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Abstract

The size of the giant component in the configuration model is given by a well-known expression involving the generating function of the degree distribution. In this note, we argue that the size of the giant is not determined by the tail behavior of the degree distribution but rather by the distribution over small degrees. Upper and lower bounds for the component size are derived for an arbitrary given distribution over small degrees $d \leq L$ and given expected degree, and numerical implementations show that these bounds are very close already for small values of L. On the other hand, examples illustrate that, for a fixed degree tail, the component size can vary substantially depending on the distribution over small degrees. Hence the degree tail does not play the same crucial role for the size of the giant as it does for many other properties of the graph.

Keywords: Configuration model, component size, degree distribution.

AMS 2010 Subject Classification: 05C80.

1 Introduction and results

The configuration model is one of the simplest and most well-known models for generating a random graph with a prescribed degree distribution. It takes a probability distribution with support on the non-negative integers as input and gives a graph with this degree distribution as output. The model is very well studied and there are precise answers to most questions concerning properties of the model such as the threshold for the occurrence of a giant component [13, 11], the size of the largest component [14, 11], diameter and distances in the supercritical regime [7, 8, 9], criteria for the graph to be simple [10] etc; see [5, Chapter 7] and [6, Chapters 4-5] for detailed overviews. Empirical networks often exhibit power law distributions, that is, the number of vertices with degree d decays as an inverse power of d for large degrees. For this reason, there has been a lot of attention on properties of the configuration model with this type of degree distribution. Here we focus on the size of the largest component in the supercritical regime as a functional of the degree distribution. Our main message is that the size of the largest supercritical component is not determined by the tail behavior of the degree distribution, but by the distribution over small degrees. While this is not surprising, in view of the general focus on degree tails in the literature, we think it deserves to be pointed out and elaborated on.

The model and its phase transition

To define the model, fix the number n of vertices in the graph and let $F = \{p_d\}_{d\geq 0}$ be a probability distribution with support on the non-negative integers. Assign a random number D_i of half-edges independently to each vertex $i = 1, \ldots, n$, with $D_i \sim F$. If the total number of half-edges is odd, one extra half-edge is added to a uniformly chosen vertex. Then pair half-edges uniformly at random to create edges, that is, first pick two half-edges uniformly at random and

join them into an edge, then pick two half-edges from the set of remaining half-edges and create another edge, and so on until all half-edges have been paired. The construction allows for self-loops and multiple edges between the same pair of vertices. However, if the degree distribution has finite mean, such edges can be removed without changing the asymptotic degree distribution, and if the second moment is finite, there is a strictly positive probability that the graph is simple; see e.g. [2, 10].

Write $\mu = \mathbb{E}[D]$ and $\nu = \mathbb{E}[D(D-1)]/\mu$. It is well-known that the threshold for the occurrence of a giant component in the configuration model is given by ν: if $\nu > 1$, then there is with high probability a unique giant component occupying a positive fraction ξ of the vertices as $n \to \infty$, while if $\nu < 1$, then the largest component grows sublinearly in n; see [13, 11]. To see this, consider an exploration of the graph starting from a uniformly chosen vertex and then proceeding via nearest neighbors. For large n, such an exploration can be approximated by a branching process, where the offspring (=degree) of the first vertex has distribution F. For vertices in later generations, their degrees are distributed according to a size biased version of F. Indeed, by construction of the graph, the vertices constitute the end-points of uniformly chosen half-edges, and the probability of encountering a vertex with degree d is therefore proportional to d. Since we arrive at a vertex from one neighbor, the remaining number of neighbors – corresponding to the offspring of the vertex – has a down-shifted size biased distribution $\tilde{F} = \{\tilde{p}_d\}_{d\geq 0}$, defined by

$$\tilde{p}_d = \frac{(d+1)p_{d+1}}{\mu}. \tag{1}$$

Infinite survival in the approximating branching process corresponds to a giant component in the graph, and the critical parameter ν is easily identified as the mean of the distribution (1). The asymptotic size ξ of the giant component is given by the survival probability in the two-stage branching process. Write $G(s)$ for the probability generating function for the degree distribution F and note that the probability generating function for $\tilde{F}$ is given by $G'(s)/\mu$. Let $\tilde{z}$ denote the probability that a branching process with offspring distribution $\tilde{F}$ goes extinct. Then

