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Abstract

In this article we introduce a dynamic Erdős-Rényi graph model, in which, independently for each vertex pair, edges appear and disappear according to a Markov on-off process.

In studying the dynamic graph we present two main results. The first being on how long it takes for the graph to reach stationarity. We give an explicit expression for this time, as well as proving that this is the fastest time to reach stationarity among all strong stationary times.

The second result concerns the time it takes for the dynamic graph to reach a certain number of edges. We give an explicit expression for the expected value of such a time, as well as study its asymptotic behavior. This time is related to the first time the dynamic Erdős-Rényi graph contains a cluster exceeding a certain size.

1 Introduction

The Erdős-Rényi graph, in this text called the *static* Erdős-Rényi graph, is a well-studied model for random graphs, which is either (i) consisting of n vertices and k edges, where the edges are assigned uniformly to the $\binom{n}{2}$ vertex pairs—this graph model is denoted $G(n,m)$; or (ii) consisting of n vertices where edges are assigned independently between vertex pairs with probability p—this graph model is denoted $G(n,p)$, see [2] for more details and many properties of the model. In this article we introduce a natural dynamic version of such a model: the dynamic Erdős-Rényi graph.

Before moving on we set some notation: Throughout $N = \binom{n}{2}$. Furthermore. we use the asymptotic order notation: $f(n) = O(g(n))$ if and only if $|f(n)| \leq M|g(n)|$ for large n and some $M < \infty$; $f(n) = \Theta(g(n))$ if and only if both $f(n) = O(g(n))$ and $g(n) = O(f(n))$; finally, $f(n) = o(g(n))$ if and only if $\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0$. Throughout, all asymptotic's are for the limit $n \to \infty$.

1.1 The dynamic Erdős-Rényi graph

For $\alpha, \beta > 0$ and n a positive integer, the dynamic Erdős-Rényi graph $\{G(t),\ t \geq 0\}$ is a stochastic process evolving according to the following dynamics,

(i) The number of vertices is fixed at n.

(ii) Independently for each vertex pair, if no edge is present an edge is added after an $\mathrm{Exp}(\frac{\beta}{n-1})$-distributed time; if an edge is present, the edge is removed after an $\mathrm{Exp}(\alpha)$-distributed time.

Note that (ii) can be replaced by,

(iia) independently for each vertex pair, the state of an edge (present or not present) is updated at the points of a Poisson process with intensity $\lambda = \alpha + \frac{\beta}{n-1}$. Independently of the Poisson process and previous states of the edges, with probability $p = \frac{\beta}{\beta+(n-1)\alpha}$ an edge will be present after the update, and with probability $q = 1 - p$ it will not be present.

The choice of birth rate $\frac{\beta}{n-1}$ and not $\frac{\beta}{n}$ is because if the birth rate equals $\frac{\beta}{n-1}$ then if $\beta = \alpha$, the dynamic graph converges to a *critical* Erdős-Rényi graph ($G(n,p)$ with $p = \frac{1}{n}$). However, for large n it makes no difference which of the two birth rates is chosen.

Let $\{\chi_{u,v}(t),\ t \geq 0\}$ denote the indicator process representing if an edge is present between vertex u and v. We refer to this process as an edge process. This is, by definition, a birth-death process on $\{0, 1\}$ with birth-rate $\lambda = \frac{\beta}{n-1}$ and death-rate $\mu = \alpha$, also known as an on-off process. We think of the dynamic Erdős-Rényi graph $\{G(t),\ t \geq 0\}$ as being composed of these i.i.d. processes, with $G(t) = (\chi_{1,2}(t), \ldots, \chi_{n-1,n}(t))$.

Throughout we assume that the underlying probability space has enough structure so that (iia) holds, i.e. that the edge processes are generated according to (iia). This means that the probability space has a filtration $\{\mathcal{F}_t,\ t \geq 0\}$, where $\mathcal{F}_t$ is the information generated by the update times and corresponding edge updates up to time t, and that $\{G(t),\ t \geq 0\}$ is adapted to this filtration. We shall see that, for our purposes, this assumption can be made without loss of generality.

1.2 The fastest time to stationarity.

Below, we see that the distribution of $\{G(t),\ t \geq 0\}$ converges to the distribution of a static Erdős-Rényi graph with edge probability $p = \frac{\beta}{\beta+(n-1)\alpha}$, which is also the stationary distribution of the dynamic graph. In Section 2 we construct a time T_s, called the *fastest time to stationarity* for $\{G(t),\ t \geq 0\}$ [5], with the following properties,

(i) $G(T_s)$ is distributed according to the stationary distribution of $\{G(t),\ t \geq 0\}$, and is independent of T_s;

(ii) $\{G(T_s + s);\ s \geq 0\}$ is a stationary process, and is independent of T_s;

(iii) if T' is any other random variable satisfying (i) and (ii) then for all $t > 0$,
$$\mathbb{P}(T < t) \geq \mathbb{P}(T' < t)$$
i.e. T is the stochastically smallest time satisfying (i) and (ii).

A key part in constructing such a time to stationarity is noting that when an edge process $\{\chi_{u,v}(t),\ t \geq 0\}$ enters stationarity it stays there. Since the dynamic graph, $G(t) = (\chi_{1,2}(t), \ldots, \chi_{n-1,n}(t))$ is composed of edge processes and the processes are independent, the dynamic graph should be in stationarity if all edge processes have entered stationarity. Hence we proceed by finding the fastest times to stationarity for the underlying edge processes, i.e. $\{T_{u,v},\ \forall (u,v)\}$, and then show that the maximum of these is indeed the fastest time to stationarity for the dynamic graph.

In order to derive the stationary distribution for the dynamic graph process, we note that that following result for the underlying edge processes is immediate from defining property (iia) of the dynamic Erdős-Rényi graph:

Lemma 1. *For $u, v \in V$, the edge processes $\{\chi_{u,v}(t),\ t \geq 0\}$ are independent ergodic Markov processes on $\{0, 1\}$, with probability transition functions equal to,*

$$\mathbb{P}(\chi_{u,v}(t) = 1 | \chi_{u,v}(0) = 0) = p_{0,1}(t) = \frac{\beta}{\beta + (n-1)\alpha}\left(1 - e^{-(\alpha + \frac{\beta}{n-1})t}\right)$$
$$\mathbb{P}(\chi_{u,v}(t) = 1 | \chi_{u,v}(0) = 1) = p_{1,1}(t) = e^{-(\alpha + \frac{\beta}{n-1})t} + \frac{\beta}{\beta + (n-1)\alpha}\left(1 - e^{-(\alpha + \frac{\beta}{n-1})t}\right)$$

and stationary distribution π equal to,

$$\pi(1) = \frac{\beta}{\beta + (n-1)\alpha}$$
$$\pi(0) = \frac{(n-1)\alpha}{\beta + (n-1)\alpha}.$$

Because of the independence of the edge processes, we also have

Proposition 1. *The dynamic Erdős-Rényi graph* $\{G(t),\ t \geq 0\}$ *is an ergodic Markov process with finite state space and with unique stationary and limiting distribution equal to that of a static Erdős-Rényi graph* $G(n,p)$ *with edge probability* $p = \frac{\beta}{\beta+(n-1)\alpha}$.

By (iia) in the definition of $\{G(t),\ t \geq 0\}$, we immediately see that after the first update, i.e. after an exponentially distributed time with parameter $\alpha + \frac{\beta}{n-1}$, an edge process is in stationarity. It will be proven in Section 2 that this is also the fastest time to stationarity for an edge process given that it starts in state 0 or 1. We deduce that the fastest time to stationarity T_s for the dynamic graph is distributed as the maximum of $N = \binom{n}{2}$ (the number of edge processes) independent exponentially distributed random variables with parameter $\alpha + \frac{\beta}{n-1}$. The following is our first main result:

Theorem 1. *Let* $\{G(t),\ t \geq 0\}$ *be the dynamic Erdős-Rényi graph starting in an arbitrary state. Let* $T_{u,v}$ *be the fastest time to stationarity for* $\{\chi_{u,v}(t),\ t \geq 0\}$*. Then,*

$$T_s = \max_{u,v}\{T_{u,v}\}$$

is the fastest time to stationarity for the dynamic Erdős-Rényi graph. Furthermore, its distribution function is given by,

$$\mathbb{P}(T_s < t) = \left(1 - e^{-(\alpha+\frac{\beta}{n-1})t}\right)^N. \tag{1}$$

Concerning the asymptotic behavior of (1), we show that it is very likely that the graph enters stationarity roughly at time $\frac{2\log(n)}{\alpha}$. The following result will be proven in Section 2:

Corollary 1. *Let* T_s *be the fastest time to stationarity for the dynamic Erdős-Rényi graph. Then for any function* $w(n)$ *such that* $w(n) \to \infty$ *as* $n \to \infty$ *the following holds,*

$$\mathbb{P}(T_s < t(n)) \to 1 \text{ as } n \to \infty, \qquad \text{if} \qquad t(n) > \frac{2\log(n)}{\alpha} + w(n),$$
$$\mathbb{P}(T_s < t(n)) \to 0 \text{ as } n \to \infty, \qquad \text{if} \qquad t(n) < \frac{2\log(n)}{\alpha} - w(n).$$

Furthermore,

$$\mathbb{E}(T_s) = O(\log(n)).$$

Hence for large n and time $t >> \frac{2\log(n)}{\alpha}$, we conclude that in the time period $[0,t]$ the process is in stationarity most of that time. So in studying certain properties of the dynamic graph one may be able to reduce the problem to study properties of the graph when in stationarity—something that is often more tractable, and is indeed exploited in Section 3.

