping coefficients of equal powers of ε we get the expansions and formulas given in Lemma 7.2. □

Lemma 7.3. *For any $j \neq 0$, we have the following asymptotic expansion,*

$$\pi^{(\varepsilon)}_j = \pi^{(0)}_j + \pi_j[1]\varepsilon + \cdots + \pi_j[k]\varepsilon^k + o(\varepsilon^k). \tag{7.21}$$

The coefficients $\pi_j[n]$, $n = 1, \ldots, k$, $j \neq 0$, are for any $i \neq 0$ given by the following recursive formulas,

$$\pi_j[n] = \frac{1}{e_i[0]}\left(d_{ij}[n] - \sum_{q=0}^{n-1} e_i[n-q]\pi_j[q]\right), \ n = 1, \ldots, k,$$

where $\pi_j[0] = \pi^{(0)}_j$, $d_{ij}[0] = \omega^{(0)}_{iij}(\rho^{(0)})$, and $e_i[n] = \sum_{j \neq 0} d_{ij}[n]$, $n = 0, \ldots, k$.

Proof. It follows from formula (7.3) and Lemma 7.2 that we for all $i, j \neq 0$ have

$$\pi^{(\varepsilon)}_j = \frac{d_{ij}[0] + d_{ij}[1]\varepsilon + \cdots + d_{ij}[k]\varepsilon^k + o(\varepsilon^k)}{e_i[0] + e_i[1]\varepsilon + \cdots + e_i[k]\varepsilon^k + o(\varepsilon^k)}. \tag{7.22}$$

Since $e_i[0] > 0$, it follows from (7.22) that the expansion (7.21) exists. From this and (7.22) we get the following equation,

$$\begin{aligned}&(e_i[0] + e_i[1]\varepsilon + \cdots + e_i[k]\varepsilon^k + o(\varepsilon^k)) \\ &\qquad \times (\pi_j[0] + \pi_j[1]\varepsilon + \cdots + \pi_j[k]\varepsilon^k + o(\varepsilon^k)) \\ &\qquad = d_{ij}[0] + d_{ij}[1]\varepsilon + \cdots + d_{ij}[k]\varepsilon^k + o(\varepsilon^k).\end{aligned} \tag{7.23}$$

By expanding the left hand side of (7.23) and then equating coefficients of equal powers of ε in the left and right hand sides, we obtain the coefficients given in Lemma 7.3. □

8 Perturbed Markov Chains

In this section it is shown how the results of the present paper can be applied in the special case of perturbed discrete time Markov chains. As an illustration, we present a simple numerical example.

For every $\varepsilon \geq 0$, let $\eta_n^{(\varepsilon)}$, $n = 0, 1, \dots$, be a homogeneous discrete time Markov chain with state space $X = \{0, 1, \dots, N\}$, an initial distribution $p_i^{(\varepsilon)} = \mathsf{P}\{\eta_0^{(\varepsilon)} = i\}$, $i \in X$, and transition probabilities

$$p_{ij}^{(\varepsilon)} = \mathsf{P}\{\eta_{n+1}^{(\varepsilon)} = j \,|\, \eta_n^{(\varepsilon)} = i\},\ i, j \in X.$$

This model is a particular case of the semi-Markov process described in Section 2 with transition probabilities given by

$$Q_{ij}^{(\varepsilon)}(n) = p_{ij}^{(\varepsilon)} \chi(n = 1),\ n = 1, 2, \dots,\ i, j \in X.$$

In this case, mixed power-exponential moment functionals for transition probabilities take the following form,

$$p_{ij}^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} Q_{ij}^{(\varepsilon)}(n) = e^{\rho} p_{ij}^{(\varepsilon)},\ \rho \in \mathbb{R},\ r = 0, 1, \dots,\ i, j \in X. \quad (8.1)$$

Conditions **A**–**D** and $\mathbf{P_k}$ imposed in Section 2 now hold if the following conditions are satisfied:

A′: $g_{ij}^{(0)} > 0$, $i, j \neq 0$.