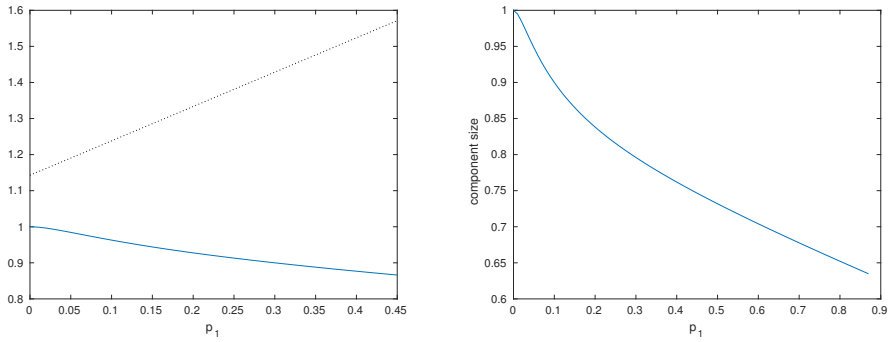
$$\xi = 1 - G(\tilde{z}), \tag{2}$$

where $\tilde{z}$ is the smallest non-negative solution to the equation $s = G'(s)/\mu$. A comprehensive description of the above exploration process can be found e.g. in [6, Chapter 4]. As for notation, when we want to emphasize the role of the distribution F for the above quantities, we write ξ_F and $\tilde{z}_F$ etc. Furthermore, we always equip quantities related to down-shifted size biased distributions with a wiggle-hat.

Basic examples

We will be interested in how the size ξ of the giant component depends on properties of the degree distribution F. Despite the large interest in the configuration model in the context of network modeling, there has been surprisingly little work on this issue. One recent example however is [12], where component sizes are compared when degree distributions are ordered according to various concepts of stochastic domination. We also mention [3], where a distribution is identified that maximizes the size of the largest component for a given mean degree: this is achieved by putting all mass at 0 and two consecutive integers. Here, we will throughout restrict to the class of distributions with $p_0 = 0$, that is, to graphs without isolated vertices. We hence require that all vertices have a chance of being included in a giant component (if such a component exists), and rule out cases where the component size can be tuned by removing some fraction of the vertices. We remark however that the bounds in Theorem 1.1 are valid also when $p_0 > 0$.

First note that, when the mean μ is fixed, the critical parameter ν increases as the variance of the distribution increases, making it easier to form a giant component. This might lead one to



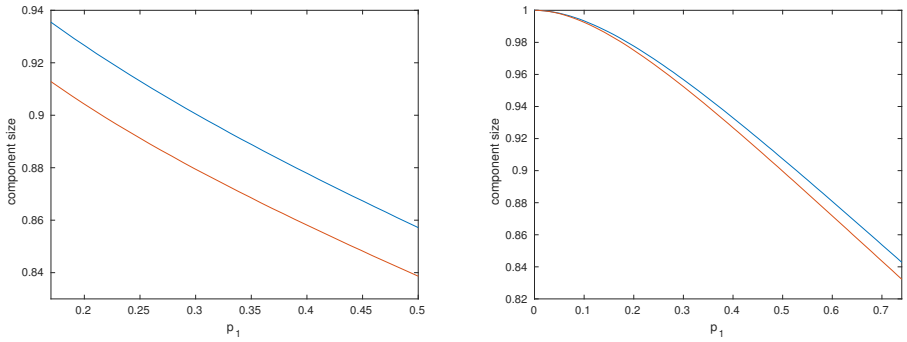
(a) $D \in \{1, 2, 3\}$ (b) $D \in \{1, 2, 10\}$

Figure 1: Asymptotic size ξ of the giant (solid line) and critical parameter ν (dotted line in (a)) plotted against p_1 with mean fixed at $\mu = 2.1$.