1.3 Hitting times.

In Section 3 we present a result on the expected time it takes for the graph, starting with j edges, to reach a fixed number of i edges, where $j < i$. We give an explicit expression for the expected value of this time, as well as study its asymptotic properties. Special care is given to the case when $j = 0$, i.e. when the dynamic graph starts without any edges.

Let $\eta(t)$ denote the number of edges at time t in $\{G(t),\ t \geq 0\}$. Then $\{\eta(t), t \geq 0\}$ is a birth and death process on the integers $\{0, 1, \ldots, N\}$, with birth rates $\lambda_k = (N-k)\beta/(n-1)$ and death rates $\mu_k = k\alpha$.

For such processes it is well known that the hitting time of i is distributed as the sum of i independent exponentially distributed random variables, whose parameters are given by the nonzero eigenvalues of the matrix $-Q$, where Q is the generator matrix of the process $\{\eta(t), t \geq 0\}$, with i turned into an absorbing state. So, Q is given through $Q_{00} = -\lambda_0$, $Q_{01} = \lambda_0$ and for $k = 1, 2, \ldots, i-1$: $Q_{kk} = -(\lambda_k + \mu_k)$, $Q_{k,k+1} = \lambda_k$ for $k = 0, 1, \ldots, i-1$ and $Q_{k,k-1} = \mu_k$, while all other elements of Q are 0. (see e.g. [6, Thm. 1.1]).

Because the eigenvalues of a matrix are typically hard to find, we use another approach in deriving the expected hitting time of i. The obtained expression is difficult to compute for large n so we also give bounds on the expected time it takes for the dynamic graph to go from 0 to $i = [cn]$ edges, where c is a constant and $[x]$ denotes the closest integer to x. The main reason this particular scaling is studied is its connection to the size of the largest component. Namely, if $\mathcal{G}(n, i(n))$ is a static Erdős-Rényi graph with a prescribed number of edges and $|C(n, i(n))|$ is the size of the largest component of such a graph, it is possible to show that: for every $0 < \epsilon < 1$ there exist a $c > 1/2$ such that if $i(n) = [cn]$ then,

$$\frac{|C(n, i(n))|}{n} \xrightarrow{p} \epsilon \text{ as } n \to \infty,$$

where $\xrightarrow{p}$ denotes convergence in probability. Furthermore,

$$c = \frac{-\log(1-\epsilon)}{2\epsilon}.$$

Hence, for given $\epsilon \in (0,1)$ we know how many edges are needed in the static Erdős-Rényi graph for the fraction of vertices in the largest component to be roughly equal to ϵ with high probability, namely $i = [cn]$ where $c = \frac{-\log(1-\epsilon)}{2\epsilon}$. This can be used for the dynamic graph. Since, if we wait until that many edges are present it is very likely that the size of the largest component in the dynamic graph has already exceeded ϵn. This will be discussed further in Section 3.1.

The expected time to go from 0 to $i = [cn]$ exhibits three different behaviors depending on the value of c. For $c < \frac{\beta}{2\alpha}$ the graph reaches i edges after a constant time; for $c = \frac{\beta}{2\alpha}$ the graph reaches i edges after an logarithmic time, which follows from the time to stationarity being $O(\log(n))$; while for $c > \frac{\beta}{2\alpha}$ the graph reaches i edges after en exponentially large time. In Section 3 we prove the following:

Theorem 2. *Let $\tau_j(i)$ be the time it takes, starting with j edges, for the dynamic Erdős-Rényi graph to reach $i = [cn]$ edges, where $c > 0$. Then,*

(a) If $c < \frac{\beta}{2\alpha}$ then,

$$\mathbb{E}(\tau_0(i)) \to \frac{-\log(1 - \frac{2\alpha}{\beta}c)}{\alpha} \quad \text{as } n \to \infty.$$

(b) If $c = \frac{\beta}{2\alpha}$ then,

$$\mathbb{E}(\tau_j(i)) = O(\log(n)) \qquad \forall j.$$

(c) If $c > \frac{\beta}{2\alpha}$ then,

$$\Theta(n^{-1})e^{n\left(c\log(\frac{2\alpha}{\beta}c)-c+\frac{\beta}{2\alpha}\right)} \leq \mathbb{E}(\tau_0(i)) \leq \Theta(n^{-1/2})e^{n\left(c\log(\frac{2\alpha}{\beta}c)-c+\frac{\beta}{2\alpha}\right)}$$

where $c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha} > 0$.

This theorem may be used to provide bounds for the expected time the dynamic Erdős-Rényi graph needs to first contain a component of a desired size. In particular, in the critical case ($\alpha = \beta$)—for which the typical size of the largest cluster is $o(n)$—we can find an upper bound for the expected time until the fraction of vertices in the largest component exceeds $\epsilon > 0$.

Corollary 2. *Let $\hat{\tau}(\epsilon n)$ be the first time the dynamic Erdős-Rényi graph, starting with no edges, has a component of size at least ϵn. Then for all $\hat{\epsilon} \in (\epsilon, 0.7968)$,*

$$\mathbb{E}[\hat{\tau}(\epsilon n)] = O(n^{-1/2})e^{n(\hat{\epsilon}^2/16 + O_\epsilon(\hat{\epsilon}^3))}.$$

2 The fastest time to stationarity

In constructing the fastest time to stationarity for $\{G(t),\ t \geq 0\}$ we shall find the fastest times $\{T_{u,v}\}$ to stationary for the underlying edge processes $\{\chi_{u,v}(t),\ t \geq 0\}$, and taking T_s to be the maximum of these. Waiting until all the edge processes have entered stationarity should ensure that the dynamic graph is in stationarity, since $G(t) = (\chi_{1,2}(t), \ldots, \chi_{n-1,n}(t))$. In order to show that this time to stationarity is indeed the fastest time to stationarity we need the concepts of a *strong stationary time* and of *separation*, as defined in [5].

2.1 Separation and strong stationary times

Roughly speaking, a strong stationary time T for a stochastic process X is a stopping time for X with some extra external randomness such that $X(T)$ has the stationary distribution and is independent of T. In order to define a strong stationary time for a process X one needs the concept of a *randomized stopping time*.

Definition 1. ([5]) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space. Let $\mathcal{F}_\infty$ be the smallest σ-algebra containing $\mathcal{F}_t$ for all t.
Furthermore, let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ independent of $\mathcal{F}_\infty$. We say that $T : \Omega \to [0, \infty]$ is a randomized stopping time relative to $\{\mathcal{F}_t,\ t \geq 0\}$ if for each $t \geq 0$,

$$\{T \leq t\} \in \sigma(\mathcal{F}_t, \mathcal{G}),$$

where $\sigma(\mathcal{F}_t, \mathcal{G})$ is the smallest σ-algebra containing both $\mathcal{F}_t$ and $\mathcal{G}$.
If the process X is adapted to $\{\mathcal{F}_t,\ t \geq 0\}$ we say that T is a randomized stopping time for X.

We are now ready to define the strong stationary time and the fastest time to stationarity.

Definition 2. ([5]) Let X be a stochastic process, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P)$ and adapted to $\{\mathcal{F}_t,\ t \geq 0\}$, taking values in some state space S. Assume that X has a unique stationary distribution π. Furthermore let T be a randomized stopping time relative to $\{\mathcal{F}_t,\ t \geq 0\}$. Then, T is said to be a *strong stationary time* for X if: $X(T)$ has the stationary distribution and is independent of T given that $\{T < \infty\}$, i.e. if,

$$\begin{aligned}\mathbb{P}(T \leq t, X(T) = y | T < \infty) &= \mathbb{P}(T \leq t | T < \infty)\mathbb{P}(X(T) = y | T < \infty) \\ &= \mathbb{P}(T \leq t | T < \infty)\pi(y)\end{aligned}$$

for all $0 \leq t < \infty$ and $y \in S$.

If, for any other strong stationary time T', we have $\mathbb{P}(T > t) \leq \mathbb{P}(T' > t)$ then we say that T is the *fastest time to stationarity.*

Remark. We shall only be concerned with strong stationary times T such that $\mathbb{P}(T < \infty) = 1$, hence we can drop the conditioning on $\{T < \infty\}$ in Definition 2 above.

Fill [5] provides us with an important proposition regarding strong stationary times.

Proposition 2 ([5, Prop. 2.4])**.** *Let X be an ergodic Markov chain with right-continuous paths on some finite or countable state space S. Then the following are equivalent for a randomized stopping time relative to X.*

(i) T is a strong stationary time;

(ii) $\mathbb{P}(T \leq t, X(t) = y) = \mathbb{P}(T \leq t)\pi(y), \qquad$ *for* $0 \leq t < \infty$*;*

(iii) $\mathbb{P}(T \leq t, X(u) = y) = \mathbb{P}(T \leq t)\pi(y), \qquad$ *for* $0 \leq t < u < \infty$*.*

Remark. We note that, if T is a strong stationary time then, by (iii), $T + s$ is also a strong stationary time. Hence X stays in stationarity upon entering it.