B′: $g_{ii}^{(0)}(n)$ is non-periodic for some $i \neq 0$.

$\mathbf{P'_k}$: $p_{ij}^{(\varepsilon)} = p_{ij}^{(0)} + p_{ij}[1]\varepsilon + \cdots + p_{ij}[k]\varepsilon^k + o(\varepsilon^k)$, $i, j \neq 0$, where $|p_{ij}[n]| < \infty$, $n = 1, \dots, k$, $i, j \neq 0$.

Let us here remark that in order to construct an asymptotic expansion of order k for the quasi-stationary distribution of a Markov chain, it is sufficient to assume that the perturbation condition holds for the parameter k, and not for $k + 1$ as needed for semi-Markov processes. The stronger perturbation condition with parameter $k + 1$ is needed in order to construct asymptotic expansions for the functions $\varphi_i^{(\varepsilon)}(\rho, r)$ defined in Section 6. However, for Markov chains these functions take the form $\varphi_i^{(\varepsilon)}(\rho, r) = \chi(r = 0)$ which make the asymptotic expansions trivial.

It follows from (8.1) and $\mathbf{P'_k}$ that the coefficients in the perturbation condition $\mathbf{P^*_k}$ to be used in Lemmas 6.1, 6.2, and 6.4 are given by

$$p_{ij}[\rho, r, n] = e^{\rho} p_{ij}[n],\ r = 0, \dots, k,\ n = 0, \dots, k - r,\ i, j \neq 0.$$

Let us illustrate the remarks made above by means of a simple numerical example where we compute the second order asymptotic expansion for the quasi-stationary distribution of a Markov chain with four states. We consider the more simple case where transitions to state 0 is not possible for the limiting Markov chain. In this case, exact computations can be made and we can focus on the algorithm itself and need not need to consider possible numerical issues.

We consider a perturbed Markov chain $\eta_n^{(\varepsilon)}$, $n = 0, 1, \ldots$, on the state space $X = \{0, 1, 2, 3\}$ with matrix of transition probabilities given by

$$\|p_{ij}^{(\varepsilon)}\| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - e^{-\varepsilon} & 0 & e^{-\varepsilon} & 0 \\ 1 - e^{-\varepsilon} & 0 & 0 & e^{-\varepsilon} \\ 1 - e^{-2\varepsilon} & \frac{1}{2}e^{-2\varepsilon} & \frac{1}{2}e^{-2\varepsilon} & 0 \end{bmatrix}, \ \varepsilon \geq 0. \tag{8.2}$$

First, the root of the characteristic equation for the limiting Markov chain needs to be found. Since $\phi_{ii}^{(0)}(0) = \mathsf{P}_i\{\nu_0^{(0)} > \nu_i^{(0)}\} = 1$, we have $\rho^{(0)} = 0$. In the case where transitions to state 0 is possible also for the limiting Markov chain, the root $\rho^{(0)}$ needs to be computed numerically. Then, the system of linear equations (3.6) can be used to calculate $\phi_{ii}^{(0)}(\rho)$.

Next step is to determine the coefficients in the expansions given in equations (7.1) and (7.2) for the case where $k = 2$ and i is some fixed state which we can choose arbitrarily. Let us choose $i = 1$. In order to compute these coefficients we apply the results of Section 6 with $\rho = 0$ and $j = 1$.

It follows from (8.1) and (8.2) that the vectors and matrices defined by equations (6.5) and (6.7) take the following forms, respectively,

$$\mathbf{p}_1^{(\varepsilon)}(0, r) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}e^{-2\varepsilon} \end{bmatrix}, \ {}_1\mathbf{P}^{(\varepsilon)}(0, r) = \begin{bmatrix} 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & e^{-\varepsilon} \\ 0 & \frac{1}{2}e^{-2\varepsilon} & 0 \end{bmatrix}, \ r = 0, 1, 2.$$