suspect that the size of the giant component is also increasing in ν. This however is not true, in fact it is typically the other way around, as elaborated on in [12]. To understand this, note that fixing the mean and increasing the variance implies that there will be more vertices with small degree in the graph. Vertices with small degree are those that may not be included in the giant component, which then becomes smaller. Consider a very simple example with $D \in \{1, 2, 3\}$ where the probability p_1 of degree 1 is varied and the probabilities p_2 and p_3 are tuned so that the mean is kept fixed. As p_1 increases, hence also the probability p_3 increases, implying a larger variance. Figure 1(a) shows a plot of the component size and the critical parameter against p_1 when $\mu = 2.1$, and we see that the giant component shrinks from occupying all vertices to a fraction 0.85 of them, while the critical parameter increases linearly. Figure 1(b) shows a similar plot (with only the component size) when $D \in \{1, 2, 10\}$ and again $\mu = 2.1$, and we see that the component size decreases from 1 to less than 0.65. Note that these examples also illustrate that the mean in itself does not determine the component size, since the mean is constant in both pictures.

In the example we see that the component size ξ decreases as the fraction of degree 1 vertices increases. This is very natural since degree 1 vertices serve as dead ends in the component. If $\mathbb{P}(D \geq 2) = 1$, then the extinction probability $\tilde{z}$ equals 0, implying that $\xi = 1$. The size of the giant is hence determined by the balance between degree 1 vertices and vertices of larger degree. Increasing the variance in a distribution with a fixed mean typically implies an increase in the number of low degree vertices, and our main message is that the distribution over small degrees is in fact more important for the size of the giant component than the precise distribution over very large degrees. In particular, the tail behavior of the degree distribution does not play the same crucial role for the size of the giant as it does for certain other quantities such as e.g. the scaling of the distances in the giant component [7, 8].

That the distribution over small degrees can play a significant role is illustrated in Figure 2, where the degrees have a fixed tail distribution and the remaining probability is allocated at small degrees in different mean-preserving ways. In Figure 2(a), the degrees distribution is fixed for $d > 3$ (we consider a Poisson(2) distribution and a power-law with exponent -3) and the remaining probability is allocated at the degrees 1, 2 and 3. Specifically, the probability p_1 is varied and p_2 and p_3 are then adjusted so that the mean is kept fixed at $\mu = 2.2$. Figure 2(a) shows plots of the component size against p_1 and we see that, although the tails remain the same, the component size changes with p_1 in both cases. Figure 2(b) shows as similar plot when



(a) Blue: $p_d = \mathbb{P}(\mathrm{Po}(2) = d)$
Red: $p_d = 2d^{-3}$

(b) Blue: $p_d = \mathbb{P}(\mathrm{Po}(7) = d)$
Red: $p_d = 5d^{-2.5}$

Figure 2: Asymptotic size ξ of the giant plotted against p_1 with (a): p_d fixed for $d \geq 4$ and mean $\mu = 2.2$ (b): and p_d fixed for $d \geq 10$ and mean $\mu = 3.5$ in (b).

the tail is fixed for $d > 10$ (Poisson and power-law) and the mean is equal to 3.5.

Bounds for a given distribution over small degrees

We also argue that, conversely, fixing the distribution over small degrees typically leaves little room for controlling the component size by tuning the tail. This requires bounds for the component size for a given distribution over small degrees. To formulate our results here, let $\mathbf{p}_L = \{p_1, \ldots, p_L\}$ denote a fixed set of probabilities associated with degrees $1, \ldots, L$ for some $L \geq 1$, and write $\mathcal{F}(\mu, \mathbf{p}_L)$ for the set of all distributions having mean μ and those specific probabilities up to L. It turns out that a lower bound for the component size for distributions in $\mathcal{F}(\mu, \mathbf{p}_L)$ is obtained by placing all remaining mass $p_{>L} = 1 - \sum_{i=1}^{L} p_i$ at the point $L+1$ and, under a mild technical condition, an upper bound is obtained by placing all remaining mass at two specific consecutive integers. The upper bound is optimal in the sense that any smaller bound is violated by some distribution in $\mathcal{F}(\mu, \mathbf{p}_L)$, and the lower bound can be modified into an optimal one under a similar technical condition as for the upper bound.

