Mathematical Statistics
Stockholm University

A Note on the Connection Between Some Classical Mortality Laws and Proportional Frailty

Mathias Lindholm

Research Report 2016:15

ISSN 1650-0377

Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se

Mathematical Statistics
Stockholm University
Research Report **2016:15**,
http://www.math.su.se

A Note on the Connection Between Some Classical Mortality Laws and Proportional Frailty

Mathias Lindholm*

September 2016

Abstract

We provide a simple frailty argument that produces the Gompertz-Makeham mortality law as the population hazard rate under the assumption of proportional frailty given a common exponential hazard rate. Further, based on the result for the Gompertz-Makeham law a heuristic argument provides a slight change of proof which immediately produces a version of Perks mortality law. Moreover, we give conditions for which functional forms of the baseline hazard that will yield proper frailty distributions given that we want to retrieve a certain overall population hazard within the proportional frailty framework.

Keywords: Survival analysis, Frailty, Makeham, Perks.

*Stockholm University, Dept. Mathematics.

1 Main Results

The main result concerns the so-called Gompertz-Makeham mortality law proposed by William Makeham as an extension to the Gompertz law proposed by Benjamin Gompertz. This mortality law is one of a number of standard mortality models used in demographics and actuarial science in countries that have the tradition of modelling mortality in continuous time.

The Gompertz-Makeham mortality law, $h(t)$, expressed as a hazard rate is given by

$$h_M(t) = a + be^{ct}, \ a, b, c \geq 0, \tag{1}$$

and the standard Gompertz law is obtained by setting $a \equiv 0$, see e.g. [6, Eq. (3.9.36)]. Note that the formulation of the Gompertz-Makeham law in (1) is in terms of e^{ct}, whereas the original formulation is in terms of $\tilde{c}^x$. We have chosen to use the above formulation for convenience.

In the present note we will make use of known relations between population hazard rates (or laws) and the concept of proportional frailty. We will now give a brief summary of notation and results needed to prove our main results stated below: The population hazard rate $h(t)$ is related to the (absolutely continuous) life time distribution T via the following relations:

$$S(T > t) = \exp\{-\int_0^t h(u)du\} = \exp\{-H(t)\}, \tag{2}$$

where $S(t)$ is the survival function of T, see e.g. [1, Ch. 1.1.2]. The concept of proportional frailty given a common baseline hazard $\alpha(t)$ and frailty distribution Z is then given by

$$h(t|Z) := \alpha(t)Z, \tag{3}$$

and

$$S(t|Z) := \exp\{-Z\int_0^t \alpha(s)ds\} = \exp\{-ZA(t)\}, \tag{4}$$

which yields the total population survival function

$$S(t) := \mathrm{E}[S(t|Z)] = \mathcal{L}_Z(A(t)), \tag{5}$$

where $\mathcal{L}_Z(t)$ is the Laplace transform of Z, see e.g. [1, Ch. 6.2.1].

Given the above together with that the Gompertz-Makeham law is a generalisation of the Gompertz law, one can ask if it is possible to describe the Gompertz-Makeham law by using a simple Gompertz hazard common to all individuals and adding individual frailty. The answer is affirmative and the main result is as follows:

Proposition 1. *The Gompertz-Makeham mortality law defined by the hazard rate in* (1) *can be expressed as the population hazard when all individuals are following a common baseline hazard rate* $\alpha(t) = \exp\{ct\}$ *mixed with an individual proportional frailty* Z *given by*

$$Z := X + Y,$$

where $X \sim \Gamma(a/c, 1/c)$ *in terms of rate, i.e.* $\mathrm{E}[X] = a$ *and* $\mathrm{Var}(X) = ac$*, and* $Y \sim \delta(b)$ *(degenerate distribution).*

Remark: The distribution of Z can be seen as a translation of a Γ-distribution.

Note that Proposition 1. tells us that we can generate a Gompertz-Makeham law at population level when each individual follows a common baseline hazard rate e^{ct} combined with individual frailty which follows a certain translated Γ-distribution. One can also note that the special case with the Gompertz law, which corresponds to that $a = 0$, gives us a constant (degenerate) frailty distribution corresponding to a single heterogeneous population. This is perhaps not surprising since the baseline hazard e^{ct} is a special case within the Gompertz family.

Other results concerning frailty that relate to the Gompertz-Makeham law are e.g. [3], [2] and [4] and [5] and the references therein.

