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# The value of a liability cash flow in discrete time subject to capital requirements

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**Abstract**

The aim of this paper is to define the market-consistent value of a liability cash flow in discrete time subject to repeated capital requirements, and explore its properties. Our multi-period market-consistent valuation approach is based on defining a criterion for selecting a static replicating portfolio and defining the value of the residual liability, whose cash flow is the difference between the original liability cash flow and that of the replicating portfolio. The value of the residual cash flow is obtained as a solution to a backward recursion that is implied by the procedure for financing the repeated capital requirements, and no-arbitrage arguments. We show that the liability value resulting from no-arbitrage pricing of the dividends to capital providers may be expressed as a multi-period cost-of-capital valuation. Explicit valuation formulas are obtained under Gaussian model assumptions.

## 1 Introduction

The aim of this paper is to define the market-consistent value of a liability cash flow in discrete time subject to repeated capital requirements, and explore its properties. The liability should be interpreted as the aggregate liability of a company, i.e. at the level on which capital requirements are imposed. Our multi-period valuation approach is based on defining a criterion for selecting a static replicating portfolio and defining the value of the residual liability whose cash flow is the difference between the original liability cash flow and that of the replicating portfolio. For defining the value of the residual cash flow we do not impose a particular valuation functional. Instead we derive the value as a solution to a backward recursion that is implied by the procedure for financing the repeated capital requirements, and no-arbitrage arguments.

The approach to market-consistent liability valuation presented in [11] has been the main source of inspiration for the current paper. Similarly to what is advocated in [11], and as is explicitly stated in current insurance market regulation, we consider a hypothetical transfer of the liability to a so-called reference undertaking whose only purpose is to manage the runoff of the liability. The repeated capital requirements are financed by capital providers with limited liability. In [11], a valuation framework based on dynamic replication and cost-of-capital arguments was presented. In [7] a valuation framework, inspired by [11], based on dynamic monetary risk measures and dynamic monetary utility functions was presented and explicit valuation formulas were derived under Gaussian model assumptions. An essential difference between [11] and [7] is that initial static replication, instead of dynamic replication, of the liability cash flow is considered in [7]. The static replicating portfolio is transferred to the reference undertaking together with the liability. Static replication is a reasonable assumption since sophisticated dynamic hedging may be unrealistic for an entity only designed to manage a liability in runoff.

In [7], the static replicating portfolio was assumed to be given and the analysis only focused on the multi-period valuation of the residual liability cash flow. Criteria for selection of a replicating portfolio were not analyzed. A large part of the current paper focuses on presenting properties of a particular criterion for selection of the replicating portfolio that forms the basis for defining the value of the liability. Moreover, in the current paper the value of the residual liability is implied by no-arbitrage pricing of a derivative security with optionality written on the cumulative cash flow to capital provider. We demonstrate that there is a correspondence between the choice of pricing measure used for pricing the derivative security and an adapted process of cost-of-capital rates that defines the capital providers' acceptability criteria for providing solvency capital throughout the runoff of the liability.

Replicating portfolio theory for capital requirement calculation has attracted much interest in recent years. There, the value of a liability cash flow at a future time is modeled as a conditional expected value with respect to the market's pricing measure of the sum of discounted future liability cash flows. Since computation of this liability value is typically not feasible, one seeks an accurate approximation by replacing the liability cash flow (or its value) by that of a portfolio of traded replication instruments. Then, a risk measure is applied to the approximation of the liability value yielding an approximation of the capital requirement. In [2], [12], [13] and [14] various aspects of this replicating portfolio approach to capital calculations are clarified. A fact that somewhat complicates the analysis is that risk measures defining capital requirements are defined with respect to the real-world probability measure $\mathbb{P}$, whereas the replication criteria are usually expressed in terms of the market's pricing measure $\mathbb{Q}$. Comparisons of properties and effects of different replication criteria are presented in [12], [13] and [14]. In [2], it is shown how replicating portfolio theory can be formulated in order to allow for efficient replication of liability values exhibiting path-dependence. Common to the works [2], [12], [13] and [14] is that the liability value is defined as a conditional expected value of the sum of discounted liability cash flows. This is very different from the approach presented here. As explained above, we do not define the value of the liability from to outset. Rather we consider the dividends to the capital provider that finances the capital requirements of the residual liability cash flow that remains after imperfect initial replication.

Dynamic risk measures and dynamic risk-adjusted values have been analyzed in great detail during the last decade, see e.g. [1], [3], [4], [5] and the references therein for important contributions. Much of the research in this area has been aimed at establishing properties and representation results for dynamic risk measures in general functional analytic settings, particularly for bounded stochastic processes and under convexity requirements for the risk measures. We want to allow for models for unbounded liability cash flows. Moreover, limited liability for the capital providers in our setting implies that the dynamic valuation mappings appearing here will not be concave even when the conditional risk measures are convex. We will only assume very basic properties of the conditional risk measures defining capital requirements, namely, so-called translation invariance, monotonicity and normalization. In particular, we want to allow for conditional versions of a risk measure such as Value-at-Risk that is extensively used in practice.

Another approach to market-consistent liability valuation is presented in [15], combining no-arbitrage valuation and actuarial valuation into a general framework. Both the current paper and [15] advocates a two-step valuation. However, this has a quite different meaning in [15] compared to the approach presented here. In [15], an actuarial pricing principle is used to price the residual risk. However, the residual risk in [15] does not correspond to our residual liability cash flow. Moreover, as described above, we do not price the residual liability cash flow by a given pricing operator; in our setting the value of the residual liability cash flow is implied by no-arbitrage valuation of the cumulative dividends to capital providers. See also [9], [10] and the references therein for other approaches the market-consistent valuation of insurance liability cash flows.

The paper is organized as follows: The liability valuation framework is presented in Section 2 which is divided into three subsections. Section 2.1 presents the procedure for financing the repeated capital requirements, imposed on the reference undertaking, by capital injections from capital providers. In particular, it is shown that no-arbitrage pricing of the derivative security written on the cumulative cash flows to the capital providers leads to a backward recursion for the value of the residual liability cash flow. Section 2.2 presents the mathematical framework for valuation of the residual liability cash flow when capital requirements are expressed in terms of a dynamic monetary risk measure. Section 2.3 focuses on criteria for selecting the static replicating portfolio, proposes a particular criterion and explores its properties. Based on this criterion and the framework presented in Section 2.2 the value of the liability is defined. Explicit valuation formulas are obtained in Section 3 under Gaussian model assumptions. The proofs are found in Section 4.

## 2 The valuation framework

We consider time periods $1, \dots, T$, corresponding time points $0, 1, \dots, T$, and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ with $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_T = \mathcal{F}$, and $\mathbb{P}$ denotes the real-world measure. We write $L^p(\mathcal{F}_t, \mathbb{P})$ for the normed linear space of $\mathcal{F}_t$-measurable random variables $X$ with norm $\mathbb{E}^{\mathbb{P}}[|X|^p]^{1/p}$. Equalities and inequalities between random variables should be interpreted in the $\mathbb{P}$-almost sure sense. We assume a given numéraire process $(N_t)_{t=0}^T$ and that all financial values are discounted by this numéraire. Although the choice of numéraire is irrelevant for the analysis, we take the numéraire to be the bank account numéraire: $N_0 = 1$ and $N_t$ is the amount at time $t$ from rolling forward an initial unit investment in one-period risk-free bonds. A value $N_t Y$ at time $t$ has discounted value $Y$ at time $t$.

We assume that there exist a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale $(D_t)_{t=0}^T$ with $\mathbb{E}^{\mathbb{P}}[D_T] = 1$ defining the pricing measure $\mathbb{Q}$ of an arbitrage-free financial market via $D_t = d\mathbb{Q}/d\mathbb{P} \mid \mathcal{F}_t$, i.e. for $u > t$ and an $\mathcal{F}_u$-measurable $Z$,

$$\mathbb{E}_t^{\mathbb{Q}}[Z] = \frac{1}{D_t}\mathbb{E}_t^{\mathbb{P}}[D_u Z],$$

where subscript $t$ means conditioning on $\mathcal{F}_t$. Given the incomplete-market setting, the pricing measure $\mathbb{Q}$ is not uniquely determined by no-arbitrage arguments. The pricing measure $\mathbb{Q}$ is neutral to financial trading risk. Moreover, $\mathbb{Q}$ should be chosen so values assigned to non-hedgeable (insurance) cash flows reflect risk averseness of market participants towards such risks. Demands from market participants of compensation for providing capital for financing capital requirements for non-hedgeable risks may be viewed as risk averseness.

### 2.1 The liability derivative instrument

A discounted liability cash flow corresponds to an $\mathbb{F}$-adapted stochastic process $X^o = (X_t^o)_{t=1}^T$ interpreted as a discounted cash flow from an aggregate insurance liability in runoff. Our aim is to give a precise meaning to the market-consistent value of the liability and provide results that allow this value to be computed.

As is done in e.g. [11] and prescribed by EIOPA, see [6, Article 38], we take the point of view that an aggregate liability cash flow should be valued by considering a hypothetical transfer of the liability and its associated replicating portfolio to a separate entity, a so-called reference undertaking, whose assets have the purpose of matching the value or cash flow of the liability as well as possible.

We will give a meaning to the liability value by a particular valuation procedure. At time 0, a replicating portfolio is purchased with the aim replicating the liability value or its cash flow at all times. Let $X^r = (X^r_t)_{t=1}^T$ denote the discounted cash flow of the replicating portfolio and note that, from standard assumptions of no arbitrage, its initial price is $\sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X^r_t]$. The cash flow $X^r$ is allowed to be any discounted cash flow with finite expectation that can be generated from a given set of replication instruments by a trading strategy fixed at time 0 that rules out borrowing and capital injections.

