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Friendly frogs, stable marriage, and the magic of invariance

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Abstract

We introduce a two-player game involving two tokens located at points of a fixed set. The players take turns to move a token to an unoccupied point in such a way that the distance between the two tokens is decreased. Optimal strategies for this game and its variants are intimately tied to Gale-Shapley stable marriage. We focus particularly on the case of random infinite sets, where we use invariance, ergodicity, mass transport, and deletion-tolerance to determine game outcomes.

Keywords: Combinatorial game; random game; stable marriage; Poisson process.

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1 Friendly Frogs

Here is a simple two-player game, which we call **friendly frogs**. A pond contains several lily pads. (Their locations form a finite set L of points in Euclidean space $\mathbb{R}^d$). There are two frogs. The first player, **Alice**, chooses a lily pad and places a frog on it. The second player, **Bob**, then places a second frog on a distinct lily pad. The players then take turns to move, starting with Alice. A move consists of jumping either frog to another lily pad, in such a way that the distance between the two frogs is strictly decreased, but they are not allowed to occupy the same lily pad. (The frogs are friends, so do not like to be moved further apart, but a lily pad is not large enough to support them both.) A player who cannot move **loses** the game (and the other player **wins**). See Figure 1 for an example game.

We are interested in optimal play. A **strategy** for a player is a map that assigns a legal move (if one exists) to each position, and a **winning** strategy is one that results in a win for that player whatever strategy the other player uses. (In friendly frogs, a position consists of the locations of 0, 1 or 2 frogs.) If there exists a winning strategy for a player, we say that the game is a **win** for that player (and a **loss** for the other player).

Since there are only finitely many possible positions, and the distance between the frogs decreases on each move, the game must end after a finite number of moves. Consequently, for any set L, the game is a win for exactly one player. Is it Alice or Bob? Surprisingly, the answer depends only on the size of L.

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