The separation, $s(t) = \sup_y \left(1 - \frac{\mathbb{P}(X(t)=y)}{\pi(y)}\right)$, for a stochastic process is a function in time which measures the "distance" between the distribution at time t and its stationary distribution, and has an intimate connection with strong stationary times.

Strong stationary times are well-understood for ergodic Markov processes on countable state spaces, see [5]. The main result of [5] is that for such processes the following holds,

(I) If T is a strong stationary time, then for all $0 \leq t < \infty$,

$$s(t) \leq \mathbb{P}(T > t) \tag{2}$$

i.e. the separation at time t is a lower bound for the probability that the process has not yet entered stationarity at time t.

(II) If the state space of the process is finite (and the underlying probability space rich enough to support an uniformly distributed random variable on $(0, 1)$ independent of the process), there exist a strong stationary time T such that (2) holds with equality. We call such a time *the fastest time to stationarity.*

Remark. It follows that knowing the distribution of the fastest time to stationarity, equation (2) gives a way of quantifying the rate of convergence of the dynamic graph to stationarity, since the separation measures the distance between the distribution of a process at time t and its stationary distribution.

As stated before, we assume that the underlying probability space is rich enough to support (iia) in the definition of $\{G(t),\ t \geq 0\}$—for the purpose of finding the distribution of the fastest time to stationarity this assumption can be made without loss of generality, since if such a time exists on a probability space its distribution, is by (II), determined by $s(t)$ (which is not dependent on the probability space).

Fill also gives the algorithm for constructing the fastest time to stationarity for an ergodic Markov chain on a finite state space. However, the proof of this is technical and not very intuitive. Nevertheless, we can still use the above mentioned results from [5] to show that our candidate time to stationarity for $\{G(t),\ t \geq 0\}$ is indeed the stochastically smallest one.

2.2 The fastest time to stationarity

In order to construct the fastest time to stationarity for the dynamic graph we proceed by constructing such times for the underlying edge processes.

Lemma 2. *Let $\{\chi_{u,v}(t),\ t \geq 0\}$ be an edge process starting with 0 or 1 edges. Then the fastest time to stationarity $T_{u,v}$ for $\{\chi_{u,v}(t),\ t \geq 0\}$ is distributed as,*

$$T_{u,v} \sim \mathit{Exp}\left(\alpha + \frac{\beta}{n-1}\right).$$

Proof. After the time to the first update $T_{u,v}$ of an edge—which is an exponential distributed time with rate $(\alpha + \frac{\beta}{n-1})$—the edge process is in stationarity, since $\mathbb{P}(\chi_{u,v}(T_{u,v} + s) = 1) = \mathbb{P}(\text{edge added at last update}) = p$. It is also clear that $T_{u,v}$ is a (randomized) stopping time relative the filtration which $\{\chi_{u,v}(t),\ t \geq 0\}$ is adapted to (information generated by update times and corresponding edge updates).

Also, $T_{u,v}$ satisfies,

$$\begin{aligned}\mathbb{P}(T_{u,v} \leq t, \chi_{u,v}(T_{u,v}) = 1) &= \mathbb{P}(T_{u,v} \leq t, \text{edge added last update}) \\ &= \mathbb{P}(T_{u,v} \leq t)\mathbb{P}(\text{edge added last update}) = \mathbb{P}(T_{u,v} \leq t)p.\end{aligned}$$

By Definition 2, $T_{u,v}$ is a strong stationary time for $\{\chi_{u,v}(t),\ t \geq 0\}$.

Furthermore, it is easily shown that $\mathbb{P}(T_{u,v} > t)$ equals the separation

$$s(t) = \sup_{i \in \{0,1\}} \left(1 - \frac{\mathbb{P}(\chi_{u,v}(t) = i)}{\pi(i)}\right)$$

of the process $\{\chi_{u,v}(t),\ t \geq 0\}$ since,

$$\chi_{u,v}(0) = 0 \implies s(t) = \sup_{i \in \{0,1\}} \left(1 - \frac{\mathbb{P}(\chi_{u,v}(t) = i)}{\pi(i)}\right) = \left(1 - \frac{p_{0,1}(t)}{\pi(1)}\right) = \mathbb{P}(T_{u,v} > t),$$

$$\chi_{u,v}(0) = 1 \implies s(t) = \sup_{i \in \{0,1\}} (1 - \frac{\mathbb{P}(\chi_{u,v}(t) = i)}{\pi(i)}) = \left(1 - \frac{p_{1,0}(t)}{\pi(0)}\right) = \mathbb{P}(T_{u,v} > t).$$

Hence, by (II), $T_{u,v}$ is the fastest time to stationarity for $\{\chi_{u,v}(t),\ t \geq 0\}$, if the process starts with 0 or 1 edges. □

Remark. If the initial distribution of $\{\chi_{u,v}(t),\ t \geq 0\}$ is arbitrary then the time in Lemma 2 is still a strong stationary time for the process—however it need not be the fastest one: clearly, if the initial distribution is the stationary distribution then the fastest time to stationarity is $T = 0$.

By Proposition 2, an ergodic Markov process stays in stationarity after entering it. Therefore we expect that a vector of i.i.d. ergodic Markov chains for which all the components have entered stationarity, is also in stationarity, and that the time at which all the components have entered stationarity is also the fastest time to stationarity. The following lemma shows that this is indeed the case.

Lemma 3. *Let $X_1, X_2, \ldots X_n$ be independent ergodic Markov processes, each process X_i taking values in a finite state spaces S_i. Furthermore, assume that X_i has stationary distribution π_i. Let T_i be the fastest time to stationarity for X_i. Assume that $T_1, T_2, \ldots, T_n$ are independent. Then $T = \max_i\{T_i\}$ is the fastest time to stationarity for the process $(X_1, X_2, \ldots, X_n)$.*

Proof. We prove the claim for $n = 2$ and then the lemma follows by induction.

Let X and Y be two processes satisfying the premises of the lemma. Note that (X, Y) is an ergodic Markov chain. Hence, the stationary distribution of (X, Y) exists and is given by,

$$\lim_{t\to\infty} \mathbb{P}(X(t) = x, Y(t) = y) = \pi_X(x)\pi_Y(y)$$

for all $(x, y) \in S_x \times S_y$.

We have that T_X is a stopping time relative $\{\sigma\{\mathcal{F}_t^X, \mathcal{G}^X\},\ t \geq 0\}$, where $\{\mathcal{F}_t^X,\ \geq 0\}$ is the filtration for which X is adapted and $\mathcal{G}^X$ is a σ-algebra independent of $\mathcal{F}_\infty^X = \sigma(\bigcup_t \mathcal{F}_t^X)$. Analogously, T_Y is a stopping time relative $\{\sigma\{\mathcal{F}_t^Y, \mathcal{G}^Y\},\ t \geq 0\}$. It then follows that $T = \max(T_X, T_Y)$ is a stopping time relative $\sigma\{\sigma\{\mathcal{F}_t^X, \mathcal{F}_t^Y\}, \sigma\{\mathcal{G}^X, \mathcal{G}^Y\},\ t \geq 0\}$, and therefore a randomized stopping time for (X, Y).

Secondly we must prove that T is a strong stationary time for (X, Y) which by Proposition 2 is equivalent to showing that,

$$\mathbb{P}(T \leq t, X(t) = x, Y(t) = y) = \mathbb{P}(T \leq t)\pi_X(x)\pi_Y(y)$$

for all $0 \leq t < \infty$, $x \in S_x,\ y \in S_y$.

Since $\{\max(Z_1, \dots, Z_n) \le t\} = \{Z_1 \le t, \dots Z_n \le t\}$ and T_X and T_Y are strong stationary times we obtain,

$$\begin{aligned}\mathbb{P}(T \le t, X(t) = x, Y(t) = y) &= \mathbb{P}(T_X \le t, T_Y \le t, X(t) = x, Y(t) = y) \\ &= \mathbb{P}(T_X \le t, X(t) = x)\mathbb{P}(T_Y \le t, Y(t) = y) \\ &= \mathbb{P}(T_X \le t)\pi_X(x)\mathbb{P}(T_Y \le t)\pi_Y(y) = \mathbb{P}(T \le t)\pi_X(x)\pi_Y(y)\end{aligned}$$

where the third equality follows from Proposition 2.