For a fixed $\mathbf{p}_L$, consider a distribution F_{L+1} with $p_{L+1} = p_{>L}$ (and $p_i = 0$ for $i \geq L+2$), write $G_{L+1}(s)$ for its probability generating function and ξ_{L+1} for the size of the giant component in a configuration graph with this degree distribution. Note that $F_{L+1} \notin \mathcal{F}(\mu, \mathbf{p}_L)$ since the mean is not equal to μ. Furthermore, denote

$$\kappa = \mathbb{E}[D | D > L],$$

where $D \sim F$, and note that fixing $\mathbf{p}_L$ and μ implies that also κ is fixed, that is, $\kappa = \kappa(\mu, \mathbf{p}_L)$ does not depend on other properties of F. Next, let $F_\kappa = \{p_i^{(\kappa)}\}$ be a distribution where all remaining mass is placed at the two integers $\lfloor \kappa \rfloor$ and $\lceil \kappa \rceil$ (or one integer if κ is an integer) in such a way that the mean is preserved, that is,

$$p_i^{(\kappa)} = \begin{cases} p_i & \text{for } i = 0, \ldots, L; \\ (\lfloor \kappa \rfloor + 1 - \kappa) p_{>L} & \text{for } i = \lfloor \kappa \rfloor; \\ (\kappa - \lfloor \kappa \rfloor) p_{>L} & \text{for } i = \lfloor \kappa \rfloor + 1. \end{cases}$$

Write G_κ for the associated generating function and ξ_κ for the component size in the corresponding configuration graph. Finally, let $\tilde{z}_{L+1}$ and $\tilde{z}_\kappa$ denote the extinction probabilities in branching processes with offspring distributions given by down-shifted size biased versions of the above

distributions. Our bounds on the component size with fixed initial probabilities $\mathbf{p}_L$ and fixed mean μ are as follows.

Theorem 1.1. *Fix $\mathbf{p}_L$ and μ.*

(a) *We have that $\xi_F \geq \xi_{L+1}$ for all $F \in \mathcal{F}(\mu, \mathbf{p}_L)$.*

(b) *If $\mathbf{p}_L$ and μ are such that $L+1 \geq |(\log \tilde{z}_{L+1})^{-1}|$, then*

$$\xi_F \geq 1 - G_{L+1}\left(\tilde{z}_{L+1}^{(\mu)}\right) \quad \text{for all } F \in \mathcal{F}(\mu, \mathbf{p}_L),$$

where $\tilde{z}_{L+1}^{(\mu)}$ is the smallest non-negative solves to the equation $s = G'_{L+1}(s)/\mu$.

(c) *If $\mathbf{p}_L$ and μ are such that $L+1 \geq 2\,|(\log \tilde{z}_\kappa)^{-1}|$, then*

$$\xi_F \leq \xi_\kappa \quad \text{for all } F \in \mathcal{F}(\mu, \mathbf{p}_L).$$

The bounds in (b) and (c) are optimal under the given conditions, that is, in (c), for any $\xi < \xi_\kappa$, there exists a distribution $F_0 \in \mathcal{F}(\mu, \mathbf{p}_L)$ such that $\xi_{F_0} > \xi$, and similarly for (b).

Remark 1. The restrictions on $\mathbf{p}_L$ and μ in (b) and (c) are imposed for technical reasons. They imply that, if the extinction probabilities $\tilde{z}_{L+1}$ and $\tilde{z}_\kappa$ are close to 1, then L has to be large, that is, a sufficiently large part of the distribution has to be fixed. We believe that this serves to avoid e.g. situations where $\mathcal{F}(\mu, \mathbf{p}_L)$ contains both subcritical and supercritical distributions. For most distributions, the conditions are mild, in the sense that they are satisfied already for moderate values of L (in relation to μ); see Table 1 for examples. □

Remark 2. The distribution F_{L+1} can be thought of as the limiting case of a distribution F_m where most of the remaining mass $p_{>L}$ is placed at $L+1$ and a vanishing amount on another integer $m \to \infty$; see the proof of Theorem 1.1(b). The mean in this distribution F_m is kept fixed at μ, and the bound in (b) differs from the component size ξ_{L+1} obtained for the distribution F_{L+1} in that the correct mean μ is used instead of the mean of F_{L+1} in the equation defining $\tilde{z}_{L+1}^{(\mu)}$ (explaining the notation). Note that the spread in the distribution of the remaining mass is maximized in the distribution F_m. In the distribution F_κ, on the other hand, the mass is concentrated as much as possible (while still keeping the mean fixed). We remark that it can be shown that the size of the giant is always, without any technical conditions, maximized by concentrating the remaining mass at $L+1$ and two consecutive integers; see [4], which generalizes a result from [3]. The general result however does not identify the two integers, and is therefore not quantitatively useful for us. □

Table 1 contains numerical values of the bounds in Theorem 1.1 for a number of different distributions (that all fulfill the technical conditions). We note that, in all cases, the upper and lower bound on the size of the giant are very close, supporting the claim that, if the distribution over low degrees is fixed, then the size of the giant is not affected much by the tail of the distribution.