Thus, we have the following asymptotic expansions,

$$\mathbf{p}_1^{(\varepsilon)}(0, r) = \mathbf{p}_1[0, r, 0] + \mathbf{p}_1[0, r, 1]\varepsilon + \mathbf{p}_1[0, r, 2]\varepsilon^2 + \mathbf{o}(\varepsilon^2), \ r = 0, 1, 2,$$

where

$$\mathbf{p}_1[0, r, 0] = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}, \ \mathbf{p}_1[0, r, 1] = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{p}_1[0, r, 2] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{8.3}$$

and

$${}_1\mathbf{P}^{(\varepsilon)}(0, r) = {}_1\mathbf{P}[0, r, 0] + {}_1\mathbf{P}[0, r, 1]\varepsilon + {}_1\mathbf{P}[0, r, 2]\varepsilon^2 + \mathbf{o}(\varepsilon^2), \ r = 0, 1, 2,$$

where

$$
{}_1\mathbf{P}[0,r,0] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}, \qquad {}_1\mathbf{P}[0,r,1] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},
$$

$$
{}_1\mathbf{P}[0,r,2] = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}. \tag{8.4}
$$

It follows from Lemma 6.1 that the matrix ${}_1\mathbf{U}^{(\varepsilon)}(0) = (\mathbf{I} - {}_1\mathbf{P}^{(\varepsilon)}(0))^{-1}$ has the asymptotic expansion

$$
{}_1\mathbf{U}^{(\varepsilon)}(0) = {}_1\mathbf{U}[0,0] + {}_1\mathbf{U}[0,1]\varepsilon + {}_1\mathbf{U}[0,2]\varepsilon^2 + \mathbf{o}(\varepsilon^2),
$$

where

$$
\begin{aligned}
{}_1\mathbf{U}[0,0] &= (\mathbf{I} - {}_1\mathbf{P}^{(0)}(0))^{-1}, \\
{}_1\mathbf{U}[0,1] &= {}_1\mathbf{U}[0,0]{}_1\mathbf{P}[0,0,1]{}_1\mathbf{U}[0,0], \\
{}_1\mathbf{U}[0,2] &= {}_1\mathbf{U}[0,0]({}_1\mathbf{P}[0,0,1]{}_1\mathbf{U}[0,1] + {}_1\mathbf{P}[0,0,2]{}_1\mathbf{U}[0,0]).
\end{aligned} \tag{8.5}
$$

Using (8.4) and (8.5), the following numerical values are obtained,

$$
{}_1\mathbf{U}[0,0] = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \qquad {}_1\mathbf{U}[0,1] = \begin{bmatrix} 0 & -8 & -10 \\ 0 & -6 & -8 \\ 0 & -5 & -6 \end{bmatrix},
$$

$$
{}_1\mathbf{U}[0,2] = \begin{bmatrix} 0 & 34 & 43 \\ 0 & 27 & 34 \\ 0 & 43/2 & 27 \end{bmatrix}. \tag{8.6}
$$

From Lemma 6.2 we now get the following asymptotic expansions,

$$
\begin{aligned}
\Phi_1^{(\varepsilon)}(0,0) &= \Phi_1[0,0,0] + \Phi_1[0,0,1]\varepsilon + \Phi_1[0,0,2]\varepsilon^2 + \mathbf{o}(\varepsilon^2), \\
\Phi_1^{(\varepsilon)}(0,1) &= \Phi_1[0,1,0] + \Phi_1[0,1,1]\varepsilon + \mathbf{o}(\varepsilon), \\
\Phi_1^{(\varepsilon)}(0,2) &= \Phi_1[0,2,0] + \mathbf{o}(1),
\end{aligned} \tag{8.7}
$$

where the coefficients in these expansions can be calculated from (8.3), (8.4), (8.6), and the formulas given in Lemma 6.2. This yields

$$
\Phi_1[0,0,0] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \Phi_1[0,0,1] = \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix}, \quad \Phi_1[0,0,2] = \begin{bmatrix} 67/2 \\ 27 \\ 43/2 \end{bmatrix},
$$

$$
\Phi_1[0,1,0] = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad \Phi_1[0,1,1] = \begin{bmatrix} -47 \\ -36 \\ -27 \end{bmatrix}, \quad \Phi_1[0,2,0] = \begin{bmatrix} 33 \\ 24 \\ 17 \end{bmatrix}. \tag{8.8}
$$