It remains to show that, if T' is any other strong stationary time, we have that $\mathbb{P}(T' \le t) \le \mathbb{P}(T \le t)$. Now since T_X and T_Y are the fastest times to stationarity and since the state space is finite, see (II) in Section 2.1,

$$\begin{aligned}\mathbb{P}(T_X > t) = s_X(t) &\iff \mathbb{P}(T_X \le t) = a_X(t) = \inf_x \frac{\mathbb{P}(X(t) = x)}{\pi_X(x)} \\ \mathbb{P}(T_Y > t) = s_Y(t) &\iff \mathbb{P}(T_Y \le t) = a_Y(t) = \inf_y \frac{\mathbb{P}(Y(t) = y)}{\pi_Y(y)}.\end{aligned}$$

Using this we conclude that,

$$\begin{aligned}\mathbb{P}(T \le t) &= \mathbb{P}(T_X \le t)\mathbb{P}(T_Y \le t) = a_X(t)a_Y(t) \\ &= \inf_x \frac{\mathbb{P}(X(t) = x)}{\pi_X(x)} \inf_y \frac{\mathbb{P}(Y(t) = y)}{\pi_Y(y)} = \inf_{(x,y)} \frac{\mathbb{P}(X(t) = x)\mathbb{P}(Y(t) = y)}{\pi_X(x)\pi_Y(y)}.\end{aligned}$$

Now let T' be any other strong time to stationary. We have that $\forall (x, y) \in S_x \times S_y$,

$$\mathbb{P}(T' \le t) = \frac{\mathbb{P}(T' \le t, X(t) = x, Y(t) = y)}{\pi_X(x)\pi_Y(y)} \le \frac{\mathbb{P}(X(t) = x)\mathbb{P}(Y(t) = y)}{\pi_X(x)\pi_Y(y)}.$$

Since this holds for all states (x, y) we can conclude that $\mathbb{P}(T' \le t) \le \mathbb{P}(T \le t)$, and we see that T is stochastically smallest among all strong stationary times.

The full lemma follows by induction. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Recall that $G(t) = (\chi_{1,2}(t), \dots, \chi_{n-1,n}(t))$.
Assume $\{G(t),\ t \ge 0\}$ starts with an arbitrary number of edges—i.e. $G(0) = k$. By Lemma 2, $T_{u,v} \sim \text{Exp}(\alpha + \frac{\beta}{n-1})$ is the fastest time to stationary for $\{\chi_{u,v},\ t \ge 0\}$ whether we start with or without an edge. Now apply Lemma 3 to conclude that $T_s = \max_{u,v}\{T_{u,v}\}$ is the fastest time to stationarity for the dynamic Erdős-Rényi graph. Furthermore,

$$\mathbb{P}(T_s < t) = \mathbb{P}(\max(T_{1,2}, \dots, T_{n-1,n}) < t) = \mathbb{P}(T_{u,v} < t)^N = \left(1 - e^{-(\alpha + \frac{\beta}{n-1})t}\right)^N.$$

□

Remark. If the initial distribution of $\{G(t),\ t \geq 0\}$ is arbitrary then the time in Lemma 1 is still a strong stationary time for the process, however it need not be the fastest one (see Remark 2.2).

2.3 Asymptotics.

The distribution for the fastest time to stationarity in Theorem 1 is exact but not very insightful. Here we provide the proof of Corollary 1, in which we deal with the asymptotic of the distribution function.

Proof of Corollary 1. Let all limits be for $n \to \infty$. Recall for $a \in \mathbb{R}$, the standard limit $(1 - a/n)^n \to e^{-a}$. This implies that for positive functions $f(n)$, $(1 - f(n)/n)^n \to 1$ if $f(n) \to 0$; and $(1 - f(n)/n)^n \to 0$ if both $f(n) \to \infty$ and $f(n)/n \to 0$.

By Theorem 1 we know.

$$\mathbb{P}(T_s < t) = \left(1 - e^{-(\alpha + \frac{\beta}{n-1})t}\right)^N = \left(1 - \frac{e^{\log(N) - (\alpha + \frac{\beta}{n-1})t}}{N}\right)^N.$$

If $t - 2(\log(n))/\alpha \to \infty$, then

$$\log(N) - \left(\alpha + \frac{\beta}{n-1}\right)t = \log(n) + \log(n-1) - \log 2 - \left(\alpha + \frac{\beta}{n-1}\right)t \to -\infty,$$

and thus

$$e^{\log(N) - (\alpha + \frac{\beta}{n-1})t} \to 0,$$

which in turn implies

$$\mathbb{P}(T_s < t) = \left(1 - \frac{e^{\log(N) - (\alpha + \frac{\beta}{n-1})t}}{N}\right)^N \to 1.$$

While if $t - 2(\log(n))/\alpha \to -\infty$, then

$$\log(N) - \left(\alpha + \frac{\beta}{n-1}\right)t = \log(n) + \log(n-1) - \log 2 - \left(\alpha + \frac{\beta}{n-1}\right)t \to \infty.$$

and thus

$$e^{\log(N) - (\alpha + \frac{\beta}{n-1})t} \to \infty,$$

which in turn implies

$$\mathbb{P}(T_s < t) = \left(1 - \frac{e^{\log(N) - (\alpha + \frac{\beta}{n-1})t}}{N}\right)^N \to 0.$$

To prove that $\mathbb{E}(T_s) = O(\log(n))$ we note that,

$$1 - (1 - e^{-(\alpha + \frac{\beta}{n-1})t})^N \leq \min\{1, Ne^{-(\alpha+\frac{\beta}{n-1})t}\}$$
$$\log(N) \leq 2\log(n).$$

Hence,

$$\mathbb{E}(T_s) = \int_0^\infty \mathbb{P}(T_s > t)dt = \int_0^\infty \left(1 - \left(1 - e^{-(\alpha+\frac{\beta}{n-1})t}\right)^N\right) dt$$
$$= \int_0^{\log(N)/\alpha} \left(1 - \left(1 - e^{-(\alpha+\frac{\beta}{n-1})t}\right)^N\right) dt + \int_{\log(N)/\alpha}^\infty \left(1 - \left(1 - e^{-(\alpha+\frac{\beta}{n-1})t}\right)^N\right) dt$$
$$\leq \frac{\log(N)}{\alpha} + \int_{\log(N)/\alpha}^\infty Ne^{-(\alpha+\frac{\beta}{n-1})t} dt \leq 2\frac{\log(n)}{\alpha} + (\alpha + \frac{\beta}{n-1})^{-1} = O(\log(n)).$$

This completes the prove of the corollary. □

3 Hitting times for a fixed number of edges

In this section we study $\mathbb{E}(\tau_j(i))$, where $\tau_j(i)$ is the time it takes for the dynamic graph to reach i edges, given that it starts with j edges, i.e.

$$\tau_j(i) = \inf\{t > 0;\ \eta(t) = i,\ \eta(0) = j\}.$$

We derive an exact expression for $\mathbb{E}(\tau_j(i))$, $(j < i)$, and give special attention to the case $\mathbb{E}(\tau_0(i))$, when $i = [cn]$ and $c > 0$, where we provide (asymptotic) bounds for $\mathbb{E}(\tau_0(i))$.

As mentioned in the introduction, it is known (see e.g. [6]), that $\tau_0(i)$ is distributed as the sum of i independent exponentially distributed random variables with as rate parameters, the nonzero eigenvalues of the negative generator matrix of the variant of the process $\{\eta(t), t \geq 0\}$ restricted to states $\{0, 1, \ldots, i\}$ in which i is turned into an absorbing state. However, finding those eigenvalues is difficult, and therefore we put effort in finding expressions for the expected hitting time of i and asymptotics for it.

In deriving an exact expression for $\mathbb{E}(\tau_j(i))$ we shall need to exploit the strong Markov property of the dynamic graph. For our purposes we say that a Markov process X has the strong Markov property if for any *a.s. finite* stopping time τ for X we have that $X_\tau = \{X(t+\tau),\ t \geq 0\}$ is a probabilistic copy of X starting in $X(\tau)$, as well as being independent of X up to time τ, given $X(\tau)$.

First we compute $\mathbb{E}(\tau_k(k+1))$ for $k < N$. Then, by the strong Markov property of $\{\eta(t),\ t \geq 0\}$,

$$\mathbb{E}(\tau_j(i)) = \sum_{k=j}^{i-1} \mathbb{E}(\tau_k(k+1)). \tag{3}$$

This leads us to the following proposition.

Proposition 3. *Let $\tau_j(i)$ be the time it takes for the dynamic Erdős-Rényi graph, starting with j edges, to reach i edges, where $j < i$. Then,*

$$\mathbb{E}(\tau_i(i+1)) = \frac{(n-1)(N-i-1)!i!}{\beta N!} \sum_{k=0}^{i} \binom{N}{i-k} \left(\frac{\alpha}{\beta}(n-1)\right)^k \tag{4}$$

and

$$\mathbb{E}(\tau_j(i)) = \sum_{m=j}^{i-1} \frac{(n-1)(N-m-1)!m!}{\beta N!} \sum_{k=0}^{m} \binom{N}{m-k} \left(\frac{\alpha}{\beta}(n-1)\right)^k. \tag{5}$$

Proof. Recall that $\{\eta(t),\ t \geq 0\}$ is an ergodic Markov chain on a finite state space, this ensures that the process has the strong Markov property, see [3, Thm. 4.1]. For notational convenience let $\lambda_k = (N-k)\beta/(n-1)$ be the birth rate and $\mu_k = \alpha k$ be the death rate in state k of $\{\eta(t),\ t \geq 0\}$.