The valuation of the liability cash flow is based on a simple observation. Externally imposed capital requirements imply that the reference undertaking needs capital injections throughout the liability runoff. A capital provider is the owner of the reference undertaking for as long as the necessary capital injections are provided. The capital provider may at any time choose to stop providing capital and, in this case, has no further obligations towards the reference undertaking. From the capital provider's perspective, ownership of the reference undertaking is equivalent to holding a derivative security with optionality, described in detail below, written on the residual liability cash flow $X^o - X^r$.

We will define the value at time 0 of the liability cash flow $X^o$ as

$$\sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X^r_t] + V_0(X^o - X^r) = \mathbb{E}_0^{\mathbb{Q}}\Big[\sum_{t=1}^T X^r_t + V_0(X^o - X^r)\Big],$$

where $X^r$ is the discounted cash flow of a particular replicating portfolio and $V_0 := V_0(X^o - X^r)$ is the size of a position in the numéraire asset. $V_0$ will be determined from $X^o$ and $X^r$ by solving a non-linear backward recursion that appears as the consequence of capital requirements and the procedure for handling the residual liability cash flow $X := X^o - X^r$ from imperfect replication.

The $\mathbb{F}$-adapted sequence $(V_t)_{t=0}^T$, whose terms are the discounted values of the residual liability cash flows $(X_s)_{s=t+1}^T$, $t = 0, \dots, T-1$, will be determined from no-arbitrage pricing of a particular derivative security written on the discounted residual cash flow $X$.

- At time 0, the liability, replicating portfolio and a position $V_0$ in the numéraire asset are transferred to a reference undertaking. From this point, it is sufficient to only consider the residual liability with discounted cash flow $X$ and the position in the numéraire asset. We will define the derivative instrument which is intimately connected with a trading strategy in the numéraire asset. The holder of the derivative may exercise the right to stop at any time $t \in \{1, \dots, T\}$.

- At time $t$, provided that the holder of the derivative has not exercised the right to stop, denote the discounted available capital, all invested in the numéraire asset, by $V_t$ and the discounted solvency capital requirement by $R_t$, required for the residual liability cash flow in runoff. The holder of the derivative is required to offset the difference by paying an amount with discounted value $R_t - V_t$. The position $R_t$ in the numéraire asset is held until time $t+1$.

- At time $t+1$ the discounted value of the payoff to the holder of the derivative is $(R_t - X_{t+1} - V_{t+1})_+ := \max(R_t - X_{t+1} - V_{t+1}, 0)$ upon stopping, and $R_t - X_{t+1} - V_{t+1}$ upon not stopping.

- The random sequence $(V_t)_{t=0}^T$ is determined from the requirement that the market price of the derivative is zero at all times. Let $H_{t,t'}$, $t' \geq t+1$, be the discounted gain for the holder of the derivative from time $t$ to time $t'$ upon stopping at time $t'$. Notice that

$$\begin{aligned} H_{t,t+1} &= -(R_t - V_t) + (R_t - X_{t+1} - V_{t+1})_+, \\ H_{t,t'} &= -(R_t - V_t) + (R_t - X_{t+1} - V_{t+1}) + H_{t+1,t'}, \quad t' > t+1. \end{aligned}$$

The holder of the derivative is the owner of the reference undertaking. An essential feature is that the owner of the reference undertaking neither pays nor is paid anything for the ownership, i.e. for the transfer of the liability, the replicating portfolio and the position in the numéraire asset. Moreover, the position $V_t$ in the numéraire asset at any time $t$ is such that the value of continued ownership is zero. This requirement, and its consequences, are given in the following result. Here ess sup refers to the essential supremum with respect to $\mathbb{P}$, see Appendix A.4 in [8]. Details on arbitrage-free pricing of American contingent claims can be found in Section 6.3 in [8].

**Theorem 1.** *For $t \in \{0, \dots, T-1\}$, let $\mathcal{S}_{t+1,T}$ be the set of stopping times in $\{t+1, \dots, T\}$, and set*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau}] := 0.$$

*Then, for $t \in \{0, \dots, T-1\}$,*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau}] = \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau_{t+1}^*}],$$

*where $\tau_{t+1}^* := \min\{s \in \{t+1, \dots, T\} : R_{s-1} - X_s - V_s < 0\}$, and*

$$V_t = R_t - \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+], \quad V_T := 0. \tag{1}$$

**Remark 1.** *Notice that*

$$\begin{aligned} V_t &= R_t - \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+] \\ &= R_t - \frac{1}{1+\eta_t}\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+], \end{aligned}$$

*where*

$$\frac{1}{1+\eta_t} := \frac{\mathbb{E}_t^{\mathbb{P}}[D_{t+1}(R_t - X_{t+1} - V_{t+1})_+]}{D_t\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+]}$$

*In order for an interpretation of $\eta_t$ we need more structure, such as the so-called Basic Actuarial Model in Section 6.2 in [16]. There, $\mathbb{F}$ is defined in terms of a financial filtration $\mathbb{A} = (\mathcal{A}_t)_{t=0}^T$ and insurance technical filtration $\mathbb{T} = (\mathcal{T}_t)_{t=0}^T$ such that $\mathcal{F}_t$ is generated by $\mathcal{A}_t$ and $\mathcal{T}_t$ for all $t$, and $\mathbb{A}$ and $\mathbb{T}$ are independent with respect to $\mathbb{P}$. Moreover, modified to our setting, $D_t := D_t^A D_t^T$, where $(D_t^A)_{t=0}^T$ is $\mathbb{A}$-adapted and a $(\mathbb{P},\mathbb{A})$-martingale with $D_0^A = 1$, and $(D_t^T)_{t=0}^T$ is $\mathbb{T}$-adapted and a $(\mathbb{P},\mathbb{T})$-martingale with $D_0^T = 1$. Under these assumptions and if $X$ is independent of $\mathbb{A}$,*

$$\frac{1}{1+\eta_t} = \frac{\mathbb{E}_t^{\mathbb{P}}[(D_{t+1}^T/D_t^T)(R_t - X_{t+1} - V_{t+1})_+]}{\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})_+]}.$$

*Notice that multiplication by the factor $D_{t+1}^T/D_t^T$ represents a probability distortion, or change of measure, which, for $\eta_t > 0$, should be interpreted as a required increase in the expected return on capital above the risk-free return to compensate for the risk aversion of the capital provider.*

## 2.2 The value of the residual liability cash flow

In order to ensure that the value of the liability, to be defined, is a sensible object we require three basic properties of the risk measures quantifying solvency capital requirements.

By conditional and dynamic monetary risk measures quantifying one-period capital requirements we mean the following:

**Definition 1.** *For $p \in [0,\infty]$ and $t \in \{0,\dots,T-1\}$, a conditional monetary risk measure is a mapping $R_t : L^p(\mathcal{F}_{t+1},\mathbb{P}) \to L^p(\mathcal{F}_t,\mathbb{P})$ satisfying*

$$\text{if } \lambda \in L^p(\mathcal{F}_t,\mathbb{P}) \text{ and } Y \in L^p(\mathcal{F}_{t+1},\mathbb{P}), \text{ then } R_t(Y+\lambda) = R_t(Y) - \lambda, \tag{2}$$

$$\text{if } Y, \widetilde{Y} \in L^p(\mathcal{F}_{t+1},\mathbb{P}) \text{ and } Y \le \widetilde{Y}, \text{ then } R_t(Y) \ge R_t(\widetilde{Y}), \tag{3}$$

$$R_t(0) = 0. \tag{4}$$

*A sequence $(R_t)_{t=0}^{T-1}$ of conditional monetary risk measures is called a dynamic monetary risk measure.*

For $t \geq 0$, $x \in \mathbb{R}$, $u \in (0,1)$ and an $\mathcal{F}_{t+1}$-measurable $Z$, let

$$\begin{aligned} F_{t,-Z}(x) &:= \mathbb{P}(-Z \leq x \mid \mathcal{F}_t), \\ F^{-1}_{t,-Z}(1-u) &:= \min\{m \in \mathbb{R} : F_{t,-Z}(m) \geq 1-u\}, \end{aligned}$$

and define conditional versions of Value-at-Risk and Expected Shortfall as

$$\begin{aligned} \mathrm{VaR}_{t,u}(Z) &:= F^{-1}_{t,-Z}(1-u), \\ \mathrm{ES}_{t,u}(Z) &:= \frac{1}{u}\int_0^u \mathrm{VaR}_{t,v}(Z)dv. \end{aligned}$$

$\mathrm{VaR}_{t,u}$ and $\mathrm{ES}_{t,u}$ are spacial cases of the following more general type of conditional monetary risk measure.

**Definition 2.** *Let $t \in \{0, \dots, T-1\}$ and let $M$ be a probability distribution on the Borel subsets of $(0,1)$ such that either $M$ has a bounded density with respect to the Lebesgue measure or the support of $M$ is bounded away from $0$ and $1$. Define*

$$R_t(Z) := \int_0^1 F^{-1}_{t,-Z}(u)dM(u).$$

**Theorem 2.** *For $p \in [1,\infty]$, $R_t$ in Definition 2 is a conditional risk measures in the sense of Definition 1. In particular, for $p \in [1,\infty]$, $\mathrm{VaR}_{t,u}$ and $\mathrm{ES}_{t,u}$ are conditional monetary risk measures in the sense of Definition 1.*

The statement of Theorem 2 follows from combining Proposition 4 (i) and Remark 5 in [7]; the proof is therefore omitted. Notice that $\mathrm{VaR}_{t,v}$ is obtained by choosing $M$ such that $M(\{1-v\}) = 1$, and $\mathrm{ES}_{t,v}$ is obtained by choosing $M$ with density $u \mapsto v^{-1}1_{(1-v,1)}(u)$.