Another mortality law to which the Gompertz-Makeham law is a special case is Perks law defined as

$$h_P(t) := \frac{\nu e^{\beta x} + \gamma}{\delta e^{\beta x} + 1}, \tag{6}$$

where $\beta, \delta, \gamma, \nu \geq 0$, see e.g. [6, Eq. (3.9.41)]. A slight change of proof of Proposition 1. gives us the following corollary which is a version of Perks law:

Corollary 1. *The following version of Perks law*

$$h_{P'}(t) = \delta\nu\left(\frac{\lambda}{\theta + \delta(e^{\nu t} - 1)} + \zeta\right)e^{\nu t}, \tag{7}$$

can be expressed as the population hazard when all individuals are following a common baseline hazard rate $\alpha(t) = \delta\nu\exp\{\nu t\}$ *mixed with an individual proportional frailty* Z *given by*

$$Z := X + Y,$$

where $X \sim \Gamma(\lambda, \theta)$ *in terms of rate, i.e.* $\mathrm{E}[X] = \lambda/\theta$ *and* $\mathrm{Var}(X) = \lambda/\theta^2$ *and* $Y \sim \delta(\zeta)$ *(degenerate distribution).*

It is straightforward to identify the Gompertz-Makeham law from (7) and again see that the Gompertz law is a degenerate case of (7). The version of Perks law given in (7) is similar to the one obtained in [3], but the result of [3] is mainly focused on the mortality for high ages and does not retrieve the Gompertz-Makeham law. The same is the case with the approach in e.g. [2] and [4, 5].

Another comment with respect to (6) is that by noting that the denominator is close to being a primitive function to the numerator, one can see that it is possible to retrieve yet another version of Perks law. More precisely: let $\alpha(t) = \nu/\delta$ denote the common constant baseline hazard and let the frailty distribution be given by the following two-point distribution

$$Z := \begin{cases} 1 & \text{with prob. } \delta/(1+\delta), \\ 1 + \beta\delta/\nu & \qquad\quad 1/(1+\delta). \end{cases} \tag{8}$$

Then, by using definition (5) we get the population survival function which can be differentiated in order to obtain the following version of Perks law:

$$h_{P''}(t) := \frac{\nu e^{\beta x} + \beta + \nu/\delta}{\delta e^{\beta x} + 1}, \tag{9}$$

where $\beta, \nu, \delta \geq 0$. Consequently, by comparing with (7) we see that we for some parametrisations now have two representations of the same population hazard. One can also note that (9) can be seen as a mixture of two sub-populations: one baseline population and one that is extra frail.

More generally, the above results are all versions of the following inverse problem: given a common baseline hazard and a known overall population hazard, is it possible to retrieve a proper (probability) frailty distribution? A partial answer to this question is given by the following result:

Proposition 2. *Given a population hazard rate* $h(t)$*, with corresponding* $H(t) = \int h(s)ds$*, and a common baseline hazard rate* $\alpha(t)$*, with corresponding* $A(t) = \int \alpha(s)ds$*, the mean and variance of the induced frailty distribution is given by*

$$\mathrm{E}[Z] = \frac{h'(A^{-1}(0))}{\alpha(A^{-1}(0))}, \tag{10}$$

$$\mathrm{Var}(Z) = \frac{h(A^{-1}(0))\alpha'(A^{-1}(0)) - h'(A^{-1}(0))\alpha(A^{-1}(0))}{(\alpha(A^{-1}(0)))^3}, \tag{11}$$

where

$$\alpha'(A^{-1}(0)) := \alpha'(t)|_{t=A^{-1}(0)}$$

and

$$h'(A^{-1}(0)) := h'(t)|_{t=A^{-1}(0)},$$

given that $A^{-1}(t)$ *exists.*

From (10) and (11) it is possible to deduce which forms of baseline hazard rates $\alpha(t)$ that will yield a non-degenerate frailty distribution. As an example it follows that $\alpha(t) \equiv \alpha$ can not generate a proper Gompertz-Makeham law.

2 Proofs

2.1 Proof of Proposition 1

By combining (1) and (2) we get

$$S(t) = \exp\{at + \frac{b}{c}(e^{ct} - 1)\}. \tag{12}$$

Now, let the baseline hazard be $\alpha(t) = e^{\nu t}$ which gives us that $A(t) = (e^{\nu t} - 1)/\nu$, which by equating (12) and (5) yields

$$\mathcal{L}_Z(t) = S(A^{-1}(t)) = \frac{1}{(1 + \nu t)^{a/\nu}} \exp\{-\frac{b}{c}((1 + \nu t)^{c/\nu})\}, \tag{13}$$

and the claimed result follows by setting $\nu = c$ and noting that

$$\mathcal{L}_Z(t) = \frac{1}{(1 + ct)^{a/c}} \exp\{-bt\} \tag{14}$$

$$= \mathcal{L}_X(t)\mathcal{L}_Y(t), \tag{15}$$

where $\mathcal{L}_X(t)$ is the Laplace transform of $X \sim \Gamma(a/c, 1/c)$ in terms of rate, i.e. $\mathrm{E}[X] = a$ and $\mathrm{Var}(X) = ac$, and $\mathcal{L}_Y(t)$ is the Laplace transform of $Y \sim \delta(b)$ (degenerate distribution).