We begin by deriving a recursive formula for $\mathbb{E}(\tau_i(i+1))$. Since $\{\eta(t),\ t \geq 0\}$ is ergodic and therefore positively recurrent we have that $\mathbb{E}(\tau_i(i+1)) < \infty$. We derive a recursive formula for $\mathbb{E}(\tau_i(i+1))$ by conditioning on the first jump. Let $p(i, i+1) = \frac{\lambda_i}{\lambda_i+\mu_i}$ be the probability that the process moves from i edges to $i+1$ edges, and let $i \to (i+1)$ indicate such an event. Define $p(i, i-1) = \frac{\mu_i}{\lambda_i+\mu_i}$ and $i \to (i-1)$ in an analogous way. Also let $H_i \sim \text{Exp}(\lambda_i + \mu_i)$ be the holding time in state i. Then,

$$\begin{aligned}
&\mathbb{E}(\tau_i(i+1)) \\
&= p(i, i+1)\mathbb{E}(\tau_i(i+1)|i \to (i+1)) + p(i, i-1)\mathbb{E}(\tau_i(i+1)|i \to (i-1)) \\
&= p(i, i+1)\mathbb{E}(H_i) + p(i, i-1)\mathbb{E}(\tau_i(i+1)|i \to (i-1)) \\
&\overset{(i)}{=} p(i, i+1)\mathbb{E}(H_i) + p(i, i-1)\mathbb{E}(H_i) + p(i, i-1)\mathbb{E}(\tau_{i-1}(i+1)) \\
&\overset{(ii)}{=} \mathbb{E}(H_i) + p(i, i-1)(\mathbb{E}(\tau_{i-1}(i)) + \mathbb{E}(\tau_i(i+1))).
\end{aligned}$$

For (i) we used that when entering state $i-1$ the process probabilistically restarts itself, this is by the strong Markov property as well as $\tau_i(i-1)$ being a stopping time for $\{\eta(t),\ t \geq 0\}$ starting in i. Equality (ii) follows since,

$$\tau_{i-1}(i+1) = \tau_{i-1}(i) + \tau_i'(i+1)$$

where $\tau_i'(i+1)$ is the time it takes for the process, starting with $i-1$ edges, to go from i edges (when it eventually reaches i edges) to $i+1$ edges. This is then, again by the strong Markov property, distributed as $\tau_i(i+1)$. Hence,

$$\mathbb{E}(\tau_{i-1}(i+1)) = \mathbb{E}(\tau_{i-1}(i)) + \mathbb{E}(\tau_i(i+1)).$$

Solving the above equation for $\mathbb{E}(\tau_i(i+1))$ we obtain,

$$\mathbb{E}(\tau_i(i+1)) = \frac{\mathbb{E}(H_i) + p(i,i-1)\mathbb{E}(\tau_{i-1}(i))}{p(i,i+1)}, \qquad i \in \{1,2,\ldots,N-1\}. \quad (6)$$

For $i=0$ we have $\mathbb{E}(\tau_0(1)) = \mathbb{E}(H_0)$.

To prove (4) we use (6) together with induction. The following holds for the birth-death process, $\{\eta(t),\ t \geq 0\}$.

$$\mathbb{E}(H_i) = \frac{1}{\lambda_i + \mu_i} = \frac{n-1}{(N-i)\beta + (n-1)i\alpha}$$
$$p(i,i-1) = \frac{\mu_i}{\lambda_i + \mu_i} = \frac{i(n-1)\alpha}{(N-i)\beta + (n-1)i\alpha}$$
$$p(i,i+1) = \frac{\lambda_i}{\lambda_i + \mu_i} = \frac{(N-i)\beta}{(N-i)\beta + (n-1)i\alpha}$$

Inserting this in (6), we obtain

$$\mathbb{E}(\tau_i(i+1)) = \frac{n-1}{(N-i)\beta} + \frac{(n-1)i}{N-i}\frac{\alpha}{\beta}\mathbb{E}(\tau_{i-1}(i)), \qquad i \in \{1,2,\ldots,N-1\}.$$

For $i=0$, we have by (4) that

$$\mathbb{E}(\tau_0(1)) = \frac{n-1}{\beta N},$$

which is indeed equal to $\mathbb{E}(H_0)$.

Assume that (4) holds for arbitrary $i < N-1$. Then,

$$\mathbb{E}(\tau_{i+1}(i+2)) = \frac{\mathbb{E}(H_{i+1}) + p(i+1,i)\mathbb{E}(\tau_i(i+1))}{p(i+1,i+2)}$$
$$= \frac{n-1}{(N-i-1)\beta} + \frac{(n-1)(i+1)}{N-i-1}\left(\frac{(n-1)(N-i-1)!i!}{\beta N!}\sum_{k=0}^{i}\binom{N}{i-k}\left((n-1)\frac{\alpha}{\beta}\right)^k\right)$$

which after standard, but tedious algebra, equals

$$\frac{(n-1)(N-i-2)!(i+1)!}{\beta N!}\sum_{k=0}^{i+1}\binom{N}{i+1-k}\left((n-1)\frac{\alpha}{\beta}\right)^k.$$

This proves equation (4), and (5) follows from (3).

□

The expression (5) is exact but not very insightful. In order to get a better understanding of how $\mathbb{E}(\tau_0(i))$ behaves, we study how this expectation grows when $i = [cn]$. To do this for the case $c < \frac{\beta}{2\alpha}$, consider $\bar{\eta}(t) = \frac{\eta(t)}{n}$ and note that $\bar{\eta}(t)$ increases by $\frac{1}{n}$ at rate $\beta\frac{n}{2} - \beta\frac{n}{n-1}\bar{\eta}(t)$ and decreases by $\frac{1}{n}$ at rate $\alpha n\bar{\eta}(t)$. This implies that a candidate for a deterministic approximation of $\bar{\eta}(t)$ satisfies

$$\frac{d\bar{\eta}(t)}{dt} = \frac{\beta}{2} - \frac{\beta\bar{\eta}(t)}{n-1} - \alpha\bar{\eta}(t) \qquad \text{and} \qquad \bar{\eta}(0) = 0.$$

As $n \to \infty$ this reads

$$\frac{d\bar{\eta}(t)}{dt} = \frac{\beta}{2} - \alpha\bar{\eta}(t) \qquad \text{and} \qquad \bar{\eta}(0) = 0.$$

This differential equation is solved by $\bar{\eta}(t) = \frac{\beta}{2\alpha}(1 - e^{-\alpha t})$.

We are now ready to formulate our next lemma.

Lemma 4. *Let $\tau_0(i)$ be the time it takes for the dynamic Erdős-Rényi graph, starting with 0 edges, to reach $i = [cn]$ edges, where $c < \frac{\beta}{2\alpha}$. Then,*

$$\tau_0(i) \to \frac{-\log\left(1 - \frac{2\alpha}{\beta}c\right)}{\alpha} \qquad \textit{in probability as } n \to \infty.$$

Proof. Let all limits be for $n \to \infty$. In this proof we use $\{\bar{\eta}^{(n)}(t), t \geq 0\}$ and $\{\eta^{(n)}(t), t \geq 0\}$ to denote the dependence on the number of vertices in the graph.

We prove that $\{\bar{\eta}^{(n)}(t), t \geq 0\}$ converges pointwise in distribution to the deterministic process $\{\frac{\beta}{2\alpha}(1 - e^{-\alpha t}), t \geq 0\}$ as $n \to \infty$. That is, for given t,

$$\bar{\eta}^{(n)}(t) \xrightarrow{d} \frac{\beta}{2\alpha}(1 - e^{-\alpha t})$$

in distribution. Note that $\eta^{(n)}(t)$ is binomially distributed with parameters N and $p^{(n)}(t) = \frac{\beta}{\beta + (n-1)\alpha}(1 - e^{-(\alpha + \frac{\beta}{n-1})t})$. So we obtain that the moment generating function (MGF)

$$M^{(n)}(\sigma) = \mathbb{E}[e^{\sigma\bar{\eta}^{(n)}(t)}] = \mathbb{E}[e^{\frac{\sigma}{n}\eta^{(n)}(t)}]$$

is given by

$$M^{(n)}(\sigma) = (1 - p^{(n)}(t)(1 - e^{\sigma/n}))^N.$$

Noting that $1 - e^{\sigma/n} = -\sigma/n + o(1/n)$ and $p^{(n)}(t) = \frac{\beta(1-e^{-\alpha t})}{\alpha(n-1)} + o(1/n)$, we obtain that

$$M^{(n)}(\sigma) = \left(1 + \frac{\sigma\beta(1-e^{-\alpha t})}{\alpha n(n-1)} + o(1/n^2)\right)^N$$
$$= \left(1 + \frac{\sigma\beta(1-e^{-\alpha t})}{2\alpha N} + o(1/n^2)\right)^N \to e^{\sigma\frac{\beta(1-e^{-\alpha t})}{2\alpha}}.$$

The latter expression is indeed the MGF of the constant $\frac{\beta}{2\alpha}(1-e^{-\alpha t})$ and since convergence of MGFs implies convergence in distribution, we obtain pointwise convergence in distribution of the stochastic process to the deterministic process.

In particular for $t_c = \frac{-\log\left(1-\frac{2\alpha}{\beta}c\right)}{\alpha}$, $\bar{\eta}^{(n)}(t_c) \xrightarrow{d} c$. This implies that if $t > t_c$, then $\mathbb{P}(\eta^{(n)}(t) \geq cn)$ is arbitrary close to 1 for large enough n. By $\eta^{(n)}(t) \leq \max_{s\leq t}\eta^{(n)}(s)$, this immediately implies that $\mathbb{P}(\max_{s\leq t}\eta^{(n)}(s) \geq cn)$ is arbitrary close to 1 for large enough n.