From (1) follows that $V_t$ is determined recursively from $X_{t+1}$ and $V_{t+1}$ as follows:

$$V_t := W_t(X_{t+1} + V_{t+1}), \quad V_T := 0, \tag{5}$$

$$W_t(Y) := R_t(-Y) - \mathbb{E}^{\mathbb{Q}}_t\big[\big(R_t(-Y) - Y\big)_+\big]. \tag{6}$$

It remains to define the mappings $W_t$ properly. This can be done in various ways and we will focus on the ones that fit our purposes. From the fact that

$$\mathbb{E}^{\mathbb{Q}}_t\big[\big(R_t(-Y) - Y\big)_+\big] = \frac{1}{D_t}\mathbb{E}^{\mathbb{P}}_t\big[D_{t+1}\big(R_t(-Y) - Y\big)_+\big]$$

applications of Hölder's and Minkowski's inequalities allow us to define $W_t$.

**Theorem 3.** *(i) Fix $t \in \{0, \dots, T-1\}$ and $p \in [1,\infty]$. Suppose $D_{t+1}/D_t \in L^\infty(\mathcal{F}_{t+1}, \mathbb{P})$ and that $R_t$ is a conditional monetary risk measure according* *to Definition 1. Then $W_t$ in (6) is a mapping from $L^p(\mathcal{F}_{t+1}, \mathbb{P})$ to $L^p(\mathcal{F}_t, \mathbb{P})$ having the properties*

$$\text{if } \lambda \in L^p(\mathcal{F}_t, \mathbb{P}) \text{ and } Y \in L^p(\mathcal{F}_{t+1}, \mathbb{P}), \text{ then } W_t(Y + \lambda) = W_t(Y) + \lambda, \quad (7)$$

$$\text{if } Y, \widetilde{Y} \in L^p(\mathcal{F}_{t+1}, \mathbb{P}) \text{ and } Y \leq \widetilde{Y}, \text{ then } W_t(Y) \leq W_t(\widetilde{Y}), \quad (8)$$

$$W_t(0) = 0. \quad (9)$$

*(ii) Fix $t \in \{0, \dots, T-1\}$ and $1 \leq p_1 < p_2$. Suppose $D_{t+1}/D_t \in L^r(\mathcal{F}_{t+1}, \mathbb{P})$ for every $r \geq 1$. Suppose further that for any $p \in [p_1, p_2]$, $R_t$ is a conditional monetary risk measure according to Definition 1. Then, for any $\epsilon > 0$ such that $p - \epsilon \geq p_1$, $W_t$ in (6) can be defined as a mapping from $L^p(\mathcal{F}_{t+1}, \mathbb{P})$ to $L^{p-\epsilon}(\mathcal{F}_t, \mathbb{P})$ having the properties (7)-(9).*

The requirement $D_{t+1}/D_t \in L^\infty(\mathcal{F}_{t+1}, \mathbb{P})$ in statement (i) of Theorem 3 leads to a cleaner definition of the mapping $W_t$. However, the boundedness of $D_{t+1}/D_t$ may be a too restrictive requirement. Finiteness of all moments of $D_{t+1}/D_t$, as in statement (ii), will be an appropriate requirement for the subsequent analysis here.

Under the assumptions of Theorem 3 (i) or (ii), it follows from (5) that

$$V_t = W_t \circ \cdots \circ W_{T-1}(X_{t+1} + \cdots + X_T), \quad (10)$$

where $W_t \circ \cdots \circ W_{T-1}$ denotes the composition of mappings $W_t, \dots, W_{T-1}$, and that $V_t \in L^p(\mathcal{F}_t, \mathbb{P})$ in case of Theorem 3 (i) applies or, for any $\epsilon > 0$ such that $p - \epsilon > 0$, $V_t \in L^{p-\epsilon}(\mathcal{F}_t, \mathbb{P})$ in case Theorem 3 (ii) applies.

In statements involving $V_t$ we will in what follows, in order to avoid irrelevant lengthy technical statements, assume that suitable conditions are satisfied ensuring that $V_t = V_t(X)$ is well-defined from (10) as a mapping from $L^p((\mathcal{F}_t)_{t=1}^T, \mathbb{P})$ to $L^{p-\epsilon}(\mathcal{F}_t, \mathbb{P})$ for relevant values of $p$ and $\epsilon$ such that (7)-(9) hold.

The following result is an immediate consequence of the representation (10) combined with Theorem 3. The proof is therefore omitted.

**Theorem 4.** *(i) Let $b = (b_s)_{s=1}^T$ with $b_s \in L^p(\mathcal{F}_t, \mathbb{P})$ for each $s$, and let $X_s \leq \widetilde{X}_s$ for each $s$. Then, for every $t < T$,*

$$V_t(0) = 0, \quad V_t(X + b) = V_t(X) + \sum_{s=t+1}^{T} b_s, \quad V_t(X) \leq V_t(\widetilde{X}).$$

*(ii) For every pair of times $(s, t)$ with $s \leq t$, the two conditions $(X_u)_{u=1}^t = (\widetilde{X}_u)_{u=1}^t$ and $V_t(X) \leq V_t(\widetilde{X})$ together imply $V_s(X) \leq V_s(\widetilde{X})$.*

### 2.3 The value of the liability cash flow

In order to define the value of the liability we need to specify the available replication instruments and their cash flows. Consider $m$ discounted cash flows $X^{f,k} = (X_t^{f,k})_{t=1}^T$, $k = 1, \dots, m$ of available financial instruments and denote by $X^f$ the $\mathbb{R}^m$-valued process such that $X_t^f$ denotes the (column) vector of time-$t$ discounted cash flows of the $m$ instruments. A portfolio with portfolio-weight vector $v \in \mathbb{R}^m$, representing the number of units of the $m$ instruments, generates the discounted cash flow $v^{\mathrm{T}} X_t^f$ at time $t$.

Various criteria for selection of replicating portfolio have been considered in the literature. The optimization problem

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}} \Big[ (X_t^o - v^{\mathrm{T}} X_t^f)^2 \Big]^{1/2} \tag{11}$$

is referred to as cash flow matching in [14]. Under mild conditions, it is shown in Theorems 1 (and 2) in [14] that an optimal (unique optimal) solution exists. An alternative cash-flow-matching problem is

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}} \Big[ (X_t^o - v^{\mathrm{T}} X_t^f)^2 \Big]. \tag{12}$$

Comparisons between (11) and (12) are found in [12]. The optimization problem

$$\inf_{v \in \mathbb{R}^m} \mathbb{E}_0^{\mathbb{Q}} \Big[ \Big( \sum_{t=1}^T (X_t^o - v^{\mathrm{T}} X_t^f) \Big)^2 \Big]^{1/2} \tag{13}$$

is referred to as terminal-value matching in [12], [13] and [14]. It is a standard quadratic optimization problem with explicit solution

$$\widehat{v} = \mathbb{E}_0^{\mathbb{Q}} \left[ \begin{pmatrix} X_\cdot^{f,1} X_\cdot^{f,1} & \dots & X_\cdot^{f,1} X_\cdot^{f,m} \\ \vdots & & \vdots \\ X_\cdot^{f,m}, X_\cdot^{f,1} & \dots & X_\cdot^{f,m} X_\cdot^{f,m} \end{pmatrix} \right]^{-1} \mathbb{E}_0^{\mathbb{Q}} \left[ \begin{pmatrix} X_\cdot^o X_\cdot^{f,1} \\ \vdots \\ X_\cdot^o X_\cdot^{f,m} \end{pmatrix} \right]$$

provided that the matrix inverse exists, where the subscript $\cdot$ means summation over the index $t$.

A replicating portfolio selection criterium should have the property that if perfect replication is possible, then the discounted optimal replicating portfolio cash flow $\widehat{v}^{\mathrm{T}} X^f$ satisfies $X^o = \widehat{v}^{\mathrm{T}} X^f$. This requirement ensures market-consistent liability values: $L_0 = \sum_{t=1}^T \mathbb{E}_0^{\mathbb{Q}}[X_t^o]$ for a replicable liability cash flow.