The rest of the paper is organized so that the proof of Theorem 1.1 is given in Section 2, and some suggestions of further work in Section 3.

2 Proof of Theorem 1.1

Assume throughout this section that $\mathbf{p}_L$ and μ are fixed.

$\mathbf{p}_L = (p_1, \ldots, p_L)$	L	$p_{>L}$	μ	ξ_{L+1}	$1 - G_{L+1}(\tilde{z}_{L+1}^{(\mu)})$	ξ_κ
$(0.31, 0.31, 0.21) = \mathrm{Po}(2)_{>0}$	3	0.17	3	0.9140	0.9504	0.9508
$(0.43, 0.32) = \mathrm{Po}(1.5)_{>0}$	2	0.25	3	0.5896	0.9019	0.9103
$(0.2, 0.4)$	2	0.4	2.5	0.9537	0.9665	0.9695
$(0.2, 0.4)$	2	0.4	5	0.9537	0.9894	0.9896
$(0.7, 0, 0)$	3	0.3	2.5	0.7023	0.7247	0.7318
$(0.7, 0, 0)$	3	0.3	3	0.7023	0.8319	0.8366
$(0.5, 0.25, 0.125)$	3	0.125	2	0.7047	0.7553	0.7680
$(0.5, 0.25, 0.125)$	3	0.125	3	0.7047	0.8836	0.8851
$(0.9, 0, 0, 0)$	4	0.1	2	0.2848	0.5827	0.5948
$(0.86, 0.03, 0.02, 0.01)$	4	0.1	2	0.2674	0.6101	0.6215

Table 1: Bounds for the size of the giant component from Theorem 1.1. The first two examples are Poisson probabilities conditional on the degree being strictly positive. All distributions satisfy the technical conditions in Theorem 1.1(b) and (c).

Proof of Theorem 1.1(a). Fix a distribution $F \in \mathcal{F}(\mu, \mathbf{p}_L)$. Since the component size ξ is given by (2), and ξ_{L+1} by the analogous expression for the distribution F_{L+1}, we need to show that $G(\tilde{z}) \leq G_{L+1}(\tilde{z}_{L+1})$. It is clear that $G(s) \leq G_{L+1}(s)$ for any s, and hence, since generating functions are increasing, this follows if we show that $\tilde{z} \leq \tilde{z}_{L+1}$. Let $\{\tilde{p}_i^{(L+1)}\}_{i=0}^{L}$ denote the probabilities defining the down-shifted size biased version $\tilde{F}_{L+1}$ of F_{L+1} and recall that $\{\tilde{p}_i\}$, defined in (1), denote the corresponding probabilities for F. It is not hard to see that $\tilde{p}_i \leq \tilde{p}_i^{(L+1)}$ for all $i = 0, \ldots, L$ (and $\tilde{p}_i^{(L+1)} = 0$ for $i \geq L+1$). Hence $\tilde{F}_{L+1}$ is stochastically smaller than $\tilde{F}$, implying that $\tilde{z} \leq \tilde{z}_{L+1}$, as desired. □

Proof of Theorem 1.1(b). We begin by defining a sequence of distributions $\{F_m\}_{m \geq \kappa}$ where a vanishing (as $m \to \infty$) fraction of the remaining mass is placed at m and the rest at $L+1$, in such a way that the mean of the distribution is fixed. Specifically, $F_m = \{p_i^{(m)}\}_{i \geq 1}$ is defined by

$$p_i^{(m)} = \begin{cases} p_i & \text{for } i = 0, \ldots, L; \\ (1 - r_m) p_{>L} & \text{for } i = L+1 \text{ where } r_m = \frac{\kappa - (L+1)}{m - (L+1)}; \\ r_m p_{>L} & \text{for } i = m. \end{cases}$$