2.2 Proof of Corollary 1

The proof of Corollary 1 follows by noting that the overall population hazard $h(t)$ can be expressed as

$$h(t) = \frac{-S'(t)}{S(t)},$$

due to (2), combined with the definition of survival function under proportional frailty from (5) when using

$$\alpha(t) = \delta\nu \exp\{\nu t\},$$

and that

$$Z := X + Y,$$

where $X \sim \Gamma(\lambda, \theta)$ in terms of rate, i.e. $\mathrm{E}[X] = \lambda/\theta$ and $\mathrm{Var}(X) = \lambda/\theta^2$, $Y \sim \delta(\zeta)$ (degenerate distribution) and re-using the Laplace transforms from the proof of Proposition 1.

2.3 Proof of Proposition 2

From (2) and (5) we get that

$$S(t) = \mathcal{L}_Z(A(t)) = \exp\{-H(t)\},$$

which gives us

$$\mathcal{L}_Z(t) = S(A^{-1}(t)),$$

where we assume that $A^{-1}(t)$ exists. Since $\mathcal{L}_Z(t) := \mathrm{E}[\exp\{-tZ\}]$, it follows that

$$\mathrm{E}[Z] = -\frac{d}{dt} \log \mathcal{L}_Z(t)|_{t=0}, \tag{16}$$

and

$$\mathrm{Var}(Z) = \frac{d^2}{dt^2} \log \mathcal{L}_Z(t)|_{t=0}. \tag{17}$$

Hence, we want to differentiate $\log \mathcal{L}_Z(t)$, which gives us

$$\frac{d}{dt} \log \mathcal{L}_Z(t) = \frac{S'(A^{-1}(t))\frac{d}{dt}A^{-1}(t)}{S(A^{-1}(t))}, \tag{18}$$

$$\begin{aligned} \frac{d^2}{dt^2} \log \mathcal{L}_Z(t) = &-\frac{(S'(A^{-1}(t))\frac{d}{dt}A^{-1}(t))^2}{(S(A^{-1}(t)))^2} \\ &+ \frac{S''(A^{-1}(t)(\frac{d}{dt}A^{-1}(t))^2}{S(A^{-1}(t))} \\ &+ \frac{S'(A^{-1}(t)\frac{d^2}{dt^2}A^{-1}(t)}{S(A^{-1}(t))}. \end{aligned} \tag{19}$$

Further, from (2) we get that

$$S'(t) = -h(t)S(t), \tag{20}$$
$$S''(t) = -h'(t)S(t) + (h(t))^2 S(t), \tag{21}$$

and that

$$\frac{d}{dt}A^{-1}(t) = \frac{1}{\alpha(A^{-1}(t))}, \tag{22}$$
$$\frac{d^2}{dt^2}A^{-1}(t) = \frac{-\alpha'(A^{-1}(t))}{(\alpha(A^{-1}(t)))^3} = \frac{-\alpha'(A^{-1}(t))}{(\alpha(A^{-1}(t)))^2}\frac{d}{dt}A^{-1}(t). \tag{23}$$

By combining (18) and (19) with (20)-(23) and substituting the resulting expressions into (16) and (17) yields the desired result.

References

[1] Aalen O., Borgan Ø., Gjessing H.K. (2008) *Survival and Event History Analysis: A Process Point of View*. Springer.

[2] Butt Z., Haberman S. (2004) Application of Frailty-Based Mortality Models Using Generalized Linear Models. *ASTIN Bulletin*, Vol. 34, Issue 01, pp. 175 – 197

[3] Martinelle S. (1987) Bakgrundsmaterial Från Demografiska Funktionen: Om dödsrisker för gamla - En metodstudie. *SCB Rapport*, 1987:1.

[4] Missov T.I., Lenart A., Nemeth L., Canudas-Romo V., Vaupel J.W. (2015) The Gompertz force of mortality in terms of the modal age at death, *Demographic Research*, Vol. 32, Article 36, pp. 1031–1048.

[5] Missov T.I., Vaupel J.W. (2015) Mortality Implications of Mortality Plateaus. *SIAM Review*, Vol. 57, No. 1, pp. 61–70.

[6] Olivieri A., Pitacco E. (2010) *Introduction to Insurance Mathematics: Technical and Financial Features of Risk Transfers*. Springer.