**Remark 2.** *The versions of the optimization problems* (11), (12) *and* (13) *obtained by replacing the expectation* $\mathbb{E}_0^{\mathbb{Q}}$ *by* $\mathbb{E}_0^{\mathbb{P}}$ *may also be reasonable. Notice that if the only available replication instruments are zero-coupon bonds in the numéraire asset of all maturities* $t = 1, \dots, T$ *(or, equivalently, European call options on the numéraire asset with maturities* $t = 1, \dots, T$ *and* *common strike price* 0*), then* $m = T$ *and* $X^f$ *is the* $T \times T$ *identity matrix. In this case,*

$$\inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}\Big[(X_t^o - v^{\mathrm{T}} X_t^f)^2\Big] = \inf_{v \in \mathbb{R}^m} \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}\Big[(X_t^o - v_t)^2\Big],$$

*and the unique optimal solution is* $\widehat{v} = \mathbb{E}_0^{\mathbb{P}}[X^o]$ *which is referred to as the actuarial best-estimate reserve.*

*Notice also that, given the above restricted set of replication instruments, any* $\widehat{v}$ *satisfying* $\sum_{t=1}^T \widehat{v}_t = \sum_{t=1}^T \mathbb{E}_0^{\mathbb{P}}[X_t^o]$ *is an optimal solution to the version of the terminal value problem* (13) *obtained by replacing the expectation* $\mathbb{E}_0^{\mathbb{Q}}$ *by* $\mathbb{E}_0^{\mathbb{P}}$*.*

In our setting, the capital provider provides the capital

$$C_t := R_t(-X_{t+1} - V_{t+1}) - V_t = \mathbb{E}_t^{\mathbb{Q}}[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+]$$

at time $t$ ensuring that the solvency capital requirement is met. Good initial replication should make the need for external capital small. Therefore, it is reasonable to select a replicating portfolio that minimizes the need for external funding of the liability runoff. We may consider the optimization problem

$$\inf_{v \in \mathbb{R}^m} \psi(v), \quad \psi(v) := \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{Q}}[C_t^v], \tag{14}$$

where $X^v := X^o - v^{\mathrm{T}} X^f$, $V_{t+1}^v := V_{t+1}(X^v)$, $R_t^v := R_t(-X_{t+1}^v - V_{t+1}^v)$, and

$$C_t^v := \mathbb{E}_t^{\mathbb{Q}}\Big[(R_t^v - X_{t+1}^v - V_{t+1}^v)_+\Big].$$

Notice that, due to the translation invariance properties (2) and (7), the objective function in the optimization problem (14) is invariant under translations of $X^v$ by constant vectors. Consequently, (14) will not have a unique optimal solution if risk-free cash flows in the numéraire asset are included as replication instruments.

The optimization problems (11)-(14) can all be expressed as

$$\inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f)$$

for a mapping $\Psi : L^p((\mathcal{F}_t)_{t=1}^T, \mathbb{Q}) \to \mathbb{R}_+$ satisfying $\Psi(0) = 0$, i.e. optimality of perfect replication.

**Remark 3.** *Notice that* (13) *and* (14) *both have the property that* $\Psi(X) = 0$ *if* $\sum_{t=1}^T X_t = 0$*. For* (13) *this is obvious. To see that this property holds for*

(14), *assume that* $\sum_{t=1}^{T} X_t = 0$ *and notice that, for all* $t \in \{1, \ldots, T\}$*, due to* (7) *and* (9),

$$\begin{aligned}\sum_{s=1}^{t} X_s + V_t &= \sum_{s=1}^{t} X_s + W_t \circ \cdots \circ W_{T-1}(X_{t+1} + ... + X_T) \\ &= W_t \circ \cdots \circ W_{T-1}(X_1 + ... + X_T) \\ &= 0.\end{aligned}$$

*Hence, for all* $t \in \{1, \ldots, T\}$, $X_t + V_t = -\sum_{s=1}^{t-1} X_s$ *is* $\mathcal{F}_{t-1}$*-measurable. This in turn, using* (2), *implies that*

$$C_t := \mathbb{E}_t^{\mathbb{Q}}[(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})_+] = 0$$

*for all* $t \in \{1, \ldots, T\}$.

Existence of a minimizer $\widehat{X}^r := \widehat{v}^{\mathrm{T}} X^f$ can be expressed as

$$\Psi(X^o - \widehat{X}^r) = \inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f).$$

Conditions for existence of a minimizer $\widehat{v}$ in (14) are presented in Theorem 7 below.

Now we define the value of the liability as the market price of a particular portfolio of financial instruments: the $\Psi$-optimal replicating portfolio and a position $V_0$ in the numéraire asset.

**Definition 3.** $\widehat{X}^r := \widehat{v}^{\mathrm{T}} X^f$ *is said to be an optimal discounted replicating portfolio cash flow with respect to the criterion* $\Psi$ *if* $\Psi(X^o - \widehat{X}^r) = \inf_{v \in \mathbb{R}^m} \Psi(X^o - v^{\mathrm{T}} X^f)$*, and then*

$$L_0 := \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[\widehat{X}_t^r] + V_0(X^o - \widehat{X}^r)$$

*is the value of the liability with replicating portfolio chosen with respect to the criterion* $\Psi$.

**Remark 4.** *Notice from* (1) *that*

$$V_t = R_t - \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+] \leq \mathbb{E}_t^{\mathbb{Q}}[X_{t+1}] + \mathbb{E}_t^{\mathbb{Q}}[V_{t+1}].$$

*In particular,* $V_0 \leq \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[X_t]$ *and* $L_0 \leq \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{Q}}[X_t^o]$ *regardless of the criterion for choosing the replicating portfolio.*

**Remark 5.** *The deterministic replicating portfolio cash flow* $\widehat{X}^r = \mathbb{E}_0^{\mathbb{P}}[X^o]$ *corresponds to a classical actuarial best-estimate reserve, and solves a cash-flow-matching problem with only risk-free cash flows in the numéraire asset as replication instruments, see Remark 2. In this case, by Theorem 4,*

$$L_0 = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}[X_t^o] + V_0\Big(X^o - \mathbb{E}^{\mathbb{P}}[X^o]\Big) = V_0(X^o).$$

*In particular, if* $V_0(X^o) \geq \sum_{t=1}^T \mathbb{E}^{\mathbb{P}}[X_t^o]$*, then* $L_0 \geq \sum_{t=1}^T \mathbb{E}^{\mathbb{P}}[X_t^o]$*. As noted in Remark 2, any deterministic cash flow* $\widehat{X}^r$ *with* $\sum_{t=1}^T \widehat{X}_t^r = \sum_{t=1}^T \mathbb{E}_0^{\mathbb{P}}[X_t^o]$ *is a optimal solution to the (alternative) terminal value problem*

$$\inf_{v \in \mathbb{R}^m} \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sum_{t=1}^T (X_t^o - v^{\mathrm{T}} X_t^f)\Big)^2\Big],$$

*with only risk-free cash flows in the numéraire asset as replication instruments. In this case, by Theorem 4,*

$$L_0 = \sum_{t=1}^T \widehat{X}_t^r + V_0\Big(X^o - \widehat{X}^r\Big) = V_0(X^o).$$

We now address two important questions: existence of a an optimal replicating portfolio according to the portfolio selection criterion (14), and continuity of the value of the liability cash flow as a function of the portfolio weights of the replicating portfolio.

For $t \in \{1, \ldots, T\}$, define $Z_t := (X_t^o, -(X_t^f)^{\mathrm{T}})^{\mathrm{T}}$ and, for $w \in \mathbb{R}^{m+1}$, $\widetilde{X}_t^w := w^{\mathrm{T}} Z_t$. Notice that a residual liability cash corresponds to $\widetilde{X}^w$ with $w_1 = 1$. The reason for introducing this notation is primarily that it allows us to formulate sufficient conditions for coerciveness that will lead to sufficient conditions for the existence of an optimal replicating portfolio, see Theorem 6 below.

**Theorem 5.** *Let* $(D_t)_{t=0}^T$ *satisfy either of the conditions (i) or (ii) in Theorem 3. Suppose that, for each* $t \in \{0, \ldots, T-1\}$*,* $R_t : L^p(\mathcal{F}_{t+1}, \mathbb{P}) \to L^p(\mathcal{F}_t, \mathbb{P})$ *in (6) is a conditional monetary risk measure in the sense of Definition 1 for every* $p \in [1, \infty]$ *that is* $L^1$*-Lipschitz continuous in the sense*

$$|R_t(-Y) - R_t(-\widetilde{Y})| \leq K \mathbb{E}_t^{\mathbb{P}}[|Y - \widetilde{Y}|], \quad Y, \widetilde{Y} \in L^1(\mathcal{F}_{t+1}, \mathbb{P}).$$

*for some* $K \in (0, \infty)$*. If* $(Z_t)_{t=1}^T \in L^p((\mathcal{F}_t)_{t=1}^T, \mathbb{P})$ *for some* $p > 1$*, then*

$$\mathbb{R}^{m+1} \ni w \mapsto W_0 \circ \cdots \circ W_{T-1}\Big(\sum_{t=1}^T \widetilde{X}_t^w\Big)$$

*and* $\mathbb{R}^m \ni v \mapsto V_0(X^v)$ *are Lipschitz continuous.*

For $t = 0, \ldots, T-1$, set

$$\begin{aligned}
\widetilde{V}_t^w &:= W_t \circ \cdots \circ W_{T-1}\Big(\sum_{s=t+1}^T \widetilde{X}_t^w\Big), \\
\widetilde{R}_t^w &:= R_t\Big(-\widetilde{X}_{t+1}^w - \widetilde{V}_{t+1}^w\Big), \\
\widetilde{C}_t^w &:= \mathbb{E}_t^{\mathbb{Q}}\Big[(\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{V}_{t+1}^w)_+\Big], \\
\widetilde{\psi}(w) &:= \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{Q}}[\widetilde{C}_t^w].
\end{aligned}$$

Under mild conditions it can be shown that $\widetilde{\psi}$ and $\psi$, given by (14), are coercive, i.e.

$$\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty, \qquad \lim_{|v|\to\infty} \psi(v) = \infty.$$

**Theorem 6.** *Suppose, for $t = 0, \dots, T-1$, that $R_t$ is positively homogeneous in the sense $R_t(\lambda Y) = \lambda R_t(Y)$ for $\lambda \in \mathbb{R}_+$. Suppose further that $\inf_{|w|=1} \widetilde{\psi}(w) > 0$. Then $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ and $\lim_{|v|\to\infty} \psi(v) = \infty$, where $\psi$ is given by* (14).