Similarly, for $t < t_c$, $\mathbb{P}(\eta^{(n)}(t) \geq cn)$ is arbitrary close to 0 for large enough n. Note that for $c < \frac{\beta}{2\alpha}$ and as long as $\eta(s) < \frac{\beta}{2\alpha}n$, the process $\{\eta^{(n)}(t), t \geq 0\}$ has a upwards drift. Hence if $s < t$ then $\mathbb{P}(\eta(t) \geq cn|\eta(s) \geq cn) \geq 1/2$. This implies that

$$\mathbb{P}(\eta^{(n)}(t) \geq cn|\max_{s\leq t}\eta^{(n)}(s) \geq cn) \geq \mathbb{P}(\eta(t) \geq cn|\eta(0) \geq cn) \geq 1/2,$$

which in turn implies that

$$\mathbb{P}(\max_{s\leq t}\eta^{(n)}(s) \geq cn) \leq 2\mathbb{P}(\eta^{(n)}(t) \geq cn),$$

which is arbitrary close to 0 for large enough n. This implies that for all $\epsilon > 0$, we have $\mathbb{P}(|\tau_0([cn]) - t_c| > \epsilon) \to 0$, which proves convergence in probability.

□

Using Lemma 4 we can prove Theorem 2 (a).

Proof of Theorem 2 (a). Let $i = [cn]$ where $c < \frac{\beta}{2\alpha}$.
By Lemma 4, $\tau_0(i) \xrightarrow{p} t_c = \frac{-\log\left(1-\frac{2\alpha}{\beta}c\right)}{\alpha}$ as $n \to \infty$. Hence, for given $\epsilon > 0$ and large enough n, $\mathbb{P}(\tau_0(i) > t_c + \epsilon) < \epsilon$ and $\mathbb{P}(\tau_0(i) < t_c - \epsilon) < \epsilon$. Combining this with the fact that $\mathbb{E}(\tau_j(i)) \leq \mathbb{E}(\tau_0(i))$ for all $j < i$ and using the strong Markov property of $\{\eta(t),\ t \geq 0\}$, we obtain that for given $\epsilon > 0$ there exist

n_0 such that for all $n > n_0$,

$$\begin{aligned} &\mathbb{E}(\tau_0(i)) \\ &= \mathbb{E}(\tau_0(i)|\tau_0(i) \le t_c+\epsilon)\mathbb{P}(\tau_0(i) \le t_c+\epsilon)+\mathbb{E}(\tau_0(i)|\tau_0(i) > t_c+\epsilon)\mathbb{P}(\tau_0(i) > t_c+\epsilon) \\ &\qquad \le (t_c+\epsilon)+\epsilon(t_c+\epsilon+\mathbb{E}(\tau_{\eta(t_c+\epsilon)}(i))) \le t_c+\epsilon+\epsilon(t_c+\epsilon+\mathbb{E}(\tau_0(i))) \end{aligned}$$

This implies that,

$$\mathbb{E}(\tau_0(i)) \le \frac{1+\epsilon}{1-\epsilon}(t_c+\epsilon).$$

Using that for all $\epsilon > 0$ and large enough n we have $\mathbb{P}(\tau_0(i) > t_c-\epsilon) > 1-\epsilon$ so we get,

$$\mathbb{E}(\tau_0(i)) \ge (t_c-\epsilon)(1-\epsilon).$$

Together this implies that $\mathbb{E}(\tau_0(i)) \to t_c$ as $n \to \infty$.

□

Remark. It follows with a little extra work from $\mathbb{E}[\tau_0(i)] < \infty$ and $\tau_0(i) \xrightarrow{p} t_c$ that $\tau_0(i) \to t_c$ in expectation (see e.g. [7, Problem 5.6.6]).

The hitting time results for $c \ge \frac{\beta}{2\alpha}$, derived below provide us with bounds that are not tight, but they establish logarithmic growth for $c = \frac{\beta}{2\alpha}$; and exponential growth for $c > \frac{\beta}{2\alpha}$.

Proof of Theorem 2 (b). Let T_s be the strong stationary time for the process $\{G(t),\ t \ge 0\}$ as defined in Theorem 1, and let $i = [\frac{\beta}{2\alpha}n]$. We know that $E(T_s) = O(\log(n))$ and since $\eta(T_s)$ is binomially distributed with parameters N and $\frac{\beta}{\beta+\alpha(n-1)}$, with $\mathbb{E}(\eta(T_s)) = i + O(1)$. We know by the central limit theorem that for large enough n, $\mathbb{P}(\eta(T_s) < i) \le 2/3$. Again observing that for all $0 \le j < i$, we have $\mathbb{E}(\tau_j(i)) \le \mathbb{E}(\tau_0(i))$. Conditioning on whether $\{\eta(T_s) \ge i\}$ (equivalent to $\{\tau_0(i) \le T_s\}$) or not gives,

$$\begin{aligned} &\mathbb{E}(\tau_0(i)) = \mathbb{E}(\tau_0(i)|\eta(T_s) \ge i)\mathbb{P}(\eta(T_s) \ge i) + \mathbb{E}(\tau_0(i)|\eta(T_s) < i)\mathbb{P}(\eta(T_s) < i) \\ &= \mathbb{E}(T_s|\eta(T_s) \ge i)\mathbb{P}(\eta(T_s) \ge i) + \mathbb{E}(T_s + \tau_{\eta(T_s)}(i)|\eta(T_s) < i)\mathbb{P}(\eta(T_s) < i) \\ &\le \mathbb{E}(T_s) + \frac{2}{3}(\mathbb{E}(\tau_{\eta(T_s)}(i)|\eta(T_s) < i)) \\ &\le \mathbb{E}(T_s) + \frac{2}{3}\mathbb{E}(\tau_0(i)). \end{aligned}$$

Which implies $\mathbb{E}(\tau_0(i)) \le 3\mathbb{E}(T_s) = O(\log(n))$, as well as $\mathbb{E}(\tau_j(i)) = O(\log(n))$, for $j < i = [\frac{\beta}{2\alpha}n]$.

An analogous argument shows that $\mathbb{E}(\tau_j(i)) = O(\log(n))$ for $j > [\frac{\beta}{2\alpha}n]$.

□

Still remaining is the case when $i = [cn]$ and $c > \frac{\beta}{2\alpha}$, which is dealt with below. Before that, we need a series of lemmas.

Lemma 5. *Let $\tau_0(i)$ be as above, where $i = [cn]$, $c > \frac{\beta}{2\alpha}$.*
Let $C_{i\to s\to i} = \inf\{t > 0;\ \eta(0) = i,\ \eta(t) = i,\ \eta(t) > \tau_i(s)\}$ be the time it takes for the dynamic graph to go from i edges to s edges back to i edges, where $s = [\frac{\beta}{2\alpha}n]$. Then,

$$E(\tau_0(i)) = \mathbb{E}(C_{i\to s\to i}) + O(\log(n))$$

Proof. By the strong Markov property we have that $\mathbb{E}(C_{i\to s\to i}) = \mathbb{E}(\tau_i(s)) + \mathbb{E}(\tau_s(i))$ and $\mathbb{E}(\tau_0(i)) = \mathbb{E}(\tau_0(s)) + \mathbb{E}(\tau_s(i))$. By Theorem 2 both $\mathbb{E}(\tau_i(s))$ and $\mathbb{E}(\tau_0(s))$ are of order $O(\log(n))$. We get,

$$\begin{aligned}\mathbb{E}(\tau_0(i)) &= \mathbb{E}(\tau_0(s)) + \mathbb{E}(\tau_s(i)) \\ &= \mathbb{E}(\tau_0(s)) + \mathbb{E}(C_{i\to s\to i}) - \mathbb{E}(\tau_i(s)) = \mathbb{E}(C_{i\to s\to i}) + O(\log(n)).\end{aligned}$$

□

Lemma 5 can be used to derive bounds for $\mathbb{E}(\tau_0(i))$ as $C_{i\to s\to i}$ is a cycle time for the dynamic graph, and hence results from renewal theory can be applied. Before doing so we shall need one more lemma.

Lemma 6. *Let $i = [cn]$, $c > \frac{\beta}{2\alpha}$ and let $p_n = \frac{\beta}{\beta+(n-1)\alpha}$ be the edge probability at stationarity. Then,*

$$N \cdot D\left(\frac{i}{N}||p_n\right) = n\left(c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha}\right) + O(1)$$

where $D(a||p) = a\log(\frac{a}{p}) + (1-a)\log(\frac{1-a}{1-p})$ is the relative entropy of a Bernoulli(a) random variable with respect to a Bernoulli(p) random variable.