Note that $F_m \in \mathcal{F}(\mu, \mathbf{p}_L)$. Consider the extinction probability $\tilde{z}_m$ of a branching process with offspring distribution given by a down-shifted size biased version $\tilde{F}_m$ of F_m. We will show that (i) $\tilde{z}_m$ is increasing for large m and converges to the solution $\tilde{z}_{L+1}^{(\mu)}$ of the equation $s = G'_{L+1}(s)/\mu$, and (ii) that $G_F(\tilde{z}_F) \leq G_{L+1}(\tilde{z}_{L+1}^{(\mu)})$ for all $F \in \mathcal{F}(\mu, \mathbf{p}_L)$. First note that it follows from the proof of Theorem 1.1(a) that $\tilde{z}_m \leq \tilde{z}_{L+1}$, and the assumption $L + 1 \geq |(\log \tilde{z}_{L+1})^{-1}|$ ensures that $\tilde{z}_{L+1} < 1$. The extinction probability $\tilde{z}_m$ solves the equation $s = G'_m(s)/\mu$ and hence it follows that $\tilde{z}_m$ is increasing for large m if $G'_m(\tilde{z}_m) \leq G'_{m+1}(\tilde{z}_m)$ when m is large – indeed, the smallest solution $\tilde{z}_{m+1}$ of $s = G'_{m+1}(s)/\mu$ must then be larger than $\tilde{z}_m$. It is straightforward to bound

$$G'_{m+1}(\tilde{z}_m) - G'_m(\tilde{z}_m) \geq (L+1)(r_m - r_{m+1}) - \tilde{z}_{L+1}^{m-(L+1)},$$

which is positive for large m since $r_m - r_{m+1}$ is positive and of order m^{-2} while the last term is exponentially small. Since $\tilde{z}_m$ is increasing for large m and bounded from above by $\tilde{z}_{L+1} < 1$ it converges to some limit that is strictly smaller than 1. That this limit solves the equation $s = G'_{L+1}(s)/\mu$ follows by taking limits on both sides in the equality $\tilde{z}_m = G'_m(\tilde{z}_m)/\mu$. We denote the limit $\tilde{z}_{L+1}^{(\mu)}$.

As for (ii), fix a distribution $F \in \mathcal{F}(\mu, \mathbf{p}_L)$. Since $G_F(s) \leq G_{L+1}(s)$ for all s and generating functions are increasing, the desired conclusion follows if $\tilde{z}_F \leq \tilde{z}_{L+1}^{(\mu)}$, which in turn follows if $G'_F(\tilde{z}_F) \leq G'_{L+1}(\tilde{z}_F)$, since the smallest solution $\tilde{z}_{L+1}^{(\mu)}$ of $s = G'_{L+1}(s)/\mu$ must then be larger than $\tilde{z}_F = G'_F(\tilde{z}_F)/\mu$. The assumption $L + 1 \geq |(\log \tilde{z}_{L+1})^{-1}|$ ensures that functions of the form $f(d) = ds^{d-1}$, with $s \leq \tilde{z}_{L+1}$, are strictly decreasing for $d \geq L+1$. Since $\tilde{z}_F \leq \tilde{z}_{L+1}$ (as shown in part (a)), this means that

$$\sum_{d=L+1}^{\infty} d\tilde{z}_F^{d-1} p_d \leq (L+1)\tilde{z}_F^L p_{>L},$$

which implies that $G'_F(\tilde{z}_F) \leq G'_{L+1}(\tilde{z}_F)$, as desired.

Finally note that the derived bound is optimal: Since $G_m(\tilde{z}_m) \nearrow G_{L+1}(\tilde{z}_{L+1}^{(\mu)})$, for any $\xi > 1 - G_{L+1}(\tilde{z}_{L+1}^{(\mu)})$ there exist an m such that $\xi_{F_m} < \xi$ □

The following simple lemma will be used in the proof of Theorem 1.1. It will be applied to $N = D|D > L$ and the mean is therefore denoted by κ.