**Remark 6.** *Notice that the condition $\inf_{|w|=1} \widetilde{\psi}(w) > 0$ means that perfect replication is not possible. It also disqualifies risk-free cash flows as replication instruments. The argument is as follows. If one of the replication instruments has a risk-free cash flow $x$ so that $X^{f,k} = x$ $\mathbb{P}$-a.s., then $X^{f,k} = x$ $\mathbb{Q}$-a.s. and $w^{\mathrm{T}} Z = x$ for some $w \in \mathbb{R}^{m+1}$ with $|w| = 1$. Then $\widetilde{\psi}(w) = 0$.*

For $t \in \{0, \dots, T-1\}$, set

$$R^{\circ}_{t,T-1} := \begin{cases} R_t, & t = T-1, \\ R_t \circ (-R_{t+1}) \circ \cdots \circ (-R_{T-1}), & t < T-1. \end{cases}$$

**Theorem 7.** *Suppose, for $t = 0, \dots, T-1$, that $R_t$ is positively homogeneous in the sense $R_t(\lambda Y) = \lambda R_t(Y)$ for $\lambda \in \mathbb{R}_+$. Suppose further that $\psi$ in* (14) *is continuous, and for all $w \in \mathbb{R}^{m+1} \setminus \{0\}$ there exists $t \in \{0, \dots, T-1\}$ such that*

$$\mathbb{P}\Big( (R^{\circ}_{t,T-1} - R^{\circ}_{t+1,T-1})\big( -w^{\mathrm{T}}(Z_{t+1} + \cdots + Z_T)\big) > 0 \Big) > 0. \qquad (15)$$

*Then there exists an optimal solution $\widehat{v} \in \mathbb{R}^m$ to* (14).

**Remark 7.** *The conditions of Theorem 7 are sufficient but not necessary for the existence of an optimal solution to* (14). *For instance, including risk-free cash flows as replication instrument would violate the condition that* (15) *holds for some $t$ and all $w$ without affecting either the optimal portfolio wights in the original replication instruments or the value of the liability cash flow.*

# 3 Gaussian cash flows

Let $(\epsilon_t)_{t=1}^T$ be a sequence of $d$-dimensional independent random vectors that are standard normally distributed under $\mathbb{P}$. For, $t = 1, \dots, T$ and nonrandom $A_t \in \mathbb{R}^n$, $B_{t,1}, \dots, B_{t,t} \in \mathbb{R}^{n \times d}$, let

$$G_t := A_t + \sum_{s=1}^{t} B_{t,s} \epsilon_s.$$

Let $(\mathcal{G}_t)_{t=0}^T$, with $\mathcal{G}_0 = \{\emptyset, \Omega\}$, be the filtration generated by the Gaussian process $(G_t)_{t=1}^T$. Subscript '$t$' will in what follows mean conditioning on $\mathcal{G}_t$. In order to keep the presentation simple, we assume that the sequence $(G_t)_{t=1}^T$ is strictly increasing. See Remark 8 for comments on this assumption.

A natural interpretation of the Gaussian model in line with the so-called Basic Actuarial Model is as follows: $X^o = G^{(1)}$ is the discounted liability cash flow, $G^{(2)}, \dots, G^{(m+1)}$ represent discounted cash flows of asset of replication instruments, and $G^{(m+2)}, \dots, G^{(n)}$ represent insurance technical information flows.

For a nonrandom sequence $(\lambda_t)_{t=1}^T$, $\lambda_t \in \mathbb{R}^n$, let

$$D_t := \exp\Big\{\sum_{s=1}^t \big(\lambda_s^{\mathrm{T}}\epsilon_s - \frac{1}{2}\lambda_s^{\mathrm{T}}\lambda_s\big)\Big\}, \quad t = 1, \dots, T.$$

We let the measure $\mathbb{Q}$ be defined in terms of the $(\mathbb{P}, \mathbb{G})$-martingale $(D_t)_{t=1}^T$: For a $\mathcal{G}_t$-measurable sufficiently integrable $Z$ and $s < t$, in accordance with Section 2, $\mathbb{E}_s^{\mathbb{Q}}[Z] = D_s^{-1}\mathbb{E}_s^{\mathbb{P}}[D_t Z]$. This choice has several pleasant consequences: for arbitrary vectors $g_t \in \mathbb{R}^n$,

$$\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=1}^T g_s^{\mathrm{T}} G_s\Big] - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^T g_s^{\mathrm{T}} G_s\Big] \in \mathcal{G}_0,$$

$$\mathrm{Var}_t^{\mathbb{Q}}\Big(\sum_{s=1}^T g_s^{\mathrm{T}} G_s\Big) = \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^T g_s^{\mathrm{T}} G_s\Big) \in \mathcal{G}_0,$$

i.e. the conditional expectations with respect to $\mathbb{Q}$ and $\mathbb{P}$ only differ by a constant and the conditional variances with respect to $\mathbb{Q}$ and $\mathbb{P}$ are equal and nonrandom.

**Definition 4.** *The triple $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ is called a Gaussian model.*

The Gaussian model allows for explicit valuation formulas when combined with conditional monetary risk measures in Definition 2. The following properties considerably simplifies computations. For $u > t$,

$$\mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^u g_s^{\mathrm{T}} G_s\Big] \in \mathrm{span}\{1, G_1, \dots, G_t\},$$

$$\sum_{s=1}^u g_s^{\mathrm{T}} G_s - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^u g_s^{\mathrm{T}} G_s\Big]$$

is independent of $\mathcal{G}_t$, and, whenever $\mathrm{Var}_t^{\mathbb{P}}\big(\sum_{s=1}^u g_s^{\mathrm{T}} G_s\big) \neq 0$,

$$\mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^u g_s^{\mathrm{T}} G_s\Big)^{-1/2}\Big(\sum_{s=1}^u g_s^{\mathrm{T}} G_s - \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^u g_s^{\mathrm{T}} G_s\Big]\Big)$$

is standard normally distributed with respect to $\mathbb{P}$. Therefore, we may without loss of generality assume the existence of a $\mathbb{G}$-adapted sequence $(e_t)_{t=1}^T$ of independent standard normally distributed random variables, with respect to $\mathbb{P}$, such that

$$R_t\Big(\sum_{s=1}^{t+1} g_s^{\mathrm{T}} G_s\Big) = R_t\Big(\mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{t+1} g_s^{\mathrm{T}} G_s\Big] + \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^{t+1} g_s^{\mathrm{T}} G_s\Big)^{1/2} e_{t+1}\Big) \tag{16}$$

$$= -\mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=1}^{t+1} g_s^{\mathrm{T}} G_s\Big] + \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=1}^{t+1} g_s^{\mathrm{T}} G_s\Big)^{1/2} r_0, \tag{17}$$

where

$$r_0 := \int_0^1 \Phi^{-1}(u) dM(u). \tag{18}$$

The equality between (16) and (17) holds because a risk measure $R_t$ in the sense of Definition 2 has the additional properties:

$$\text{if } \lambda \in \mathbb{R}_+ \text{ and } Y \in L^p(\mathcal{F}_{t+1}, \mathbb{P}), \text{ then } R_t(\lambda Y) = \lambda R_t(Y),$$
$$R_t(e_{t+1}) = r_0.$$

We will first derive an explicit expression for the value of a general Gaussian liability cash flow, where the generality lies in that $X_t$ is allowed to be an arbitrary linear combination $g_t^{\mathrm{T}} G_t$, where $g_t \in \mathbb{R}^n$ may be time dependent. Then we will return to the relevant special case when $g_t = g$ for all $t$ and $g^{(1)} = 1$, $(g^{(k)})_{k=2}^{m+1} = v \in \mathbb{R}^m$ and $g^{(k)} = 0$ for $k > m+1$.

**Theorem 8.** *Let $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ be a Gaussian model and let $X_t := g_t^{\mathrm{T}} G_t$ for $t = 1, \dots, T$. For $t = 0, \dots, T-1$, let $R_t$ be conditional monetary risk measures in the sense of Definition 2 for a common probability distribution $M$. Let $r_0$ be given by* (18)*. Then*

$$V_t(X) = \sum_{s=t+1}^T \mathbb{E}_t^{\mathbb{Q}}[X_s] + K_t^{\mathbb{Q}} = \sum_{s=t+1}^T \mathbb{E}_t^{\mathbb{P}}[X_s] + K_t^{\mathbb{P}},$$

*where, with $e_1$ standard normally distributed with respect to $\mathbb{P}$,*

$$\begin{aligned}
K_t^{\mathbb{Q}} = \sum_{s=t+1}^T \Big(&\sigma_s r_0 - \sum_{u=s}^T g_u^{\mathrm{T}} B_{u,s} \lambda_s \\
&- \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^T g_u^{\mathrm{T}} B_{u,s} \lambda_s\Big)_+\Big]\Big),
\end{aligned}$$

$$K_t^{\mathbb{P}} = \sum_{s=t+1}^T \Big(\sigma_s(r_0 - e_1) - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^T g_u^{\mathrm{T}} B_{u,s} \lambda_s\Big)_+\Big]\Big),$$

$$\sigma_s^2 = \mathrm{Var}_{s-1}^{\mathbb{P}}\Big(\sum_{u=s}^T X_u\Big) - \mathrm{Var}_s^{\mathbb{P}}\Big(\sum_{u=s}^T X_u\Big) = \sum_{j=s}^T \sum_{k=s}^T g_j^{\mathrm{T}} B_{j,s} B_{k,s}^{\mathrm{T}} g_k.$$