From (8.7) and (8.8) it follows that

$$\begin{aligned}
\phi_{11}^{(\varepsilon)}(0,0) &= b_1[0,0] + b_1[0,1]\varepsilon + b_1[0,2]\varepsilon^2 + o(\varepsilon^2),\\
\phi_{11}^{(\varepsilon)}(0,1) &= b_1[1,0] + b_1[1,1]\varepsilon + o(\varepsilon),\\
\phi_{11}^{(\varepsilon)}(0,2) &= b_1[2,0] + o(1),
\end{aligned}$$

where

$$\begin{array}{lll}
b_1[0,0] = 1, & b_1[0,1] = -7, & b_1[0,2] = 67/2,\\
b_1[1,0] = 5, & b_1[1,1] = -47, & b_1[2,0] = 33.
\end{array} \tag{8.9}$$

Lemma 6.4 gives the following asymptotic expansions, for $s = 1,2,3$,

$$\begin{aligned}
\boldsymbol{\omega}_{1s}^{(\varepsilon)}(0,0) &= \boldsymbol{\omega}_{1s}[0,0,0] + \boldsymbol{\omega}_{1s}[0,0,1]\varepsilon + \boldsymbol{\omega}_{1s}[0,0,2]\varepsilon^2 + \mathbf{o}(\varepsilon^2),\\
\boldsymbol{\omega}_{1s}^{(\varepsilon)}(0,1) &= \boldsymbol{\omega}_{1s}[0,1,0] + \boldsymbol{\omega}_{1s}[0,1,1]\varepsilon + \mathbf{o}(\varepsilon),\\
\boldsymbol{\omega}_{1s}^{(\varepsilon)}(0,2) &= \boldsymbol{\omega}_{1s}[0,2,0] + \mathbf{o}(1).
\end{aligned} \tag{8.10}$$

where the coefficients can be calculated from (8.3), (8.4), (8.6), and the formulas given in Lemma 6.4. Note here that the coefficients in the expansion (6.37) needed in these formulas are given by

$$\widehat{\boldsymbol{\varphi}}_s[0,0,0] = \begin{bmatrix}\delta(1,s)\\ \delta(2,s)\\ \delta(3,s)\end{bmatrix}, \ \widehat{\boldsymbol{\varphi}}_s[0,r,n] = \begin{bmatrix}0\\0\\0\end{bmatrix}, \ (r,n) \neq (0,0).$$

From (8.10) we can extract the following expansions for $j = 1,2,3$,

$$\begin{aligned}
\omega_{11j}^{(\varepsilon)}(0,0) &= a_{1j}[0,0] + a_{1j}[0,1]\varepsilon + a_{1j}[0,2]\varepsilon^2 + o(\varepsilon^2),\\
\omega_{11j}^{(\varepsilon)}(0,1) &= a_{1j}[1,0] + a_{1j}[1,1]\varepsilon + o(\varepsilon),\\
\omega_{11j}^{(\varepsilon)}(0,2) &= a_{1j}[2,0] + o(1).
\end{aligned}$$

Numerical values for the coefficients in our example are given by

$$\begin{array}{lll}
a_{11}[0,0] = 1, & a_{12}[0,0] = 2, & a_{13}[0,0] = 2,\\
a_{11}[0,1] = 0, & a_{12}[0,1] = -8, & a_{13}[0,1] = -10,\\
a_{11}[0,2] = 0, & a_{12}[0,2] = 34, & a_{13}[0,2] = 43,\\
a_{11}[1,0] = 0, & a_{12}[1,0] = 6, & a_{13}[1,0] = 8,\\
a_{11}[1,1] = 0, & a_{12}[1,1] = -48, & a_{13}[1,1] = -64,\\
a_{11}[2,0] = 0, & a_{12}[2,0] = 34, & a_{13}[2,0] = 48.
\end{array} \tag{8.11}$$

The asymptotic expansion for the quasi-stationary distribution can now be computed from the coefficients in equations (8.9) and (8.11) by applying the lemmas in Section 7.