Lemma 2.1. *Let N be an integer valued random variable with mean κ. There exist integer valued random variables N_1 and N_2 with $\mathbb{E}[N_1] = \lfloor \kappa \rfloor$ and $\mathbb{E}[N_2] = \lfloor \kappa \rfloor + 1$ such that, with $Z \sim Be(\lfloor \kappa \rfloor + 1 - \kappa)$, we have that*

$$N \stackrel{d}{=} ZN_1 + (1 - Z)N_2.$$

Proof. Let $N_{\text{low}} = N|N \leq \lfloor \kappa \rfloor$ and $N_{\text{hi}} = N|N > \lfloor \kappa \rfloor$ and write κ_{low} and κ_{hi} for the respective means. Furthermore, let X and Y be Bernoulli variables with parameter $\frac{\kappa_{\text{hi}} - \lfloor \kappa \rfloor}{\kappa_{\text{hi}} - \kappa_{\text{low}}}$ and $\frac{\kappa_{\text{hi}} - \lfloor \kappa \rfloor - 1}{\kappa_{\text{hi}} - \kappa_{\text{low}}}$, respectively. Then set

$$N_1 = XN_{\text{low}} + (1 - X)N_{\text{hi}} \qquad N_2 = YN_{\text{low}} + (1 - Y)N_{\text{hi}}.$$

It is straightforward to confirm that $\mathbb{P}(ZN_1 + (1 - Z)N_2 = i) = \mathbb{P}(N = i)$ for all i. □

Proof of Theorem 1.1 (c). We need to show that $G_F(\tilde{z}_F) \geq G_\kappa(\tilde{z}_\kappa)$ for all $F \in \mathcal{F}(\mu, \mathbf{p}_L)$. To this end, we begin by showing that

$$G_F(s) \geq G_\kappa(s) \quad \text{for all } s \in [0, 1] \text{ and all } F \in \mathcal{F}(\mu, \mathbf{p}_L). \tag{3}$$

Pick $F \in \mathcal{F}(\mu, \mathbf{p}_L)$ and let $D \sim F$. The probability generating function $G_F(s)$ can be written as

$$G_F(s) = \mathbb{E}\left[s^D\right] = \mathbb{E}\left[s^D | D \leq L\right](1 - p_{>L}) + \mathbb{E}\left[s^D | D > L\right] p_{>L}.$$

Applying Lemma 2.1 with $N = D|D > L$, we can write

$$\mathbb{E}\left[s^D | D > L\right] = \mathbb{E}\left[s^{N_1}\right] \mathbb{P}(Z = 1) + \mathbb{E}\left[s^{N_2}\right] \mathbb{P}(Z = 0),$$

where $\mathbb{E}[N_1] = \lfloor \kappa \rfloor$, $\mathbb{E}[N_2] = \lfloor \kappa \rfloor + 1$ and $Z \sim \text{Be}(\lfloor \kappa \rfloor + 1 - \kappa)$. Jensen's inequality then yields that

$$\mathbb{E}\left[s^D | D > L\right] \geq s^{\lfloor \kappa \rfloor} \mathbb{P}(Z = 1) + s^{\lfloor \kappa \rfloor + 1} \mathbb{P}(Z = 0).$$

The probability generating function $G_\kappa(s)$ can be written as

$$G_\kappa(s) = \mathbb{E}\left[s^D | D \leq L\right](1 - p_{>L}) + \left(s^{\lfloor \kappa \rfloor} \mathbb{P}(Z = 1) + s^{\lfloor \kappa \rfloor + 1} \mathbb{P}(Z = 0)\right) p_{>L},$$

and hence (3) follows. Since generating functions are increasing, the desired bound now follows if $\tilde{z}_F \geq \tilde{z}_\kappa$, which in turn follows if $G'_F(\tilde{z}_\kappa) \geq G'_\kappa(\tilde{z}_\kappa)$. The assumption $L+1 \geq 2\,|(\log \tilde{z}_\kappa)^{-1}|$ ensures that this is the case: Let $D_\kappa \sim F_\kappa$. Note that, since the two distributions F and F_κ agree up to L, the desired inequality follows if $\mathbb{E}[D\tilde{z}_\kappa^D | D > L] \geq \mathbb{E}[D_\kappa \tilde{z}_\kappa^{D_\kappa} | D_\kappa > L]$. We have that

$$\mathbb{E}\left[D_\kappa \tilde{z}_\kappa^{D_\kappa} | D_\kappa > L\right] = \lfloor \kappa \rfloor \tilde{z}_\kappa^{\lfloor \kappa \rfloor}(\lfloor \kappa \rfloor + 1 - \kappa) + (\lfloor \kappa \rfloor + 1)\tilde{z}_\kappa^{\lfloor \kappa \rfloor + 1}(\kappa - \lfloor \kappa \rfloor).$$