*Moreover,*

$$C_t = \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}(r_0 - e_1) - \sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1}\Big)_+\Big].$$

**Remark 8.** *Theorem 8 holds without the assumption that the sequence $(G_t)_{t=1}^T$ is strictly increasing. If there exists $t$ such that $\mathcal{G}_t = \mathcal{G}_{t-1}$, then $\sigma_t = 0$ and minor changes in the proof of Theorem 8 are needed.*

**Remark 9.** *Notice that*

$$\begin{aligned} C_t &= \mathbb{E}_t^{\mathbb{Q}}\Big[\Big(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1}\Big)_+\Big] \\ &= \frac{1}{1+\eta_t}\mathbb{E}_t^{\mathbb{P}}\Big[\Big(R_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1}\Big)_+\Big], \end{aligned}$$

*where, given the setting in Theorem 8,*

$$\frac{1}{1+\eta_t} = \frac{\mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}(r_0 - e_1) - \sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1}\Big)_+\Big]}{\mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}(r_0 - e_1)\Big)_+\Big]}.$$

*In particular, $\eta_t \geq 0$ for every $t$ if $\sum_{u=t+1}^{T} g_u^{\mathrm{T}} B_{u,t+1}\lambda_{t+1} \geq 0$ for every $t$. Since*

$$\begin{aligned} \sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{Q}}[X_u] - \sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{P}}[X_u] &= \sum_{u=t+1}^{T} \sum_{s=t+1}^{u} g_u^{\mathrm{T}} B_{u,s}\lambda_s \\ &= \sum_{s=t+1}^{T} \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s \end{aligned}$$

*we see that $\eta_t \geq 0$ for every $t$ holds if $\sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{Q}}[X_u] \geq \sum_{u=t+1}^{T} \mathbb{E}_t^{\mathbb{P}}[X_u]$ for every $t$. This is completely in line with Remark 1.*

**Theorem 9.** *Let $((G_t)_{t=1}^T, (D_t)_{t=1}^T, (\mathcal{G}_t)_{t=0}^T)$ be a Gaussian model. Let $X^o = G^{(1)}$ be the discounted liability cash flow, let $X^{f,1} := G^{(2)}, \dots, X^{f,m} := G^{(m+1)}$ represent discounted cash flows of replication instruments, and let $G^{(m+2)}, \dots, G^{(n)}$ represent insurance technical information flows. For $t = 0, \dots, T-1$, let $R_t$ be conditional monetary risk measures in the sense of Definition 2 for a common probability distribution $M$. Let $r_0$ be given by (18). For $t = 1, \dots, T$, let $Z_t := (X_t^o, -(X_t^f)^{\mathrm{T}})^{\mathrm{T}}$ and assume that*

$$\textit{there is no } w \in \mathbb{R}^{m+1} \setminus \{0\} \textit{ such that } \sum_{t=1}^{T} w^{\mathrm{T}} Z_t \in \mathcal{G}_0.$$

*Then there exists an optimal solution to* (14) *and the value of the liability is given by*

$$L_0 = \sum_{t=1}^{T} \mathbb{E}_0^{\mathbb{P}}[X_t^o] + \widehat{K}_0^{\mathbb{P}} = \sum_{t=1}^{T} A_t^{(1)} + \widehat{K}_0^{\mathbb{P}},$$

*where, with* $S_t := \sum_{u=t}^{T} B_{u,t}$ *and* $e_1$ *standard normally distributed with respect to* $\mathbb{P}$*,*

$$\begin{aligned}
\widehat{K}_0^{\mathbb{P}} &= \sum_{t=1}^{T} \Big(\widehat{\sigma}_t r_0 - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\widehat{\sigma}_t(r_0 - e_1) - \widehat{g}^{\mathrm{T}} S_t \lambda_t\Big)_+\Big]\Big), \\
\widehat{\sigma}_t^2 &= \widehat{g}^{\mathrm{T}} S_t S_t^{\mathrm{T}} \widehat{g},
\end{aligned}$$

*where* $\widehat{g}$ *is the minimizer in* $\{g \in \mathbb{R}^n : g_1 = 1, g_k = 0$ *for* $k > m+1\}$ *of*

$$g \mapsto \sum_{t=0}^{T-1} \mathbb{E}_0^{\mathbb{P}}\Big[\Big((g^{\mathrm{T}} S_{t+1} S_{t+1}^{\mathrm{T}} g)^{1/2}(r_0 - e_1) - g^{\mathrm{T}} S_{t+1} \lambda_{t+1}\Big)_+\Big].$$

# 4 Proofs

*Proof of Theorem 1.*

$$\begin{aligned}
0 &= \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[H_{t,\tau}] \\
&= \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \mathbb{E}_t^{\mathbb{Q}}[\mathbb{I}\{\tau = t+1\} H_{t,t+1} + \mathbb{I}\{\tau > t+1\} H_{t,\tau}] \\
&= \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+1,T}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{\tau = t+1\} H_{t,t+1} + \mathbb{I}\{\tau > t+1\} \mathbb{E}_{t+1}^{\mathbb{Q}}[H_{t,\tau}]\Big]\Big) \\
&= \operatorname*{ess\,sup}_{A \in \mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\} H_{t,t+1} + \mathbb{I}\{A^C\} \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+2,T}} \mathbb{E}_{t+1}^{\mathbb{Q}}[H_{t,\tau}]\Big]\Big) \\
&= -R_t + V_t + \operatorname*{ess\,sup}_{A \in \mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\}(R_t - X_{t+1} - V_{t+1})_+ \\
&\quad + \mathbb{I}\{A^C\}\Big(R_t - X_{t+1} - V_{t+1} + \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t+2,T}} \mathbb{E}_{t+1}^{\mathbb{Q}}[H_{t+1,\tau}]\Big)\Big]\Big) \\
&= -R_t + V_t + \operatorname*{ess\,sup}_{A \in \mathcal{F}_{t+1}} \Big(\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{I}\{A\}(R_t - X_{t+1} - V_{t+1})_+ \\
&\quad + \mathbb{I}\{A^C\}(R_t - X_{t+1} - V_{t+1})\Big]\Big) \\
&= -R_t + V_t + \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})_+].
\end{aligned}$$

$\square$

*Proof of Theorem 3.* We prove the more involved statement (ii). Statement (i) is proved with the same arguments.

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y)-Y)_+\Big]^p\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{P}}\Big[\frac{D_{t+1}}{D_t}(R_t(-Y)-Y)_+\Big]^p\Big] \\ &\le \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p(R_t(-Y)-Y)_+^p\Big]\Big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p(R_t(-Y)-Y)_+^p\Big],\end{aligned}$$

where the inequality is due to Jensen's inequality for conditional expectations. Moreover, for every $r>1$, by Hölder's inequality,

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^p(R_t(-Y)-Y)_+^p\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\Big(\frac{D_{t+1}}{D_t}\Big)^{pr}\Big]^{\frac{1}{r}}\mathbb{E}^{\mathbb{P}}\Big[(R_t(-Y)-Y)_+^{p\frac{r}{r-1}}\Big]^{\frac{r-1}{r}}.$$

For $r>1$ sufficiently large, it follows from the assumptions that the two expectations exist finitely. Finally, it follows from Minkowski's inequality that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[\Big(R_t(-Y)-\mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y)-Y)_+\Big]\Big)^p\Big]^{\frac{1}{p}} \\ &\quad\le \mathbb{E}^{\mathbb{P}}\Big[R_t(-Y)^p\Big]^{\frac{1}{p}}+\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}_t^{\mathbb{Q}}\Big[(R_t(-Y)-Y)_+\Big]^p\Big]^{\frac{1}{p}}.\end{aligned}$$

The finiteness of the first terms follows from the assumptions and the finiteness of the second term has been proven above. This proves that the mapping is well-defined.

The remaining part of statement (ii) follows, upon minor modifications, from Proposition 1 in [7]. □

*Proof of Theorem 5.* For $w\in\mathbb{R}^{m+1}$ and $t\in\{0,\dots,T-1\}$, define

$$V_t^w := W_t\circ\cdots\circ W_{T-1}\Big(\sum_{s=t+1}^{T} w^{\mathrm{T}}Z_s\Big).$$

We prove the statement inductively. Assume that for some nonnegative $B_{t+2}\in L^1(\mathcal{F}_{t+2},\mathbb{P})$,

$$|V_{t+1}^w-V_{t+1}^v|\le ||v-w||_1\mathbb{E}_{t+1}^{\mathbb{P}}[B_{t+2}],$$

where $||\cdot||_p$ denotes the Euclidean $p$-norm in $\mathbb{R}^{m+1}$. We start by showing the induction step, noting that verifying the induction base is trivial since $V_T^w=0$.