From Lemma 7.1 we get that the asymptotic expansion for the root of the characteristic equation is given by

$$\rho^{(\varepsilon)} = c_1\varepsilon + c_2\varepsilon^2 + o(\varepsilon^2),$$

where

$$c_1 = -\frac{b_1[0,1]}{b_1[1,0]} = \frac{7}{5},\ c_2 = -\frac{b_1[0,2] + b_1[1,1]c_1 + b_1[2,0]c_1^2/2}{b_1[1,0]} = -\frac{1}{125}. \quad (8.12)$$

Then, Lemma 7.2 gives us the following asymptotic expansions,

$$\omega_{11j}^{(\varepsilon)}(\rho^{(\varepsilon)}) = d_{1j}[0] + d_{1j}[1]\varepsilon + d_{1j}[2]\varepsilon^2 + o(\varepsilon^2),\ j = 1, 2, 3,$$

where

$$\begin{aligned} d_{1j}[0] &= a_{1j}[0,0], \\ d_{1j}[1] &= a_{1j}[0,1] + a_{1j}[1,0]c_1, \\ d_{1j}[2] &= a_{1j}[0,2] + a_{1j}[1,1]c_1 + a_{1j}[1,0]c_2 + a_{1j}[2,0]c_1^2/2. \end{aligned} \quad (8.13)$$

From (8.11), (8.12), and (8.13), we calculate

$$\begin{array}{lll} d_{11}[0] = 1, & d_{12}[0] = 2, & d_{13}[0] = 2, \\ d_{11}[1] = 0, & d_{12}[1] = 2/5, & d_{13}[1] = 6/5, \\ d_{11}[2] = 0, & d_{12}[2] = 9/125, & d_{13}[2] = 47/125. \end{array} \quad (8.14)$$

Finally, let us use Lemma 7.3. First, using (8.14), we get

$$\begin{aligned} e_1[0] &= d_{11}[0] + d_{12}[0] + d_{13}[0] = 5, \\ e_1[1] &= d_{11}[1] + d_{12}[1] + d_{13}[1] = 8/5, \\ e_1[2] &= d_{11}[2] + d_{12}[2] + d_{13}[2] = 56/125. \end{aligned} \quad (8.15)$$

Then, we can construct the asymptotic expansion for the quasi-stationary distribution,

$$\pi_j^{(\varepsilon)} = \pi_j[0] + \pi_j[1]\varepsilon + \pi_j[2]\varepsilon^2 + o(\varepsilon^2),\ j = 1, 2, 3,$$

where

$$\begin{aligned} \pi_j[0] &= d_{1j}[0]/e_1[0], \\ \pi_j[1] &= (d_{1j}[1] - e_1[1]\pi_j[0])/e_1[0], \\ \pi_j[2] &= (d_{1j}[2] - e_1[2]\pi_j[0] - e_1[1]\pi_j[1])/e_1[0]. \end{aligned} \quad (8.16)$$

Using (8.14), (8.15), and (8.16), the following numerical values are obtained,

$$\begin{array}{lll} \pi_1[0] = 1/5, & \pi_2[0] = 2/5, & \pi_3[0] = 2/5, \\ \pi_1[1] = -8/125, & \pi_2[1] = -6/125, & \pi_3[1] = 14/125, \\ \pi_1[2] = 8/3125, & \pi_2[2] = -19/3125, & \pi_3[2] = 11/3125. \end{array}$$

Note here that $(\pi_1[0], \pi_2[0], \pi_3[0])$ is the stationary distribution for the limiting Markov chain. It is also worth noticing that $\pi_1[n] + \pi_2[n] + \pi_3[n] = 0$ for $n = 1, 2$, as expected.

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