Proof. We have that,

$$N \cdot D\left(\frac{i}{N}||p_n\right) = i\log\left(\frac{i/N}{p_n}\right) + (N-i)\log\left(\frac{1-i/N}{1-p_n}\right). \tag{7}$$

Since $\frac{i/N}{p_n} = \frac{c}{\beta/(2\alpha)} + O(\frac{1}{n})$ and $i = cn + O(1)$ we get that $\log\left(\frac{i/N}{p_n}\right) = \log\left(\frac{c}{\beta/(2\alpha)}\right) + O(\frac{1}{n})$. Which implies,

$$i\log\left(\frac{i/N}{p_n}\right) = cn\log\left(\frac{c}{\beta/(2\alpha)}\right) + O(1). \tag{8}$$

We also have, again by Taylor approximation,

$$\log(1 - i/N) = -\frac{i}{N} + O(n^{-2})$$
$$\log(1 - p_n) = -p_n + O(n^{-2}).$$

Together with $Np_n = \frac{\beta}{2\alpha}n + O(1)$ this implies,

$$(N - i)\log\left(\frac{1 - i/N}{1 - p_n}\right) = Np_n - i + O(1) = n\left(\frac{\beta}{2\alpha} - c\right) + O(1). \tag{9}$$

Inserting (8) and (9) in (7) shows that,

$$N \cdot D\left(\frac{i}{N}||p_n\right) = n\left(c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha}\right) + O(1).$$

□

Remark. For coming results note that, $c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha} > 0$ if $c > \frac{\beta}{2\alpha}$.

We can now prove Theorem 2 (c).

Proof of Theorem 2 (c). Observe that $\{\eta(t),\ t \geq 0\}$ is a regenerative process. Let $T_{\geq i} = \int_0^{C_{i\to s\to i}} 1\{\eta(t) \geq i\}dt$ be the time $\{\eta(t),\ t \geq 0\}$ spends above state i in a $(i \to s \to i)$ cycle, where $s = [\frac{\beta}{2\alpha}n]$. Since $C_{i\to s\to i}$ is a renewal time for $\{\eta(t),\ t \geq 0\}$ we have, by basic renewal theory,

$$\lim_{t\to\infty} \mathbb{P}(\eta(t) \geq i) = \frac{\mathbb{E}(T_{\geq i})}{\mathbb{E}(C_{i\to s\to i})}.$$

If T_s is a strong stationary time for the dynamic graph, then $\lim_{t\to\infty} \mathbb{P}(\eta(t) \geq i) = \mathbb{P}(\eta(T_s) \geq i)$. Hence,

$$\mathbb{E}(C_{i\to s\to i}) = \frac{\mathbb{E}(T_{\geq i})}{\mathbb{P}(\eta(T_s) \geq i)}$$

Since $\eta(T_s) \sim \mathrm{Bin}(N, p_n = \frac{\beta}{\beta+(n-1)\alpha})$ we can give upper and lower bounds for $\mathbb{P}(\eta(T_s) \geq i)$, see [1, p. 114].

$$(8i)^{-1/2}\exp\{-N \cdot D\left(\frac{i}{N}||p_n\right)\} \leq \mathbb{P}(\eta(T_s) \geq i) \leq \exp\{-N \cdot D\left(\frac{i}{N}||p_n\right)\}$$

where $D(a||p) = a\log(\frac{a}{p}) + (1 - a)\log(\frac{1-a}{1-p})$.

Next we show that $\mathbb{E}(T_{\geq i}) = \Theta(n^{-1})$. Let H_i be the holding time in state i. Then,

$$E(T_{\geq i}) \geq E(H_i) = \frac{n-1}{(N-i)\beta + i(n-1)\alpha} = Kn^{-1}, \qquad \text{for some } K \in \mathbb{R}.$$

By the strong Markov property of the process $\{\eta(t),\ t \geq 0\}$ we have that,

$$\mathbb{E}(T_{\geq i}) = p(i,i+1)(\mathbb{E}(H_i) + \mathbb{E}(\tau_{i+1}(i)) + E(T_{\geq i})) + p(i,i-1)(\mathbb{E}(H_i) + q_{i-1}\mathbb{E}(T_{\geq i}))$$

where $q_{i-1} = \mathbb{P}(\tau_{i-1}(i) < \tau_{i-1}(s))$ is the probability of reaching state i before state s, if $\{\eta(t),\ t \geq 0\}$ starts in state $i-1$; i.e. $q_{i-1} = \mathbb{P}(\tau_{i-1}(i) < \tau_{i-1}(s))$. We get,

$$\mathbb{E}(T_{\geq i}) = \frac{\mathbb{E}(H_i) + p(i,i+1)\mathbb{E}(\tau_{i+1}(i))}{p(i,i-1)(1-q_{i-1})}$$

Now, $\mathbb{E}(H_{i+k}) = \frac{n-1}{(N-i-k)\beta + (i+k)(n-1)\alpha} = \Theta(n^{-1})$, for fixed k and $i = [cn]$. Furthermore,

$$p(i+k, i+k+1) \to \frac{\beta}{\beta + 2c\alpha} \text{ as } n \to \infty$$
$$p(i+k, i+k-1) \to \frac{2c\alpha}{\beta + 2c\alpha} \text{ as } n \to \infty$$

again for fixed k and $i = [cn]$.

Secondly, since $\mathbb{E}(\tau_{i+2}(i+1)) < \mathbb{E}(\tau_{i+1}(i))$ we get,

$$\begin{aligned}\mathbb{E}(\tau_{i+1}(i)) &= \mathbb{E}(H_{i+1}) + p(i+1,i+2)(\mathbb{E}(\tau_{i+2}(i+1)) + \mathbb{E}(\tau_{i+1}(i))) \\ &\leq E(H_{i+1}) + 2p(i+1,i+2)\mathbb{E}(\tau_{i+1}(i))\end{aligned}$$

Since $2p(i+1,i+2) < 1$ we get,

$$\mathbb{E}(\tau_{i+1}(i)) \leq \frac{\mathbb{E}(H_{i+1})}{1 - 2p(i+1,i+2)} = O(n^{-1}).$$

It remains to show that $1 - q_{i-1} \not\to 0$ as $n \to \infty$ and it follows that $\mathbb{E}(T_{\geq i}) = \Theta(n^{-1})$. Note that $1 - q_{i-1}$ is the probability of reaching state s before state i when the process starts in state $i-1$. This is the same probability as first reaching state $j \in [s,i]$ before reaching i, and then going from j to s before going from j to i. By the strong Markov property we can write this as,

$$1 - q_{i-1} = \mathbb{P}(\tau_{i-1}(j) < \tau_{i-1}(i))\mathbb{P}(\tau_j(s) < \tau_j(i))$$

where j is any state between s and i. We take $j = [\frac{i+s}{2}]$ (round down if $\frac{i+s}{2} - j = 0.5$). The first probability $\mathbb{P}(\tau_{i-1}(j) < \tau_{i-1}(i))$ can be bounded below by a Gambler's ruin approach. We have that $p(k+1,k) \geq p(j+1,j) > \frac{1}{2}$, $\forall k \geq j$. So $\mathbb{P}(\tau_{i-1}(j) < \tau_{i-1}(i))$ is larger than the probability of non-ruin for a Gambler starting in state 1 trying to reach state $i-j-1$ with $p(\text{win}) = p(j+1,j) > \frac{1}{2}$. This non-ruin probability converges to a non-zero number as $n \to \infty$. By symmetry the second probability $\mathbb{P}(\tau_j(s) < \tau_j(i))$ is at least 1/2, since from j it is either (i) equal distance to s and i; or (ii) closer distance to s, and since $p(k+1,k) > \frac{1}{2}$, $\forall k \geq s$ the drift is downwards. We conclude that,

$$\mathbb{E}(T_{\geq i}) = \Theta(n^{-1}).$$

Combining all of the above we get,

$$\Theta(n^{-1}) \exp\left\{N \cdot D\left(\frac{i}{N}||p\right)\right\} \leq \mathbb{E}(C_{i\to s\to i}) \leq \Theta(n^{-1/2}) \exp\left\{N \cdot D\left(\frac{i}{N}||p\right)\right\}$$

By Lemma 5,

$$\mathbb{E}(C_{i\to s\to i}) - O(\log(n)) \leq \mathbb{E}(\tau_0(i)) \leq \mathbb{E}(C_{i\to s\to i}) + O(\log(n))$$

which together with Lemma 6 implies,

$$\Theta(n^{-1}) e^{n\left(c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha}\right)} \leq \mathbb{E}(\tau_0(i)) \leq \Theta(n^{-1/2}) e^{n\left(c\log(\frac{2\alpha}{\beta}c) - c + \frac{\beta}{2\alpha}\right)}.$$

□

3.1 The size of the largest component.

When studying any type of random graph the size of the largest component is often of interest. For instance, we might ask how long it will take for the size of the largest component in the dynamic graph to *exceed*, say, ϵn? We suggest that one way to approach this problem is through the edge process $\{\eta(t),\ t \geq 0\}$.