Defining $Y_{t+1}^w := w^{\mathrm{T}}Z_{t+1}+V_{t+1}^w$ and applying Hölder's inequality,

$$\begin{aligned}|Y_{t+1}^w-Y_{t+1}^v| &\le |V_{t+1}^w-V_{t+1}^v|+|w^{\mathrm{T}}Z_{t+1}-v^{\mathrm{T}}Z_{t+1}| \\ &\le ||v-w||_1\mathbb{E}_{t+1}^{\mathbb{P}}[B_{t+2}]+|w^{\mathrm{T}}Z_{t+1}-v^{\mathrm{T}}Z_{t+1}| \\ &\le ||v-w||_1\mathbb{E}_{t+1}^{\mathbb{P}}[||Z_{t+1}||_\infty+B_{t+2}]\end{aligned}$$

Now, due to the $L^1$-Lipschitz continuity of $R_t$,

$$\begin{aligned}|R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| &\le K\mathbb{E}^{\mathbb{P}}_t[|Y^w_{t+1} - Y^v_{t+1}|]\\ &\le K||v-w||_1\mathbb{E}^{\mathbb{P}}_t[||Z_{t+1}||_\infty + B_{t+2}]\end{aligned}$$

With $C^w_t := \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1})_+]$, due to subadditivity of $x \mapsto x_+ := \max(x,0)$,

$$\begin{aligned}C^w_t - C^v_t &= \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1})_+ - (R_t(-Y^v_{t+1}) - Y^v_{t+1})_+]\\ &\le \mathbb{E}^{\mathbb{Q}}_t[(R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1})_+]\\ &\le \mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|],\\ C^w_t - C^v_t &\ge \mathbb{E}^{\mathbb{Q}}_t[-(R_t(-Y^v_{t+1}) - Y^v_{t+1} - R_t(-Y^w_{t+1}) + Y^w_{t+1})_+]\\ &\ge -\mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|]\end{aligned}$$

from which it follows that

$$\begin{aligned}|C^w_t - C^v_t| &\le \mathbb{E}^{\mathbb{Q}}_t[|R_t(-Y^w_{t+1}) - Y^w_{t+1} - R_t(-Y^v_{t+1}) + Y^v_{t+1}|]\\ &\le |R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| + \mathbb{E}^{\mathbb{Q}}_t[|Y^w_{t+1} - Y^v_{t+1}|].\end{aligned}$$

The $\mathbb{Q}$-expectation remains to be analyzed.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_t[|Y^w_{t+1} - Y^v_{t+1}|] &\le \mathbb{E}^{\mathbb{Q}}_t\Big[||v-w||_1\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big]\\ &= ||v-w||_1\mathbb{E}^{\mathbb{P}}_t\Big[\frac{D_{t+1}}{D_t}\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big].\end{aligned}$$

Hence,

$$\begin{aligned}|V^w_t - V^v_t| &\le |R_t(-Y^w_{t+1}) - R_t(-Y^v_{t+1})| + |C^w_t - C^v_t|\\ &\le 2K||v-w||_1\mathbb{E}^{\mathbb{P}}_t[||Z_{t+1}||_\infty + B_{t+2}]\\ &\quad + ||v-w||_1\mathbb{E}^{\mathbb{P}}_t\Big[\frac{D_{t+1}}{D_t}\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big]\\ &= ||v-w||_1\mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big)\Big]\\ &= ||v-w||_1\mathbb{E}^{\mathbb{P}}_t\Big[B_{t+1}\Big],\end{aligned}$$

where $B_{T+1} = 0$ and otherwise

$$B_{t+1} := \mathbb{E}^{\mathbb{P}}_{t+1}\Big[||Z_{t+1}||_\infty + B_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big).$$

In particular,

$$|V^w_0 - V^v_0| \le ||v-w||_1\mathbb{E}^{\mathbb{P}}_0[B_1],$$

Now what remains is to show that $\mathbb{E}_0^{\mathbb{P}}[B_1] < \infty$. For the Euclidean norms, the inequality $||x||_p \leq ||x||_1$ holds for $p \in [1, \infty]$. In particular, for each $t = 1, \dots, T$, $0 \leq B_t \leq \widetilde{B}_t$, where

$$\widetilde{B}_{t+1} := \mathbb{E}_{t+1}^{\mathbb{P}}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big), \quad \widetilde{B}_{T+1} = 0.$$

Recall that for $t = 1, \dots, T$, $Z_t^k \in L^{p_t}(\mathcal{F}_t, \mathbb{P})$ for all $k$ and some $p_t > 1$. Also notice that if $\widetilde{B}_{t+2} \in L^{q_{t+2}}(\mathcal{F}_{t+2}, \mathbb{P})$ for $q_{t+2} > 1$, then $\mathbb{E}_{t+1}^{\mathbb{P}}[\widetilde{B}_{t+2}] \in L^{q_{t+2}}(\mathcal{F}_{t+1}, \mathbb{P})$ and, for $r_{t+1} = \min(p_{t+1}, q_{t+2})$,

$$\mathbb{E}_{t+1}^{\mathbb{P}}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big] \in L^{r_{t+1}}(\mathcal{F}_{t+1}).$$

Hence, for any $\epsilon > 0$,

$$\widetilde{B}_{t+1} = \mathbb{E}_{t+1}^{\mathbb{P}}\Big[||Z_{t+1}||_1 + \widetilde{B}_{t+2}\Big]\Big(2K + \frac{D_{t+1}}{D_t}\Big) \in L^{r_{t+1}-\epsilon}(\mathcal{F}_{t+1}).$$

Since $\widetilde{B}_{T+1} = 0$ we may choose $\epsilon > 0$ small enough so that $\widetilde{B}_t \in L^1(\mathcal{F}_t, \mathbb{P})$ for $t = 1, \dots, T$. Hence, also $B_t \in L^1(\mathcal{F}_t, \mathbb{P})$ for $t = 1, \dots, T$.

Finally, notice that

$$X_t^v := X_t^o - v^{\mathrm{T}} X_t^f = w^{\mathrm{T}} Z_t$$

if $w \in \mathbb{R}^{m+1}$ is chosen so that $w_1 = 1$ and $(w_k)_{k=2}^{m+1} = v$. Therefore, we have also shown that $v \mapsto V_0(X^v)$ is Lipschitz continuous. $\square$

*Proof of Theorem 6.* From positive homogeneity of the $R_t$s follows positive homogeneity of the $W_t$s which implies $\widetilde{V}_t^w(\lambda \widetilde{X}^w) = \lambda \widetilde{V}_t^w(\widetilde{X}^w)$ and further that $\widetilde{\psi}(\lambda w) = \lambda \widetilde{\psi}(w)$. In particular,

$$\widetilde{\psi}(w) = |w| \widetilde{\psi}(w/|w|) \geq |w| \inf_{|w|=1} \widetilde{\psi}(w)$$

from which $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ follows from the assumption $\inf_{|w|=1} \widetilde{\psi}(w) > 0$. For the second statement, notice that

$$X_t^v := X_t^o - v^{\mathrm{T}} X_t^f = w^{\mathrm{T}} Z_t$$

if $w \in \mathbb{R}^{m+1}$ is chosen so that $w_1 = 1$ and $(w_k)_{k=2}^{m+1} = v$. Therefore, $\lim_{|w|\to\infty} \widetilde{\psi}(w) = \infty$ implies $\lim_{|v|\to\infty} \psi(v) = \infty$. $\square$

*Proof of Theorem 7.* Take $w \in \mathbb{R}^{m+1} \setminus \{0\}$. Suppose that $\widetilde{C}_t^w = 0$ $\mathbb{Q}$-a.s. for all $t$. Then $\widetilde{V}_t^w = \widetilde{R}_t^w$ for all $t$ and $\widetilde{C}_t^w = 0$ is equivalent to $\widetilde{R}_t^w - \widetilde{X}_{t+1}^w - \widetilde{R}_{t+1}^w \leq$

0 $\mathbb{Q}$-a.s. which is equivalent to $\widetilde{R}^w_t - \widetilde{X}^w_{t+1} - \widetilde{R}^w_{t+1} \leq 0$ $\mathbb{P}$-a.s. since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. Notice that

$$\begin{aligned}
\widetilde{R}^w_t &= R_t(-\widetilde{X}^w_{t+1} - \widetilde{R}^w_{t+1}) \\
&= R_t(-\widetilde{X}^w_{t+1} - R_{t+1}(-\widetilde{X}^w_{t+2} - \widetilde{R}^w_{t+2})) \\
&= R_t \circ (-R_{t+1}) \circ \cdots \circ (-R_{T-1})\Big( - \sum_{s=t+1}^{T} \widetilde{X}^w_s \Big)
\end{aligned}$$

The inequality $\widetilde{R}^w_t - \widetilde{X}^w_{t+1} - \widetilde{R}^w_{t+1} \leq 0$ $\mathbb{P}$-a.s. can thus be expressed as

$$(R^\circ_{t,T-1} - R^\circ_{t+1,T-1})\big( - w^{\mathrm{T}}(Z_{t+1} + \cdots + Z_T)\big) \leq 0 \quad \mathbb{P}\text{-a.s.}$$

However, this is contradicting the assumption in the statement of the theorem. Therefore we conclude that $\widetilde{C}^w_t > 0$ $\mathbb{Q}$-a.s. for some $t$ which implies that $\widetilde{\psi}(w) > 0$. Therefore, by Theorem 6, $\psi$ is coercive so if a minimum exists it exists in some compact set in $\mathbb{R}^m$. However, a continuous function on a compact set attains its infimum. □

**Lemma 1.** *For $u < v$, $\mathbb{E}^{\mathbb{Q}}_u[G_v] = \mathbb{E}^{\mathbb{P}}_u[G_v] + \sum_{s=u+1}^{v} B_{v,s}\lambda_s$.*

*Proof.*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_u[G_v] &= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\mathbb{E}^{\mathbb{P}}_u\Big[\frac{D_v}{D_u}\epsilon_s\Big] \\
&= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\mathbb{E}^{\mathbb{P}}_0\Big[\exp\Big\{\lambda_s^{\mathrm{T}}\epsilon_1 - \frac{1}{2}\lambda_s^{\mathrm{T}}\lambda_s\Big\}\epsilon_1\Big] \\
&= A_v + \sum_{s=1}^{u} B_{v,s}\epsilon_s + \sum_{s=u+1}^{v} B_{v,s}\lambda_s \\
&= \mathbb{E}^{\mathbb{P}}_u[G_v] + \sum_{s=u+1}^{v} B_{v,s}\lambda_s.
\end{aligned}$$