By Lemma 2.1, with $N = D|D > L$, we can write

$$\mathbb{E}\left[D\tilde{z}_\kappa^D | D > L\right] = \mathbb{E}\left[N_1 \tilde{z}_\kappa^{N_1}\right](\lfloor \kappa \rfloor + 1 - \kappa) + \mathbb{E}\left[N_2 \tilde{z}^{N_2}\right](\kappa - \lfloor \kappa \rfloor),$$

where $N_1, N_2 > L$, $\mathbb{E}[N_1] = \lfloor \kappa \rfloor$ and $\mathbb{E}[N_2] = \lfloor \kappa \rfloor + 1$. The assumption $L+1 \geq 2\,|(\log \tilde{z}_\kappa)^{-1}|$ implies that $f(d) = d\tilde{z}_\kappa^d$ is convex for $d \geq L+1$. It follows from a straightforward modification of Jensen's inequality (specifically, a restriction to $[L+1, \infty)$) that

$$\mathbb{E}\left[N_1 \tilde{z}_\kappa^{N_1}\right] \geq \lfloor \kappa \rfloor \tilde{z}^{\lfloor \kappa \rfloor}, \qquad \mathbb{E}\left[N_2 \tilde{z}_\kappa^{N_2}\right] \geq (\lfloor \kappa \rfloor + 1)\tilde{z}^{\lfloor \kappa \rfloor + 1}$$

and the bound follows. That the bound is optimal follows by noting that $F_\kappa \in \mathcal{F}(\mu, \mathbf{p}_L)$, that is, the distribution defining the bound is included in the class. □

References

[1] Angel, O., van der Hofstad, R. and Holmgren, C. (2016): Limit laws for self-loops and multiple edges in the configuration model, *arxiv.org/abs/1603.07172*.

[2] Britton, T., Deijfen, M. and Martin-Löf, A. (2006): Generating simple random graphs with prescribed degree distribution, *J. Stat. Phys.* **124**, 1377-1397.

[3] Britton, T. and Trapman, P. (2012): Maximizing the size of the giant, *J. Appl.Probab.* **49**, 1156-1165.

[4] Deijfen, M., Rosengren, S. and Trapman, P. (2017): Maximizing the size of the giant for a given distribution over small degrees, work in progress (preprint available upon request).

[5] van der Hofstad, R. (2017): *Random graphs and complex networks*, Volume I, Cambridge Univeristy Press.

[6] van der Hofstad, R. (2015): *Random graphs and complex networks*, Volume II, available at http://www.win.tue.nl/∼rhofstad.

[7] van der Hofstad, R., Hooghiemstra, G. and van Mieghem, P. (2005): Distances in random graphs with finite variance degrees, *Rand. Struct. Alg.* **26** 76-123.

[8] van der Hofstad, R., Hooghiemstra, G. and Znamenski, D. (2007): Distances in random graphs with finite mean and infinite variance degrees, *Electr. J. Probab.* **12** 703-766.

[9] van der Hofstad, R., Hooghiemstra, G. and Znamenski, D. (2009): A phase transition for the diameter of the configuration model, *Internet Math.* **4** 113-128.

[10] Janson, S. (2009): The probability that a random multigraph is simple, *Comb. Probab. Computing* **18**, 205-225.

[11] Janson, S. and Luczak, M. (2009): A new approach to the giant component problem, *Rand. Struct. Alg.* **34**, 197-216.

[12] Leskelä, L. and Ngo, H. (2017): The impact of degree variability on connectivity properties of large networks, *Internet Math.* **13**.

[13] Molloy, M. and Reed, B. (1995): A critical point for random graphs with a given degree sequence, *Rand. Struct. Alg.* **6**, 161-179.

[14] Molloy, M. and Reed, B. (1998): The size of the giant component of a random graphs with a given degree sequence, *Comb. Prob. Comp.* **7**, 295-305.