In a static Erdős-Rényi graph there is an intimate connection between the number of edges in the graph and the size of the largest component. Namely, if $G(n, M(n))$ is a static Erdős-Rényi graph with n vertices and $M(n)$ edges, where $M(n) = [cn]$, then as $n \to \infty$, the size of the largest component exhibits three different behaviors depending on c. The following holds with high probability: (i) if $c < 1/2$, called the subcritical case, then the size of the largest component if of order $\log(n)$; (ii) if $c > 1/2$, called the supercritical case, it is of order n; (iii) if $c = 1/2$, called the critical case, then the largest component is of order $n^{2/3}$, see [2, p.130]. A modification of a classical result from Erdős-Rényi [4] gives the following lemma for the supercritical case:

Corollary 3. *Let $\{\mathcal{G}(n, M(n))\}$ be a sequence of Erdős-Rényi graphs with n vertices and $M(n)$ edges. Let $|C(n, M(n))|$ be the size of the largest component of $\mathcal{G}(n, M(n))$.*
Then, for every $0 < \epsilon < 1$ there exist a $c > \frac{1}{2}$ such that if $M(n) = [cn]$ then,

$$\frac{|C(n, M(n))|}{n} \xrightarrow{p} \epsilon \text{ as } n \to \infty.$$

Furthermore,

$$c = \frac{-\log(1-\epsilon)}{2\epsilon}.$$

Hence, if $c = \frac{-\log(1-\epsilon)}{2\epsilon}$, then $|C(n, [cn])| \approx n\epsilon$ with high probability. However, $\frac{|C(n,[cn])|}{n} \xrightarrow{p} \epsilon$ does not imply $P(|C(n, [cn])| \geq n\epsilon) \to 1$. Hence, for the latter to hold we need to choose $M(n) = [c'n]$ where $c' > c$.

This in combination with Theorem 2 can be used to provide a (asymptotic) bound on the expected time it takes for the size of the largest component in the dynamic Erdős-Rényi graph to *exceed* ϵn, for given $\epsilon > 0$. In a static Erdős-Rényi we would need more than $[cn]$ edges, say $[c'n]$ edges, where $c' > c = \frac{-\log(1-\epsilon)}{2\epsilon}$ for this to be very likely—e.g. we can take $c' = \frac{-\log(1-(\epsilon+\eta))}{2(\epsilon+\eta)}$, $\eta \in (0, 1-\epsilon)$, so that the fraction of vertices in the largest component converges in probability to $\epsilon + \eta$. For the dynamic graph, we can instead wait until that many edges has appeared, and be very certain that the size of the largest component has exceeded ϵn no later than that time. This leads to the following lemma:

Lemma 7. *Let $\hat{\tau}(\epsilon n)$ be the first time the dynamic Erdős-Rényi graph, starting with no edges, has a component of size at least ϵn, $\epsilon \in (0, 1)$. Let $\tau_0(i)$ be the time it takes for the dynamic graph to reach, starting with no edges, i edges. Then for all $c' > c = \frac{-\log(1-\epsilon)}{2\epsilon}$,*

$$\mathbb{E}[\hat{\tau}(\epsilon n)] = O(\mathbb{E}[\tau_0([c'n])])$$

Proof. Let A be the event that at time $\tau_0([c'n])$ the largest component of the graph is at least size ϵn and let A^c be the complement of A. Observe that A is independent of $\tau_0([c'n])$, since the edge processes are all independent and therefore all edge configurations have the same probability at time $\tau_0([c'n])$.

By Corollary 3 $P(A) \to 1$ as $n \to \infty$. We note that,

$$\begin{aligned}\mathbb{E}[\hat{\tau}(\epsilon n)] &= \mathbb{E}[\hat{\tau}(\epsilon n)|A]\mathbb{P}(A) + \mathbb{E}[\hat{\tau}(\epsilon n)|A^c]\mathbb{P}(A^c) \\ &\leq \mathbb{E}[\tau_0([c'n])]\mathbb{P}(A) + (\mathbb{E}[\hat{\tau}(\epsilon n)] + \mathbb{E}[\tau_0([c'n])])\mathbb{P}(A^c) \\ &= \mathbb{E}[\tau_0([c'n])] + \mathbb{E}[\hat{\tau}(\epsilon n)]\mathbb{P}(A^c).\end{aligned}$$

Hence,

$$\mathbb{E}[\hat{\tau}(\epsilon n)] \leq \frac{\mathbb{E}[\tau_0(c'n)]}{1 - \mathbb{P}(A^c)}.$$

It follows that $\mathbb{E}[\hat{\tau}(\epsilon n)] = O(\mathbb{E}[\tau_0([c'n])])$. □

We can now prove Corollary 2.

Proof of Corollary 2. In this proof order terms are for $\hat{\epsilon} \to 0$.
Let $c' = \frac{-\log(1-\hat{\epsilon})}{2\hat{\epsilon}}$, $\hat{\epsilon} > \epsilon$, so that $c' > c$. Note, by Taylor approximation, $c' = \frac{-\log(1-\hat{\epsilon})}{2\hat{\epsilon}} = 1/2 + \hat{\epsilon}/4 + \hat{\epsilon}^2/6 + O(\hat{\epsilon}^3) > 1/2 = \frac{\beta}{2\alpha}$. Hence, by Lemma 7 and Theorem 2 (c),

$$\mathbb{E}[\hat{\tau}(\epsilon n)] = O(n^{-1/2})e^{n(c'\log(2c')+1/2-c')}.$$

Recall, $\log(1+x) = \sum_{k=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ if $|x| < 1$. Hence,

$$\begin{aligned}\log(2c') = \log(1 + (2c'-1)) &= (2c'-1) + (2c'-1)^2/2 + O(c'^3) \\ &= \hat{\epsilon}/2 + 5\hat{\epsilon}^2/24 + O(\hat{\epsilon}^3)\end{aligned}$$

if $|2c'-1| < 1$ (which can be solved numerically and implies that $\epsilon < 0.7968$). Hence,

$$c'\log(2c') - c' + 1/2 = \hat{\epsilon}^2/16 + O(\hat{\epsilon}^3)$$

and the corollary follows. □

4 Discussion

4.1 The distribution of hitting times

In this paper we have deduced results on the time to stationarity for a dynamic Erdős-Rényi graph and on the expected time needed for the graph to contain a required number of edges.

Although we have deduced an expression for $\mathbb{E}[\tau_0([cn])]$, we have not really touched upon the distribution of $\tau_0([cn])$ for $c > \frac{\beta}{2\alpha}$ yet. However, the following heuristic argument suggests that for $n \to \infty$, $\tau_0([cn])/\mathbb{E}[\tau_0([cn])]$ is expected to be exponential distributed with mean 1.

Let $\eta(0)$ be binomially distributed with parameters N and $p = \frac{\beta}{\beta+\alpha(n-1)}$, i.e. the graph process starts in stationarity. Let $T_0 = 0$ and T_{i+1} be the smallest time after T_i, such that in the interval $I_{i+1} = (T_i, T_{i+1}]$ all edges are updated in the sense of (iia) of the defining properties of the dynamic Erdős-Rényi graph. The lengths of those intervals are trivially *i.i.d.* Furthermore, let A_i be the event $\{\max_{t \in I_i} \eta(t) > [cn]\}$. Note that A_i is dependent on A_{i-1} and A_{i+1} but independent of all other A_j's and independent of the T_i's. Furthermore, the A_i's are identically distributed.

We proceed by assuming that we can ignore the dependencies altogether and thus assume that the A_i's are *i.i.d.* We justify this approximation by observing that, using the proof of Theorem 2 (c), $T_{\geq [cn]}$ is typically $O(1/n)$, while the length of intervals is typically $\theta(\log(n))$, which implies that typically, hitting times of $[cn]$ are not near the boundaries of the I_i's.

Let $J = \min\{i : A_i \text{ occurs}\}$ and $q = \mathbb{P}(A_1)$. Note that under our assumption J is geometrically distributed with parameter q. Furthermore, by the independence of J and the length of the intervals we obtain $\mathbb{E}(T_J) = \mathbb{E}(T_1)\mathbb{E}(J) = \mathbb{E}(T_1)/q$. Since the expected interval lengths are $O(\log(n))$, we have $\mathbb{E}(T_J) = \mathbb{E}[\tau_0([cn])] + O(\log(n))$ and $\mathbb{E}(T_1) = O(\log(n))$, which implies that q is exponentially small in n. By the strong law of large numbers and $J \to \infty$ almost surely, also $T_J/J \to \mathbb{E}(T_1)$ almost surely as $n \to \infty$. Now,

$$\frac{\tau_0([cn])}{\mathbb{E}[\tau_0([cn])]} = \frac{T_J + O(\log(n))}{\mathbb{E}[T_J] + O(\log(n))} = \frac{J \cdot T_J/J + O(\log(n))}{\mathbb{E}(T_1)\mathbb{E}(J) + O(\log(n))} \approx J/\mathbb{E}(J),$$

for large n. By standard results on geometric distributed random variables $J/\mathbb{E}(J)$ converges to an exponential random variable with parameter 1, for $q \to 0$.

4.2 Configuration of the edges in the largest component.

It may be interesting to know how the largest component reaches size ϵn. Is it that many edges have occurred, and by the sheer number of edges the largest component exceeds ϵn in size; or is it with fewer edges, but with a very unlikely configuration of the edges.

If the time until the largest component exceeds ϵn in size is known, or roughly known (we currently only have upper bounds on the expected time), one can compare this with the bounds on $\mathbb{E}(\tau_0([cn]))$ in Theorem 2. If these two times agree, then it is a strong indication that the largest component exceeds ϵn in size through many edges being present—and not through an unlikely configuration of few edges. Of course, if they disagree, by the previous section the time until the largest component exceeds ϵn in size will be smaller than $\mathbb{E}(\tau_0([cn]))$—and this indicates that the largest component reaches ϵn in size through an unlikely configuration of few edges.

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