□

**Lemma 2.** *If $X_s := g_s^{\mathrm{T}} G_s$, then*

$$\mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] = \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}.$$

*Proof.* For $s \geq t+1$, with an empty sum defined as 0, it follows from Lemma

1 that

$$\mathbb{E}^{\mathbb{Q}}_{t+1}[X_s] = \mathbb{E}^{\mathbb{P}}_{t+1}[X_s] + g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u,$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] &= \sum_{s=t+1}^{T}\Big(\mathbb{E}^{\mathbb{P}}_t[\mathbb{E}^{\mathbb{P}}_{t+1}[X_s]] + g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u\Big) \\
&= \mathbb{E}^{\mathbb{P}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] + \sum_{s=t+1}^{T} g_s^{\mathrm{T}} \sum_{u=t+2}^{s} B_{s,u}\lambda_u, \\
\mathbb{E}^{\mathbb{P}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] &= \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} \sum_{u=t+1}^{s} B_{s,u}\lambda_u.
\end{aligned}$$

□

*Proof of Theorem 8.* We will prove inductively that

$$V_t = \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] + K^{\mathbb{Q}}_t, \tag{19}$$

and derive the recursive form of the constant term $K^{\mathbb{Q}}_t$ via induction. The induction base is trivial: $V_T = 0$. Now assume that (19) holds for $t+1$. Notice that

$$\begin{aligned}
V_t &= W_t\Big(X_{t+1} + \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+2}^{T} X_s\Big] + K^{\mathbb{Q}}_{t+1}\Big) \\
&= W_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big] + K^{\mathbb{Q}}_{t+1}\Big) \\
&= K^{\mathbb{Q}}_{t+1} + R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) \\
&\quad - \mathbb{E}^{\mathbb{Q}}_t\Big[\Big(R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) - \mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)_+\Big]
\end{aligned}$$

We first evaluate the risk measure part.

$$\begin{aligned}
&R_t\Big(-\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) \\
&= \mathbb{E}^{\mathbb{P}}_t\Big[\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big] + \mathrm{Var}^{\mathbb{P}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)^{1/2} r_0 \\
&= \mathbb{E}^{\mathbb{Q}}_t\Big[\sum_{s=t+1}^{T} X_s\Big] - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} + \mathrm{Var}^{\mathbb{P}}_t\Big(\mathbb{E}^{\mathbb{Q}}_{t+1}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big)^{1/2} r_0,
\end{aligned}$$

where in the final step we used Lemma 2. Moreover,

$$\begin{aligned}
\mathrm{Var}_t^{\mathbb{P}}\Big(\mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) &= \mathrm{Var}_t^{\mathbb{Q}}\Big(\mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big]\Big) \\
&= \mathrm{Var}_t^{\mathbb{Q}}\Big(\sum_{s=1}^{T} X_s\Big) - \mathrm{Var}_{t+1}^{\mathbb{Q}}\Big(\sum_{s=1}^{T} X_s\Big) \\
&=: \sigma_{t+1}^2.
\end{aligned}$$

The remaining term: if $\sigma_{t+1} \neq 0$, then there exists a random variable $e_{t+1}^*$ that is independent of $\mathcal{G}_t$ and standard normally distributed with respect to $\mathbb{Q}$ such that

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\Big[\Big(R_t\Big(-\mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] - K_{t+1}^{\mathbb{Q}}\Big) - \mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] - K_{t+1}^{\mathbb{Q}}\Big)_+\Big] \\
&= \mathbb{E}_t^{\mathbb{Q}}\Big[\Big(\sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - \sigma_{t+1} e_{t+1}^*\Big)_+\Big].
\end{aligned}$$

Notice that if $\sigma_{t+1} = 0$, then the existence of $e_{t+1}^*$ is not required but the same expression holds. Putting the pieces together now yields

$$\begin{aligned}
V_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] + K_{t+1}^{\mathbb{Q}} + \sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} \\
- \mathbb{E}_t^{\mathbb{Q}}\Big[\Big(\sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - \sigma_{t+1} e_{t+1}^*\Big)_+\Big]
\end{aligned}$$

which proves the induction step and from which it follows that

$$K_t^{\mathbb{Q}} = \sum_{s=t+1}^{T} \Big(\sigma_s r_0 - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big).$$

Finally,

$$\begin{aligned}
V_t &= \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] + K_t^{\mathbb{Q}} \\
&= \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=t+1}^{T} X_s\Big] + \sum_{s=t+1}^{T}\sum_{u=t+1}^{s} g_s^{\mathrm{T}} B_{s,u}\lambda_u + K_t^{\mathbb{Q}} \\
&= \mathbb{E}_t^{\mathbb{P}}\Big[\sum_{s=t+1}^{T} X_s\Big] + K_t^{\mathbb{P}},
\end{aligned}$$

where

$$K_t^{\mathbb{P}} = \sum_{s=t+1}^{T} \Big(\sigma_s r_0 - \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_s(r_0 - e_1) - \sum_{u=s}^{T} g_u^{\mathrm{T}} B_{u,s}\lambda_s\Big)_+\Big]\Big).$$

We now derive an expression for $\sigma_{t+1}$. Recall that $X_s := g_s^{\mathrm{T}} G_s$.

$$\begin{aligned}
\mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) &= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T}\sum_{u=t+1}^{s} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{u=t+1}^{T}\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \sum_{u=t+1}^{T} \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\epsilon_u\Big) \\
&= \sum_{u=t+1}^{T} \Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\Big)\Big(\sum_{s=u}^{T} g_s^{\mathrm{T}} B_{s,u}\Big)^{\mathrm{T}} \\
&= \sum_{u=t+1}^{T}\sum_{j=u}^{T}\sum_{k=u}^{T} g_j^{\mathrm{T}} B_{j,u} B_{k,u}^{\mathrm{T}} g_k
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{t+1}^2 &:= \mathrm{Var}_t^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) - \mathrm{Var}_{t+1}^{\mathbb{P}}\Big(\sum_{s=t+1}^{T} g_s^{\mathrm{T}} G_s\Big) \\
&= \sum_{j=t+1}^{T}\sum_{k=t+1}^{T} g_j^{\mathrm{T}} B_{j,t+1} B_{k,t+1}^{\mathrm{T}} g_k
\end{aligned}$$

We now derive the expression for $C_t$. Using the same arguments as earlier in the proof,

$$\begin{aligned}
C_t &= R_t(-X_{t+1} - V_{t+1}) - V_t \\
&= R_t\Big(-\mathbb{E}_{t+1}^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] - K_{t+1}^{\mathbb{Q}}\Big) - \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{s=t+1}^{T} X_s\Big] - K_t^{\mathbb{Q}} \\
&= \sigma_{t+1} r_0 - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1} - K_t^{\mathbb{Q}} + K_{t+1}^{\mathbb{Q}} \\
&= \mathbb{E}_0^{\mathbb{P}}\Big[\Big(\sigma_{t+1}(r_0 - e_1) - \sum_{s=t+1}^{T} g_s^{\mathrm{T}} B_{s,t+1}\lambda_{t+1}\Big)_+\Big].
\end{aligned}$$

□

*Proof of Theorem 9.* From Theorem 8 we immediately see that $\psi(w)$ is continuous. Notice that

$$\begin{aligned}
&(R_{t,T-1}^{\circ} - R_{t+1,T-1}^{\circ})\big(-w^{\mathrm{T}}(Z_{t+1} + \cdots + Z_T)\big) \\
&\quad = (R_{t,T-1}^{\circ} - R_{t+1,T-1}^{\circ})\big(-w^{\mathrm{T}}(Z_1 + \cdots + Z_T)\big) \\
&\quad = \mathbb{E}_t^{\mathbb{P}}[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] - \mathbb{E}_{t+1}^{\mathbb{P}}[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] + c
\end{aligned}$$

for some constant $c$, where the last equality follows from calculations completely analogous to the proof of Theorem 8. Now assume that for some $w \in \mathbb{R}^{m+1}$, (15) does not hold. In the current Gaussian setting, the support of a Gaussian distribution is either infinite or a singleton, this implies that

$$(R^{\circ}_{t,T-1} - R^{\circ}_{t+1,T-1})\big( - w^{\mathrm{T}}(Z_1 + \cdots + Z_T)\big) = 0 \quad \mathbb{P}\text{-a.s. for all } t$$

or, equivalently, that

$$\mathbb{E}^{\mathbb{P}}_t[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] - \mathbb{E}^{\mathbb{P}}_{t+1}[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] \in \mathcal{G}_0 \quad \text{for all } t. \tag{20}$$

For $t = 0$, (20) implies that $\mathbb{E}^{\mathbb{P}}_1[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] \in \mathcal{G}_0$ which together with (20) for $t = 1$ implies that $\mathbb{E}^{\mathbb{P}}_2[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] \in \mathcal{G}_0$. By repeating this argument we have shown that

$$w^{\mathrm{T}}(Z_1 + \cdots + Z_T) = \mathbb{E}^{\mathbb{P}}_T[w^{\mathrm{T}}(Z_1 + \cdots + Z_T)] \in \mathcal{G}_0$$

which contradicts the assumption $w^{\mathrm{T}}(Z_1 + \cdots + Z_T) \notin \mathcal{G}_0$. Hence, we conclude that there exists an optimal solution to (14). The remaining part follows immediately from Theorem 8. $\square$

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